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Investigation of a Class of Systems of Integral Equations*

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1. INTRODUCTION

A new class $\mathscr{R}$ of systems of integral equations is defined and investigated. It is of importance in the stochastic optimization theory, and in estimation problems. We give the description of the set of solutions to the system of integral equations in a class of distributions, prove the existence, uniqueness and stability in the sense defined below of the solution of minimal order of singularity. This solution is of principal interest for applications and we obtain explicit formulas for it. Convolution integral equations whose rational kernels are Fourier transform of a matrix kernel are (special) example of the equations of class $\mathscr{R}$. The theory does not require the factorization of matrix functions. The methods used is a generalization of that developed by the author in [1–7, 12–15].

2. PRELIMINARIES, ASSUMPTIONS

Consider the vector integral equation

$$Rh = \int_\Delta R(y, z)\, h(z)\, dz = f(y), \qquad y \in \Delta = [t - T, t], \quad T > 0. \tag{1}$$

Here t, T are fixed numbers, $R(y, z)$ a matrix kernel which is self-adjoint, and positive–semidefinite. Moreover it is assumed that $h, f \in \mathbb{R}^r$, f being smooth. Let us define a class $\mathscr{R}$ of kernels. Let L be a scalar self-adjoint differential operator in $H = L^2(I)$, $I = (-\infty, \infty)$,

$$\begin{aligned} Lu &= \sum_{j=0}^{s} p_j(x)\, u^{(j)}(x), \qquad p_s(x) \neq 0, \qquad x \in I, \\ u^{(j)} &= D^j u = d^j u/dx^j, \qquad s = \operatorname{order} L. \end{aligned} \tag{2}$$

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The coefficients $p_j(x)$ are assumed smooth. Let $\phi(x, y, \lambda)$, $d\rho(\lambda)$ be the spectral kernel and spectral measure of operator L and assume that Λ is the spectrum of L.

DEFINITION. We say that $R(x, y) \in \mathscr{R}$, if

$$R(x, y) = \int_\Lambda r(\lambda)\, \phi(x, y, \lambda)\, d\rho(\lambda),$$
$$r(\lambda) = (r_{ij}(\lambda)) = (P_{ij}(\lambda)\, Q_{ij}^{-1}(\lambda)), \qquad 1 \leqslant i, \quad j \leqslant d, \tag{3}$$

where the polynomials $P_{ij}(\lambda)$, $Q_{ij}(\lambda) > 0$ for $\lambda \in I$ are relatively prime; the matrix $r(\lambda)$ is ρ-almost everywhere positive definite, so that $r(\lambda)\, g(\lambda) = 0$ only on the sets with null ρ-measure provided that $g \in L^2(\Lambda, d\rho)$. For example, if $L = iD$, then $\Lambda = I$, $d\rho = d\lambda$, $\phi(x, y, \lambda) = (2\pi)^{-1} \exp\{i\lambda(x - y)\}$, so that $R(x, y)$ is a convolution kernel with a rational Fourier transform.

Let $Q(\lambda)$ be the least common multiple of the polynomials $Q_{ij}(\lambda)$, $1 \leqslant j$, $i \leqslant d$, $A_{ij}(\lambda) = P_{ij}(\lambda)\, Q_{ij}^{-1}(\lambda)\, Q(\lambda)$, $q = \deg Q(\lambda)$, and q is even. Equation (1) can be written as

$$A(L) \int_\Delta S(y, z)\, h(z)\, dz = f(y), \qquad y \in \Delta,$$
$$S(y, z) = \int_\Lambda Q^{-1}(\lambda)\, \phi(x, y, \lambda)\, d\rho(\lambda)\, E, \tag{4}$$

where $A(L)$ is the matrix differential operator with elements $A_{ij}(L)$, and E is the unit $d \times d$ matrix. Assume that $\det A(\lambda) > 0$, $\lambda \in I$; then the equation $A(L)\, v = f$ can be written as

$$\sum_{j=0}^{m} B_j(x)\, D^j v = f, \qquad \det B_m(x) \neq 0, \quad x \in \Delta. \tag{5}$$

