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Note on the Fried–Yennie Gauge*

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It is pointed out that the renormalization constants and the finite parts of the self energy and the vertex corrections in the infrared-free Fried–Yennie gauge are different, by finite amounts, from those which are computed as the limit of the general covariant gauge. This discrepancy occurs because of the appearance of an infrared divergence in the on-mass-shell renormalization. This note provides an example where extra caution is needed in handling the limiting procedure, where an infrared divergence is involved.

I. INTRODUCTION

Renormalization in quantum electrodynamics in general introduces infrared divergences due to an expansion on the mass shell. While gauge invariance guarantees the cancellation of such divergences in the computation of observable quantities, individual diagrams are subjected to this complication and must be treated with some care. An exceptional case is the Fried–Yennie (FY) gauge [1] in which infrared divergences do not appear even for individual diagrams.

The photon propagator in the FY gauge is expressed as

$$D_{\mu\nu}(k) = \frac{-i}{k^2}\left(\delta_{\mu\nu} + 2\,\frac{k_\mu k_\nu}{k^2}\right), \tag{1.1}$$

which is a special case of the propagator in the general covariant gauge

$$D^{\xi}_{\mu\nu}(k) = \frac{-i}{k^2}\left(\delta_{\mu\nu} + \xi\,\frac{k_\mu k_\nu}{k^2}\right), \tag{1.2}$$

where a nondimensional constant ξ is the gauge parameter. In this article, it is shown that, although $\xi = 2$ is not a singular point of Eq. (1.2), special care is needed when the results of the FY gauge and the general covariant gauge are compared. Some complication stems from the interplay of the infrared divergence and the one-mass-shell expansion. In Section II, the renormalization of the self energy in lowest order in α is discussed. A similar problem is considered for the vertex diagram in Section III.

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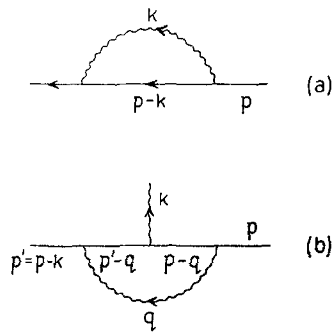

FIG. 1. The lowest-order radiative correction: (a) self energy, (b) vertex correction.

II. THE SELF ENERGY

The self energy in lowest order, shown in Fig. 1a, is expressed as

$$\Sigma(p) = \frac{-ie^2}{(2\pi)^4} \int \gamma_\mu \frac{1}{i\gamma(p-k)+m-i\epsilon} \gamma_\nu \left(\frac{\delta_{\mu\nu}}{k^2+\lambda^2-i\epsilon} + \frac{\xi k_\mu k_\nu}{(k^2+\lambda^2-i\epsilon)^2} \right)$$

$$\times \frac{d^{2\omega}k}{(2\pi\mu)^{2(\omega-2)}}$$

$$\equiv \Sigma^{(1)}(p) + \xi\Sigma^{(2)}(p), \tag{2.1}$$

where dimensional regularization [2] is used, μ is an undefined quantity of the dimension of mass and λ is the photon mass. Whenever "harmless," the limits $\omega \to 2$ and $\lambda \to 0$ are understood. Using the Feynman parameterization and the integrals in the Appendix, we obtain

$$\Sigma^{(1)}(p) = -\frac{e^2}{(4\pi)^2} \int_0^1 dx(1-x)$$

$$\times \left[\left(\frac{H}{4\pi\mu^2} \right)^{\omega-2} \frac{\Gamma(2-\omega)}{\Gamma(2)} [2i(1-\omega)(1-x)\gamma p - 2\omega m] \right] \tag{2.2}$$

and

$$\Sigma^{(2)}(p) = -\frac{e^2}{(4\pi)^2} \int_0^1 dx(1-x)$$

$$\times \left[\frac{1}{2} \left(\frac{H}{4\pi\mu^2} \right)^{\omega-2} \Gamma(2-\omega)\{[2(1-x) - 2\omega(1+x)]\, i\gamma p - 2\omega m\} \right.$$

