JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **76**, 10–17 (1980)

Perturbations Preserving Asymptotic of Spectrum*

A. G. RAMM

Department of Mathematics, The University of Michigan,
Ann Arbor, Michigan 48109

Submitted by C. L. Dolph

1. INTRODUCTION

In this paper H will be a Hilbert space with the inner product (f, g) and the norm $\|f\| = (f, f)^{1/2}$. A will denote a closed, denseley defined linear operator in H with domain $D(A)$. Its range will be denoted by $R(A)$ and its null space by $N(A) = \operatorname{Ker} A$. B will denote a linear operator which will be a perturbation of A, $B = A + T$ and it will be assumed that $D(T) \supset D(A)$ and that $D(B) = D(A)$.

By $\{L_n\}$ we shall denote a sequence of linear subspaces in H such that $L_n \subset L_{n+1}$. If $\rho(f, L)$ denotes the distance from an element $f \in H$ to the subspace L then it will also be assumed that

$$\forall f \in H, \qquad \rho(f, L_n) \to 0 \qquad \text{as} \qquad n \to \infty.$$

It will be assumed A has discrete spectrum $\sigma = \sigma(A)$ in the following sense: every point of $\sigma(A)$ is an isolated eigenvalue of the finite algebraic multiplicity. We suppose also that $0 \notin \sigma(A)$, so that A^{-1} is a bounded operator defined on all H. Let $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$ be the eigenvalues of A. If the resolvent of A is compact for $\lambda \notin \sigma(A)$ and $0 \notin \sigma(A)$, then A^{-1} is compact and $\sigma(A)$ is discrete. If A is normal and $\sigma(A)$ is discrete then A^{-1} is compact. Without the assumption about normality it seems that A^{-1} is not necessarily compact when $\sigma(A)$ is discrete. We denote the singular values of A by $s_n(A)$, $s_n(A) = s_n^{-1}(A^{-1}) = \lambda_n^{1/2}\{(A^{-1})^* A^{-1}\}$. If $A = A^* \geqslant m > 0$ we denote by H_A the Hilbert space which is the completion of $D(A)$ in the norm $\|f\| = (Af, f)^{1/2}$, the inner product in H_A being $[f, g] = (Af, g)$ for $f, g \in D(A)$, $H_A = D(A^{1/2})$. By $\sigma_c = \sigma_c(A)$ we denote the subset of the spectrum of a linear operator A which consists of the points λ, for which a bounded noncompact sequence f_n exists such that $\|Af_n - \lambda f_n\| \to 0$, $n \to \infty$, while $\sigma_r = \sigma_r(A) = \{\lambda \colon \lambda \notin \sigma_p(A), \bar{\lambda} \in \sigma_p(A^*)\}$, $\sigma_p(A)$ is the set of eigenvalues of A, $\sigma_e = \sigma_c \cup \sigma_r$. The set $\sigma \backslash \sigma_e$ consists of the eigenvalues of the operator A, $\sigma_p = \sigma_d$ if they have finite algebraic multiplicity. Denote by $\to$, $\rightharpoonup$ strong and weak convergence respectively in H. This paper discusses the following problem: when $s_n(B)\, s_n^{-1}(A) \to 1$ or $\lambda_n(B)\, \lambda_n^{-1}(A) \to 1$,

* This work was supported by AFOSR F49620-79-C-0128.

0022-247X/80/070010–08$02.00/0

as $n \to \infty$? While this problem was discussed in [1, p. 351] our results are more general and at the same time our technique is simpler. The same problem will also be discussed for perturbed quadratic forms. A quadratic form $T[f, f]$ is said to be compact relative to a positive definite quadratic form $A[f, f]$ if any sequence $\{f_n\}$, $A[f_n, f_n] \leqslant 1$ has a subsequence $\{f_m\}$ such that $T[f_m - f_k, f_m - f_k] \to 0$, $m, k \to \infty$. By $D[A]$ we denote the domain of the quadratic form A. The spectrum of a sectorial quadratic form is the spectrum of the operator, generated by the form. The connection between sectorial forms and operators is described in [2, p. 404]. Below C denotes various constants. The results were announced in [4] and applied in [5, 6].

2. Main Results

Theorem 1. *If the operator TA^{-1} is compact, then $B = A + T$ is closed. If the operators A^{-1}, TA^{-1}, $A^{-1}T$ are compact, $0 \notin \sigma(B)$, then $\sigma(B)$ is discrete.*

Theorem 2. *If $A = A^* \geqslant m > 0$, the operator $A^{-1}T$ is compact in H_A, $D(T) \supset H_A$, $B = B^*$, then $\lambda_n(B) \cdot \lambda_n^{-1}(A) \to 1$, $n \to \infty$.*

Theorem 3. *If Q, S are compact linear operators in H, such that* $\dim R(Q) = \infty$, $N(I + S) = \{0\}$, *then $s_n(Q + QS)\, s_n^{-1}(Q) \to 1$, $s_n(Q + SQ)\, s_n^{-1}(Q) \to 1$, $n \to \infty$.*