Here $B_j(x)$ are matrix coefficients, $m = sa$, $a = \max_{1 \leqslant i,j \leqslant d} \deg A_{ij}(\lambda)$. We note that the operator $Q(L)\, E$ is of the form of (5) with m replaced by $n = sq$, n even. We assume that m is even. Let ϕ_j, $1 \leqslant j \leqslant m$, be a fundamental system of matrix solutions (FSMS) to equation $A(L)\phi = 0$, and $\psi_j^{\pm}$, $1 \leqslant j \leqslant n/2$ be a FSMS of equation $Q(L)\, E\psi = 0$, $\psi_j^+(+\infty) = 0$, $\psi_j^-(-\infty) = 0$. If $L = iD$, $Q(\lambda) > 0$ it is evident that such a dichotomy is possible. If $Q(\lambda) > 0$, and L is given by (2) with $p_s(x) = 1$ the existence of such dichotomy is also known (see [9, p. 118]). Let us write Eq. (4) in the form

$$\int_\Delta S(y, z)\, h(z)\, dz = g(y) + \sum_{j=1}^{m} \phi_j(y)\, c_j\,, \qquad y \in \Delta. \tag{6}$$

Here c_j are constant vectors, and $g(y)$ is a partial solution to equation $A(L)\, v = f$. Denote by H_α, the Sobolev space $W_2^{\alpha}(\Delta)$ for $\alpha > 0$ and by $H_{-\alpha}$ its dual space,

$H_0 = L^2(\Delta)$. If $h \in H_{-\alpha}$ we say that it is a distribution with a singularity of order of α, $\sigma(h) = \alpha$, $\operatorname{supp} h \subset \Delta$.

LEMMA 1. *Let $r_{ij}(\lambda) = \delta_{ij}A^{-1}(\lambda)$. The set of solutions to Eq.* (1) *in the class of distributions having support in Δ consists of the solutions to the equation* ($\int = \int_I$)

$$\int R(y, z)\, H(z)\, dz = G(y), \qquad y \in I, \quad \operatorname{supp} H \subset \Delta, \tag{7}$$

where

$$\begin{aligned} G &= \sum_{j=1}^{n/2} \psi_j^+ b_j^+, & y > t,\\ &= f(y), & y \in \Delta;\\ &= \sum_{j=1}^{n/2} \psi_j^- b_j^-, & y < t - T. \end{aligned}$$

3. MAIN RESULTS

THEOREM 1. *Let $R(x, y) \in \mathscr{R}$, $f \in H_\alpha$, $\alpha = (n - m)/2$, and let the assumptions of n. 2 hold: namely, $P_{ij}, Q_{ij} > 0$, $\lambda \in I$,* $\det A(\lambda) > 0$ *for $\lambda \in I$, m is even, and $r(\lambda)$ is ρ-almost everywhere positive definite. Then the solution h of Eq.* (1) *exists and is unique for $\sigma(h) \leqslant \alpha$. It can be calculated from the formulas*

$$\begin{aligned} h = Q(L)\, T(y), \qquad T(y) &= \sum_{j=1}^{n/2} \psi_j^+ b_j^+, & y > t,\\ &= g(y) + \sum_{j=1}^{m} \phi_j c_j, & y \in \Delta,\\ &= \sum_{j=1}^{n/2} \psi_j^- b_j^-, & y < t - T. \end{aligned} \tag{8}$$

Here differentiation is understood in the sense of distribution theory. The vectors $b_j^\pm$, $1 \leqslant j \leqslant n/2$, c_j, $1 \leqslant j \leqslant m$, are uniquely determined by the linear system

$$D^p \sum_{j=1}^{n/2} \psi_j^- b_j^- \Big|_{y=t-T} = D^p \Big\{ g + \sum_{j=1}^{m} \phi_j c_j \Big\} \Big|_{y=t-T}, \qquad 0 \leqslant p \leqslant 0.5(n+m) - 1,$$

$$D^p \sum_{j=1}^{n/2} \psi_j^+ b_j^+ \Big|_{y=t} = D^p \Big\{ g + \sum_{j=1}^{m} \phi_j c_j \Big\} \Big|_{y=t}, \qquad 0 \leqslant p \leqslant 0.5(n+m) - 1.$$

The mapping $R^{-1}: f \to h$ is a homeomorphism of H_α onto $H_{-\alpha}$.