$$\left. - \frac{H^{\omega-3}}{(4\pi\mu^2)^{\omega-2}} [i\gamma p(1-x) - m]\, p^2 x^2 \right], \tag{2.3}$$

where

$$H/m^2 = \rho x(1-x) + x^2 + \nu(1-x) \tag{2.4}$$

and

$$\rho = \frac{p^2+m^2}{m^2}, \qquad \nu = \lambda^2/m^2. \tag{2.5}$$

Noting the formulae

$$\left(\frac{H}{4\pi\mu^2}\right)^{\omega-2} = 1 - (2-\omega)\ln\left(\frac{H}{4\pi\mu^2}\right) + O((2-\omega)^2) \tag{2.6}$$

and

$$\Gamma(2-\omega) = \frac{1}{2-\omega} + \psi(1) + O(2-\omega), \tag{2.7}$$

the coefficients in the expansion

$$\Sigma^{(i)}(p) = A_i + B_i(i\gamma p + m) + C_i(i\gamma p + m)^2, \qquad i = 1, 2 \tag{2.8}$$

can be calculated [3]

$$\begin{aligned}
A_1 &= \frac{\alpha m}{4\pi}(3D+4),\\
B_1 &= \frac{\alpha}{4\pi}(D + 4 + 2\ln\nu),\\
C_1 &= \frac{\alpha}{2\pi m}\left[\frac{1}{2}\left(\frac{1}{1-\rho} - \frac{2-3\rho}{(1-\rho)}\ln\rho\right)\right.\\
&\quad \left. + \frac{(-i\gamma p + m)}{\rho m}\left(\frac{-4+3\rho}{2(1-\rho)} + \frac{4-4\rho-\rho^2}{2(1-\rho)^2}\ln\rho - \ln\nu\right)\right]
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
A_2 &= 0,\\
B_2 &= \frac{\alpha}{4\pi}(D - \ln\nu),\\
C_2 &= \frac{\alpha}{4\pi m}\left[\frac{-1}{1-\rho} + \left(\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\right)\ln\rho\right.\\
&\quad \left. + \frac{-i\gamma p + m}{\rho m}\left(-\frac{\rho}{1-\rho} - \frac{2-2\rho+\rho^2}{(1-\rho)^2}\ln\rho + \ln\nu\right)\right],
\end{aligned} \tag{2.10}$$

where

$$\psi(1) = -\gamma = -0.57721...$$

and

$$D = \frac{1}{2-\omega} + \psi(1) + \ln\frac{4\pi\mu^2}{m^2} \tag{2.11}$$

represents the ultraviolet divergent term in the renormalization constants. The fact that $A_2 = 0$ can be seen from the identity

$$\begin{aligned}\int (\gamma k)\frac{1}{i\gamma(p-k)+m}(\gamma k)\frac{d^{2\omega}k}{k^4} &= i\int\frac{\gamma k}{k^4}d^{2\omega}k + \frac{1}{i}\int (i\gamma p + m)\frac{1}{i\gamma(p-k)+m)}(\gamma k)\frac{d^2\omega_k}{k^4}\\ &= -i(i\gamma p + m)\int\frac{1}{i\gamma(p-k)+m}(\gamma k)\frac{d^{2\omega}k}{k^4}. \end{aligned}\tag{2.12}$$