Theorem 4. *If A^{-1}, $A^{-1}T$, and TA^{-1} are compact, $0 \notin \sigma(B)$, then $\sigma(B)$ is discrete, $s_n(B)\, s_n^{-1}(A) \to 1$, $n \to \infty$. If TA^{-1}, AT^{-1} are compact and B normal, $0 \notin \sigma(B)$, and $A = A^*$ is semibounded from below, then $\lambda_n(B)\, \lambda_n^{-1}(A) \to 1$, $n \to \infty$.*

Theorem 5. *If $A[f, f]$ is a positive-definite quadratic form in H, with discrete spectrum, and a closed densely defined real valued form $T[f, f]$ is compact relative to $A[f, f]$, $D[A] \subset D[T]$, then the form $B[f, f] = A[f, f] + T[f, f]$, $D[B] = D[A]$ has discrete spectrum such that $\lambda_n(B)\, \lambda_n^{-1}(A) \to 1$, $n \to \infty$.*

Lemma 1. *Let $\{f_n\}$ be a noncompact bounded sequence in H. Then there exists a noncompact sequence ψ_m, $\psi_m \rightharpoonup 0$, such that $\psi_m = f_{n_{m+1}} - f_{n_m}$.*

Lemma 2. *The operator $(A^*A + I)^{-1}$ is compact iff $(A - \lambda I)^{-1}$ is compact.*

Lemma 3. *A linear operator T in H is compact iff* (1) $\gamma_n \equiv \sup_{h \perp L_n} | Th | \cdot | h |^{-1} \to 0$, $n \to \infty$, *or* (2) $(Tg_n, g_n) \to 0$, $n \to \infty$, *for any $g_n \rightharpoonup 0$, $n \to \infty$.*

Lemma 4. *Let A^{-1} be bounded. The operator TA^{-1} is compact iff T is A-compact, i.e., if $| f_n | + | Af_n | \leqslant C$ then Tf_n converges.*

Remark 1. If $T \geqslant 0$ then $A^{-1}T$ is compact in H_A if $W(H_A \to H_T)$ is compact. Here $W(H_A \to H_T)$ denotes the embedding operator. If $|(Tf, f)| \leqslant (Qf, f)$, $Q \geqslant 0$, and $W(H_A \to H_Q)$ is compact, then $A^{-1}T$ is compact in H_A (see [3, p. 37]).

LEMMA 5. *Under the assumptions of Theorem 5 let t be a self-adjoint and compact operator on the space H_A defined by the equality $T[f, f] = [tf, f]$, where $[u, v]$ denotes the inner product in H_A. Then the form $T[f, f]$ can be represented as follows:*

$T[f, f] = T_n[f, f] + T[f, f]$, and for an arbitrarily small ϵ, $|T[f, f]| \leqslant \epsilon[f, f]$, while $T_n[f, f]$ is a quadratic form in n-dimensional space with the property that $|T_n[f, f]| \leqslant c\,|f|^2$.

3. PROOFS

Proof of Lemma 1. If f_n is bounded but not a compact sequence it is possible to select a subsequence (denoted also by f_n) such that $f_n \rightharpoonup f$, where f_n is also not compact. To finish the proof one must construct a subsequence $\psi_m = f_{n_{m+1}} - f_{n_m}$ which is also not compact, as it is clear that $\psi_n \rightharpoonup 0$.

In the following we use the argument given in [3, p. 41]. If ψ_m converges, then $\psi_m \to 0$. But if f_n is not a compact sequence it is possible to find a subsequence $\{f_{n_m}\}$ such that $|f_{n_{m+1}} - f_{n_m}| \geqslant \epsilon > 0$. This implies that $|\psi_n| \geqslant \epsilon$ and $\psi_m \to 0$, which is a contradiction.

Proof of Lemma 2. Let $(A - \lambda I)^{-1}$ be compact. To show that $(A^*A + I)^{-1}$ is compact consider a bounded sequence $\{f_n\}$ such that $|(A^*A + I) f_n| \leqslant C$. We show that some subsequence $\{f_{n_k}\}$ converges. Since $|((A^*A + I) f_n, f_n)| = |Af_n|^2 + |f_n|^2 \leqslant c$, this implies that $|Af_n| + |f_n| \leqslant c$, $|(A - \lambda I) f_n| \leqslant c$. As $(A - I)^{-1}$ is compact a subsequence $\{f_{n_k}\}$ converges. Thus $(A^*A + I)^{-1}$ is compact. Conversely, suppose that $(A^*A + I)^{-1}$ is compact, then $(A^*A + I)^{-1/2}$ is compact. It is sufficient to show that $|(A - \lambda I) f_n| \leqslant c$, $\lambda \notin \sigma(A)$ implies that $\{f_{n_k}\}$ converges.