Remark 1. The last statement of Theorem 1 is the precise description of the stability of the solution of Eq. (1).

Remark 2. The eigenvalues of $R \in \mathscr{R}$, $R\phi_j = \lambda\phi_j$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and even those of more general operator with the kernel (3), where $r(\lambda)$ is a continuous in $\lambda \in \Lambda$ matrix function such that $R \geqslant 0$ in $L^2(I)$, have the following properties [6]:

(i) $\lambda_j(\Delta) \leqslant \lambda_j(\Delta')$ if $\Delta \subset \Delta'$;

(ii) if $\max_{x \in I} \int_I | R(x, y)| \, dy \equiv \mathscr{M} < \infty$, then the limit

$$\omega_j \equiv \lim_{\substack{t \to +\infty \\ t - T \to -\infty}} \lambda_j(\Delta), \qquad \omega_1 \leqslant \mathscr{M}, \qquad \omega_1 = \max_{\lambda \in \Lambda} \mu(\lambda)$$

exists.

Here $\mu(\lambda)$ is the maximal eigenvalue of the matrix $r(\lambda)$, $| R(x, y)| = \max_{a \neq 0} | R(x, y) \, a | \cdot | a |^{-1}$, $a \in \mathbb{R}^d$. The asymptotic behavior of the eigenvalues of the scalar kernels $R \in \mathscr{R}$ is described in [10], which also contains some abstract theorems on the asymptotic of spectrum of linear operators.

THEOREM 2. *Let* $R(x, y) = R(x - y)$, $d = 1$, $R(x) = \int_I r(\lambda) \exp(-i\lambda x) \, d\lambda$, $\sup_{\lambda \in I}\{(1 + \lambda^2)^\beta \, | r(\lambda) - P_\epsilon(\lambda) Q^{-1}(\lambda)|\} \equiv \| r - P_\epsilon Q_\epsilon^{-1} \| < \epsilon$, $\deg Q_\epsilon(\lambda) - \deg P_\epsilon(\lambda) = 2\beta > 0$, $\forall \epsilon > 0$; $f \in H_\beta$; $\inf_{\lambda \in I}\{(1 + \lambda^2)^\beta \, | r(\lambda)|\} \equiv C_1 > 0$, $C \equiv C_1^{-1}$, $\gamma \equiv 2\epsilon C < 1$, $Rh = f$, $R_\epsilon h_\epsilon = f$. *The operator* R *is defined by formula* (1), R_ϵ *is the operator of the same type corresponding to the function* $r_\epsilon(\lambda) = P_\epsilon(\lambda) Q_\epsilon^{-1}(\lambda)$. *Then*

$$| h - h_\epsilon |_{-\beta} \leqslant \gamma C (1 - \gamma)^{-1} \, | f |_\beta , \qquad | R^{-1} - R_\epsilon^{-1} | \leqslant \gamma C (1 - \gamma)^{-1}. \tag{10}$$

Here $| \cdot |_\alpha$ *is the norm in the space* H_α, *and* $| R^{-1} |$ *is the norm of linear operator* $R^{-1}\colon H_\beta \to H_{-\beta}$.

Remark 3. If $(1 + \lambda^2)^\beta r(\lambda) = A(1 + O(1))$ as $\lambda \to \infty$, $A > 0$, $r(\lambda) > 0$, $\lambda \in I$, then approximate analytic solution to equation (1) in $H_{-\beta}$ can be obtained as follows:

(i) The kernel R can be approximated by the kernel R_ϵ so that $\| r(\lambda) - P_\epsilon Q_\epsilon^{-1} \| < \epsilon$, $2\epsilon C < 1$, $\deg Q_\epsilon - \deg P_\epsilon = 2\beta > 0$.