A few comments are in order. First, we notice that the infrared divergences in Eqs. (2.9) and (2.10) are introduced by the on-mass-shell expansion, Eq. (2.8). In fact, they can be rewritten as

$$\begin{aligned}\Sigma^{(1)}(p) = \frac{\alpha}{4\pi}\Big\{ & m(3D+4) + (i\gamma p + m)(D+2)\\ & + \rho(i\gamma p + m)\Big[\frac{1}{1-\rho} + \Big(\frac{2}{1-\rho} + \frac{\rho}{(1-\rho)^2}\Big)\ln\rho\Big\}\\ & + \rho m\Big\{-\frac{1}{1-\rho} + \Big(\frac{2}{1-\rho} - \frac{\rho}{(1-\rho)^2}\Big)\ln\rho\Big\}\Big]\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\Sigma^{(2)}(p) = \frac{\alpha}{4\pi}\Big[& (i\gamma p + m)(D+2)\\ & + \rho(i\gamma p + m)\Big\{\frac{1}{1-\rho} + \Big(\frac{2}{1-\rho} + \frac{\rho}{(1-\rho)^2}\Big)\ln\rho\Big\}\\ & - \rho m\Big\{\frac{1}{1-\rho} + \Big(\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\Big)\ln\rho\Big\}\Big],\end{aligned}\tag{2.14}$$

which are free of infrared divergences. However, they cannot be expanded in powers of $(i\gamma p + m)$ because of singularities at $\rho = 0$, unless a photon mass is introduced. The logarithmic singularities in ρ in Eqs. (2.13) and (2.14) are intimately related to the infrared divergence, $\ln\nu$ terms in Eqs. (2.9) and (2.10).

An inspection of Eqs. (2.9) and (2.10) reveals further singularities at $\rho = 0$ in the expressions for C_1 and C_2, but they are due to the limit taken in the evaluation of the integrals. For example, the computation of C_1 involves the integral

$$I(p) = \frac{1}{\rho}\int_0^1 \left(\frac{x}{\rho x(1-x) + x^2 + \nu(1-x)} - \frac{x}{x^2 + \nu(1-x)}\right) dx, \tag{2.15}$$

which is regular at $\rho = 0$ as long as $\nu \neq 0$. If, however, we are interested in evaluating $I(\rho)$ for $\rho \neq 0$, the infrared parameter ν in the first term of the integrand can be set equal to zero, leading to

$$I(p) = \frac{1}{\rho}\left[\frac{-1}{1-\rho}\ln\rho + \frac{1}{2}\ln\nu\right], \qquad \rho \neq 0. \tag{2.16}$$

This is the source of the apparent singularities in C_1 (and likewise in C_2).

Finally we can easily see that the infrared singularities disappear in the renormalization expansion in the full self energy for $\xi = 2$. In other words $B_1 + 2B_2$ and $C_1 + 2C_2$ are free of infrared divergence (the FY gauge).

From Eqs. (2.9) and (2.10) or Eqs. (2.13) and (2.14) it follows that

$$\begin{aligned}\Sigma_{\mathrm{FY}} &= \Sigma^{(1)} + 2\Sigma^{(2)}\\ &= \frac{\alpha}{4\pi}\Big[(3D+4)\,m + 3D(i\gamma p + m)\\ &\quad + \frac{(i\gamma p + m)^2}{m}\left\{\frac{3i\gamma p}{m}\left(\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\ln\rho\right)\right\}\Big],\end{aligned} \tag{2.17}$$

where the identity

$$\rho m = 2(i\gamma p + m) - (i\gamma p + m)^2/m \tag{2.18}$$

is used. Defining

$$\Sigma_{\mathrm{FY}} = A_{\mathrm{FY}} + B_{\mathrm{FY}}(i\gamma p + m) + C_{\mathrm{FY}}(ip\gamma + m)^2, \tag{2.19}$$

we have

$$\begin{aligned}A_{\mathrm{FY}} &= \frac{\alpha m}{4\pi}(3D+4),\\ B_{\mathrm{FY}} &= \frac{\alpha}{4\pi}3D,\\ C_{\mathrm{FY}} &= \frac{3\alpha i\gamma p}{4\pi m^2}\left(\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\ln\rho\right).\end{aligned}$$

Note that

$$B_{\mathrm{FY}} = B_1 + 2B_2 - \frac{\alpha}{2\pi},$$
$$C_{\mathrm{FY}} = C_1 + 2C_2 + \frac{\alpha}{2\pi}(i\gamma p + m)^{-1} \tag{2.21}$$

and

$$\lim_{i\gamma p + m = 0} C_{\mathrm{FY}} \qquad \text{is finite.}$$

Of course, the equality $\Sigma_{\mathrm{FY}} = \Sigma^{(1)} + 2\Sigma^{(2)}$

$$B_{\mathrm{FY}} + (i\gamma p + m)\, C_{\mathrm{FY}} = (B_1 + 2B_2) + (i\gamma p + m)(C_1 + 2C_2) \tag{2.22}$$

is maintained. As is clear by now, the discrepancy, Eq. (2.21), is due to the limiting procedure involving the infrared divergence and the one-mass-shell expansion in the renormalization scheme. In fact, the source of the discrepancy is twofold. In order to see this point more clearly, we consider an integral in the FY gauge which is a difference of two infrared divergent terms

$$I(\nu) = \int_0^1 K(x, \nu)\, dx, \tag{2.23}$$

where

$$K(x, \nu) = \frac{2x}{x^2 + \nu(1 - x)} - \frac{2x^3}{(x^2 + \nu(1 - x))^2} = \frac{2\nu x(1 - x)}{(x^2 + \nu(1 - x))^2}\,. \tag{2.24}$$

Clearly

$$K(x, 0) = 0 \tag{2.25}$$

while [3]

$$I(\nu) = 1 - \pi\nu^{1/2}/4 + O(\nu). \tag{2.26}$$

The integral $I(\nu)$ which appears in the FY gauge should be set equal to zero. On the other hand, $I(\nu)$, which appears in the limit $\xi \to 2$, contributes a finite term.

Another source of discrepancy is the $1/\rho$ singularity in $C_1 + 2C_2$ (see Eqs. (2.9) and (2.10) which, without the infrared singularities, becomes a real singularity: From the identity

$$(i\gamma p + m)^2 \frac{1}{\rho m} = \frac{1}{2}(i\gamma p + m) + (i\gamma p + m)^2 \frac{i\gamma p + m}{2\rho m^2} \tag{2.27}$$

it follows that the first term on the r.h.s. of Eq. (2.27) should be included in the renormalization constant B_{FY}.

Incidentally, it should be noted that the expression for the self energy is remarkably simpler in the FY gauge than in any other gauge.

III. The Vertex Correction

The lowest-order vertex correction, depicted in Fig. 1b, is expressed as

$$\begin{aligned}\Gamma_\mu(p', p) = {} & ie^2 \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}\,(\mu^2)^{\omega-2}} \\ & \times \left(\gamma_\kappa \frac{1}{i\gamma(p'-q)+m-i\epsilon}\,\gamma_\mu\,\frac{1}{i\gamma(p-q)+m-i\epsilon}\,\gamma_\nu\right) \\ & \times \left(\frac{\delta_{\kappa\nu}}{q^2+\lambda^2-i\epsilon} + \frac{\xi q_\kappa q_\nu}{(q^2+\lambda^2-i\epsilon)^2}\right) \end{aligned} \tag{3.1}$$

$$= \Gamma_\mu^{(1)}(p', p) + \xi \Gamma_\mu^{(2)}(p', p), \tag{3.2}$$

where

$$\Gamma_\mu^{(1)}(p', p) = \frac{ie^2}{(2\pi)^4}\,\frac{1}{(2\pi\mu)^{2(\omega-2)}} \int d^{2\omega}q \int_0^1 2\,dx \int_0^x dy\, \frac{N_\mu^{(1)}}{(q^2+F)^3} \tag{3.3}$$

with

$$F = m^2\{x^2 + \rho(1-x)(x-y) + \rho' y(1-x) + \kappa y(x-y) + \nu(1-x)\}, \tag{3.4}$$