(ii) The solution $h_\epsilon \in H_{-\beta}$ to Eq. (1) with the kernel R_ϵ can be determined analytically by formula (8).

(iii) This h_ϵ is the analytic approximate solution to Eq. (1) and the error estimate is given by formula (10).

To prove Theorem 2 we need

LEMMA 2. *Let* $H_+ \subset H_0 \subset H_-$ *be the Hilbert spaces*, $| u_0 | \leqslant | u |_+$, *where* H_+ *is dense in* H *and* H_- *is dual to* H_+. *Let* $R\colon H_- \to H_+$ *be a linear operator. If* $|(Rh, h)| \leqslant C \, | h |_-^2$, $\forall h \in H_-$, *then* $| R | \leqslant 2C$. *If* $|(Rh, h)| \geqslant C_1 \, | h |_-$,

$\forall h \in H_-$, $C_1 > 0$, *then* $| R^{-1} | \leqslant C_1^{-1}$. *Here* $| R |$ *is the norm of the linear operator* $R: H_- \to H_+$, $| R^{-1} |$ *is the norm of* $R^{-1}: H_+ \to H_-$, *and* $| h |_-$ *is the norm in* H_-.

Remark 4. The definition and properties of the dual spaces are given, for example in [8, Chap. 1]. In our theory $H_0 = L^2(\Delta)$, $H_\pm = H_{\pm\beta}$.

4. Sketches of Proofs

Proof of Lemma 1. If H is defined as in Lemma 1 then $h = H$ is the solution to Eq. (1), with supp $h \subset \Delta$. If h is the solution to Eq. (1), supp $h \subset \Delta$, then $H = h$ is the solution to Eq. (7).

Proof of Theorem 1. The sets of solutions of Eqs. (1) and (6) are the same. By Lemma 1 the set of solutions of Eq. (6) in the class of distributions coincides with the set of solutions of Eq. (*) $\int_I S(y, z) H(z)\, dz = T(y)$, $y \in I$, when supp $H \subset \Delta$, and $T(y)$ is defined by formula (8). From definition (4) it follows that $Q(L) S(x, y) = \delta(x - y)$. Thus the set of solutions to Eqs. (*) can be described by the formula $H = Q(L)\, T(y)$, supp $H \subset \Delta$. The solution h can be found by this formula iff $T(y)$ has maximal smoothness. Inside and outside of the Δ function $T(y)$ is smooth, so that $T(y)$ will have maximal smoothness if conditions (9) are satisfied. In this case $H = h = Q(L)\, T(y) \in H_{-\alpha}$ is the solution of Eq. (1). To prove the uniqueness and existence of solutions to system (9) it is sufficient to prove that the homogeneous system (9) has only the trivial solution. Indeed every solution to the homogeneous system (9) generates a solution to equation $Rh = 0$, $h \in H_-$, supp $h \subset \Delta$. By the Parseval equality we have $0 = (Rh, h) = \int_\Delta r(\lambda)\, \tilde{h} \cdot \tilde{h}^*\, d\rho(\lambda)$. Here $\tilde{h}$ is the generalized Fourier transform (instead of plane waves eigenfunctions of the operator L are used) and the asterisk denotes the complex conjugation. As $r(\lambda)$ is ρ-almost everywhere positive definite it follows that $\tilde{h} = 0$, $h = 0$, and that $c_j = b_j^\pm = 0$, $\forall j$. As $R: H_{-\alpha} \to H_\alpha$ is a linear injective and surjective mapping it is a homeomorphism of $H_{-\alpha}$ onto H_α.

Proof of Lemma 2. The last statement of Lemma 2 is evident. The first statement is known if $R: H \to H$; see [11, p. 234]. The same proof holds in the case $R: H_- \to H_+$.

Proof of Theorem 2. Since $C_1 | h |_{-\beta}^2 = \inf_{\lambda \in I}\{(1 + \lambda^2)^\beta | r(\lambda) |\} | h |_{-\beta}^2 \leqslant (Rh, h) = \int_I r(\lambda) | \tilde{h}(\lambda) |^2\, d\lambda \leqslant \sup_{\lambda \in I}\{(1 + \lambda^2)^\beta | r(\lambda) |\} | h |_{-\beta}^2 = C_2 | h |_{-\beta}^2$, Lemma 2 implies that $| R | \leqslant 2C_2$, $| R^{-1} | \leqslant C_1^{-1}$, $R: H_{-\beta} \to H_\beta$. If $| R - R_\epsilon | < 2\epsilon$, then $| R^{-1} - R_\epsilon^{-1} | \leqslant |[I + R^{-1}(R_\epsilon - R)]^{-1} - I | \cdot | R^{-1} | \leqslant C \sum_{j=1}^\infty (2\epsilon C)^j = \gamma C(1 - \gamma)^{-1}$. It remains to note that $\| r(\lambda) - r_\epsilon(\lambda) \| < \epsilon$ implies that $| R - R_\epsilon | < 2\epsilon$ by Lemma 2.

References

1. A. G. Ramm, Filtering of nonhomogeneous random fields, *Opt. Spectrosc.* **27** (1969), 881–887; English transl., **27** (1969), 533–535.
2. A. G. Ramm, Filtering and extrapolation of some nonstationary random processes, *Radiotech. i Electron.* **16** (1971), 80–87; English transl., *Radio Engrg. Electron. Phys.* **16** (1971), 68–75.
3. A. G. Ramm, On multidimensional integral equations with the translation kernel, *Differencial'nye Uravnenija* **7** (1971), 2234–2239; *Differential Equations* **7** (1979), 1685–1687.
4. A. G. Ramm, On some class of integral equations, *Differencial'nye Uravnenija* **9** (1973), 931–941, *Differential Equations* **9** (1973), 706–713.
5. A. G. Ramm, Investigation of a class of integral equations, *Soviet Math. Dokl.* **17** (1976), 1301–1305.
6. A. G. Ramm, Discrimination of the random fields in noises, *Problems of Information Transmission* **9** (1973), 192–205.
7. A. G. Ramm, A new class of nonstationary processes and fields and its applications, *in* "Methods of Representation and Analysis of Random Processes and Fields" (Proceedings, 10th All-Union Symposium, Leningrad 3, 1978), pp. 40–43 (in Russian).
8. Ju. M. Berezanskij, "Expansions un Eigenfunctions of Self-adjoint Operators," Naukova Dumka, Kiev, 1965; Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1968.
9. M. A. Naimark, "Linear Differential Operators," Nauka, Moscow, 1969; English transl., Ungar, New York, 1967.
10. A. G. Ramm, On eigenvalues of some integral equations, *Differencial'nye Uravnenija* **14** (1978), 932–934; *Differential Equations* **14** (1978), 665–667.
11. I. M. Glazman and Ju. I. Ljubish, "Finite-Dimensional Linear Analysis," Nauka, Moscow, 1969; English transl., MIT Press, Cambridge, Mass., 1974.
12. A. G. Ramm, "Perturbations Preserving Asymptotic of Spectrum," Acc. Naz. dei Lincei, Ser. 8, Vol. 64, Fasc. 1, Jan. 1978; *J. Math. Anal. Appl.* **76** (1980), 10–17.
13. A. G. Ramm, Investigation of some classes of integral equations and their applications, *in* "Abel Inversion and Its Generalizations" (N. Preobrazhensky, Ed.), pp. 120–179, Siberian Dept. of Academy of Science, USSR, Novosibirsk, 1978 (in Russian).
14. A. G. Ramm, Theory and applications of some new classes of integral equations, Springer-Verlag, N.Y., 1980.
15. A. G. Ramm, Analytical results in random fields filtering theory, *Zeitschr. Angew. Math. Mech.*, (1980), in press.