$$\rho = \frac{p^2+m^2}{m^2}, \qquad \rho' = \frac{p'^2+m^2}{m^2}, \qquad \kappa = \frac{k^2}{m^2}, \qquad \nu = \frac{\lambda^2}{m^2} \tag{3.5}$$

and

$$\Gamma_\mu^{(2)}(p', p) = \frac{ie^2}{(2\pi)^4}\,\frac{1}{(2\pi\mu)^{2(\omega-2)}} \int d^{2\omega}q \int_0^1 dx \int_0^x dy\, \frac{6(1-x)}{(q^2+F)^4}\, N_\mu^{(2)}. \tag{3.6}$$

The numerators in the integrands $N_\mu^{(i)}$ are

$$N_\mu^{(1)} = \gamma_\nu(-i\gamma(p'-q-px+ky)+m)\,\gamma_\mu(-i\gamma(p-q-px+ky)+m)\,\gamma_\nu \tag{3.7}$$

and

$$\begin{aligned} N_\mu^{(2)} = {} & \gamma(q+px-ky)(-i\gamma(p'-q-px+ky)+m) \\ & \times \gamma_\mu(-i\gamma(p-q-px+ky)+m)\,\gamma(q+px-ky). \end{aligned} \tag{3.8}$$

Expanding

$$\Gamma_\mu^{(i)}(p', p) = L^{(i)}\gamma_\mu + \Lambda_{\mu f}^{(i)}(p', p), \qquad i = 1, 2, \tag{3.9}$$

we obtain

$$L^{(1)} = \frac{ie^2}{(2\pi)^4} \frac{1}{(2\pi\mu)^{2(\omega-2)}} \int d^{2\omega}q \int_0^1 2\,dx \int_0^x dy$$
$$\times \frac{-(1 + \frac{3}{2}(\omega - 2))\, q^2 + 4m^2(1 - x - \frac{1}{2}x^2)}{(q^2 + F)^3}, \tag{3.10}$$

where

$$F = m^2(x^2 + \nu(1 - x))$$

and

$$L^{(2)} = \frac{-ie^2}{(2\pi)^4 (2\pi\mu)^{2(\omega-2)}} \int d^{2\omega}q \int_0^1 dx \int_0^y dy \frac{6(1 - x)}{(q^2 + F)^4}$$
$$\times [(q^2)^2 - m^2q^2(1 - 6x + 3x^2) + m^4x^2(2 - x)^2]. \tag{3.11}$$

In obtaining (3.10) and (3.11), we use $p_\mu = p'_\mu$, $k_\mu = 0$ and the mass shell condition

$$(i\gamma p + m)\, u(p) = 0, \qquad \bar{u}(p)(i\gamma p + m) = 0,$$

and (3.12)

$$\bar{u}(p)\, p_\mu u(p) = \mathrm{im}\, \bar{u}(p)\, \gamma_\mu u(p).$$

The integrals in the Appendix enable us to calculate $L^{(i)}$, giving

$$L^{(1)} = \frac{\alpha}{4\pi} \left\{ \omega \left(1 + \frac{3}{2}(\omega - 2)\right) \right.$$
$$\times \int_0^1 dx\, x \left[(1 - (2 - \omega) \ln \left(\frac{m^2(x^2 + \nu(1 - x))}{4\pi\mu^2}\right)\right] \Gamma(2 - \omega)$$
$$\left. - \int_0^1 dx\, x \frac{4(1 - x - (1/2)\, x^2)}{x^2 + \nu(1 - x)} \right\} = \frac{\alpha}{4\pi} [D + 4 + 2 \ln \nu] \tag{3.13}$$

and

$$L^{(2)} = \frac{\alpha}{4\pi} \left\{ \omega(\omega + 1) \int_0^1 dx\, x(1 - x) \right.$$
$$\times \left[1 - (2 - \omega) \ln \left(\frac{m^2(x^2 + \nu(1 - x))}{4\pi\mu^2}\right)\right] \Gamma(2 - \omega)$$
$$\left. - 2 \int_0^1 dx \frac{x(1 - x)(1 - 6x + 3x^2)}{x^2 + \nu(1 - x)} + \int_0^1 dx \frac{x^3(1 - x)(2 - x)^2}{(x^2 + \nu(1 - x))^2} \right\}$$
$$= \frac{\alpha}{4\pi} (D - \ln \nu). \tag{3.14}$$

Comparing these with the results of the previous section, we recognize that the Ward identities

$$L^{(i)} = B^{(i)}, \qquad i = 1, 2, \tag{3.15}$$

are satisfied.

For the FY gauge, it is not enough to consider the special limit, such as Eq. (2.26), in the expression of $L^{(1)} + 2L^{(2)}$. As was discussed at the end of the preceding section, we have to consider the whole expression for $\Gamma_\mu^{(1)} + 2\Gamma_\mu^{(2)}$, in order to decide the renormalization constant L_{FY}. Instead we use the Ward identity to coinclude that

$$L_{\text{FY}} = B_{\text{FY}} = (3\alpha/4\pi)\, D. \tag{3.16}$$

This relationship was used in simplifying the calculation of radiative corrections to positronium decay [4] in the FY gauge. Although the final result for the decay rate is the same for any gauge, the individual diagram contribution is gauge dependent and in particular, different for the case where the limit $\lim_{m_\gamma \to 0} \lim_{\xi \to 2}$ is taken and the FY gauge where $m_\gamma = 0$ and $\xi = 2$ is assumed from the beginning. A nice feature of the FY gauge is the complete cancellation between the Coulomb subtracted term for the binding diagram and the term due to an iteration of the Bethe–Salpeter equation for the zeroth-order diagram because of the absence of infrared divergence. As explained in Ref. [4], this presents the clearest discussion of the binding diagram of the positronium decay calculation.

APPENDIX: Formulae Used in the Text

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

$$\gamma_\mu\gamma_\mu = 2\omega$$

$$\gamma_\nu\gamma_\mu\gamma_\nu = -2(\omega - 1)\,\gamma_\mu$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\nu = 4\delta_{\mu\nu} + 2(\omega - 2)\,\gamma_\lambda\gamma_\mu$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu = -2\gamma_\rho\gamma_\mu\gamma_\lambda - 2(\omega - 2)\,\gamma_\lambda\gamma_\mu\gamma_\rho$$

$$\int \frac{d^{2\omega}k}{(k^2 + H)^\alpha} = \frac{i\pi^\omega H^{\omega-\alpha}\Gamma(\alpha - \omega)}{\Gamma(\alpha)}$$

$$\int \frac{k_\mu k_\nu d^{2\omega}k}{(k^2 + H)^\alpha} = \delta_{\mu\nu} \frac{i\pi^\omega H^{\omega-\alpha+1}\Gamma(\alpha - \omega - 1)}{2\Gamma(\alpha)}$$

$$\int \frac{k^2 d^{2\omega}k}{(k^2 + H)^\alpha} = \frac{i\pi^\omega \omega H^{\omega-\alpha+1}\Gamma(\alpha - \omega - 1)}{\Gamma(\alpha)}$$

$$\int \frac{k^4 d^{2\omega}k}{(k^2 + H)^\alpha} = \frac{i\pi^\omega \omega(\omega + 1)\, H^{\omega-\alpha+2}\Gamma(\alpha - \omega - 2)}{\Gamma(\alpha)}$$

$$\int_0^1 \frac{x}{x^2 + \nu(1-x)}\, dx = -\frac{1}{2}\ln \nu + \frac{\pi\nu^{1/2}}{4} + O(\nu)$$

$$\int_0^1 \frac{x^3}{(x^2 + \nu(1-x))^2}\, dx = -\frac{1}{2} - \frac{1}{2}\ln \nu + \frac{3\pi\nu^{1/2}}{8} + O(\nu)$$

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