If $|(A - \lambda I) f_n| \leqslant c$, $\lambda \notin \sigma(A)$, then $|Af_n| + |f_n| \leqslant c$, $(Af_n, Af_n) + (f_n, f_n) \leqslant c$. But $D(A) = D\{(A^*A + I)^{1/2}\}$. Thus, $|(A^*A + I)^{1/2} f_n| \leqslant c$. As $(A^*A + I)^{-1/2}$ is compact a subsequence $\{f_{n_k}\}$ converges.

Proof of Lemma 3. (1) Let $h_1, \ldots, h_n$ be an orthonormal basis in L_n, and T compact. It is clear that $0 \leqslant \gamma_{n+1} \leqslant \gamma_n$, so $\lim \gamma_n = \gamma$, $n \to \infty$. If $\gamma > 0$ then there exists a sequence $\{f_n\}$, such that $f_n \perp L_n$, $|f_n| = 1$, $|Tf_n| \geqslant \gamma > 0$. Without loss of generality it can be assumed that $f_n \rightharpoonup 0$, as $\rho(f, L_n) \to 0$, $\forall f \in H$, $n \to \infty$. As T is compact, $Tf_n \to 0$. This contradiction shows that $\gamma = 0$. To prove the sufficiency we let $g_n \equiv h - \psi_n$, $\psi_n \equiv \sum_1^n (h, h_j)\, h_j$, $\psi_n \in L_n$, $g_n \perp L_n$, $T_n h \equiv T\psi_n$.

Then

$$| T - T_n | = \sup_{|h|=1} |(T - T_n) h | = \sup_{g_n \perp L_n, |g_n|^2 = 1 - |\psi_n|^2} | Tg_n | \leqslant \sup_{g \perp L_n, |g| \leqslant 1} | Tg |$$

$$= \gamma_n \to 0, \qquad n \to \infty.$$

As T_n has the finite rank, T is compact.

(2) The necessity of condition (2) is trivial. From the known polarization identity $(Tf, g) = 0.25\{(T(f+g), f+g) - (T(f-g), f-g) + i[(T(f+ig), f+ig) - (T(f-ig), f-ig)]\}$ and condition (2) it follows, that $(Tf_n, g_n) \to 0$ when $f_n \rightharpoonup 0$, $g_n \rightharpoonup 0$. So T is compact.

Proof of Lemma 4. If $| f_n | \leqslant c$ and TA^{-1} is compact then $TA^{-1}f_{n_k}$ converges. Let $A^{-1}f_n = g_n$, $f_n = Ag_n$. As A^{-1} is bounded $| g_n | \leqslant c$. Thus $| g_n | + | Ag_n | \leqslant c$ implies that Tg_{n_k} converges. This implies that T is A-compact. If T is A-compact, $| f_n | \leqslant c$ then $TA^{-1}f_n = Tg_n$ and $| g_n | \leqslant c$, $| Ag_n | \leqslant c$. Thus $\{Tg_{n_k}\}$ converges.

Proof of Theorem I. (a) Let $f_n \to f$, $Bf_n = Af_n + Tf_n \to g$. Then $(*)$ $| Af_n | \leqslant c$, as shown below, $\{Tf_{n_k}\}$ converges by Lemma 4 and $\{Af_{n_k}\}$ converges. As A is closed $f \in D(A) = D(B)$ and B is closed. To prove $(*)$ suppose that $| Af_n | \to \infty$. Then setting $f_n' = f_n | Af_n |^{-1}$ we obtain that $| f_n' | \to 0$, $| Af_n' | = 1$, $Af_n' + Tf_n' \to 0$. Thus Tf_{n_k}' converges. Thus Af_n' converges and as A is closed and $| f_n' | \to 0$, we have $| Af_n' | \to 0$. But this is impossible as $| Af_n' | = 1$. Thus $(*)$ is valid and B is closed. (b) To demonstrate that the spectra $\sigma(B)$ is discrete it must be proved that if $\lambda \in \sigma(B)$ then $\lambda \notin \sigma_c(B) \cup \sigma_r(B)$ and $\sigma_p(B) = \sigma_d(B)$. Let $\lambda \in \sigma_c(B)$. Then a non-compact bounded sequence $\{f_n\}$ exists, such that $Af_n + Tf_n - \lambda f_n \to 0$. By Lemma 1, a non-compact sequence $\{\psi_m\}, \{\psi_m\} \rightharpoonup 0$ exists, such that $A\psi_m - \lambda\psi_m + T\psi_m \to 0$, $\psi_m - \lambda A^{-1}\psi_m + A^{-1}T\psi_m \to 0$. As $A^{-1}T$ is compact and $\psi_m \rightharpoonup 0$ it follows that $\psi_m - \lambda A^{-1}\psi_m \to 0$. Without using the compactness of A^{-1} we prove that $\lambda \notin \sigma_c(B)$ follows from the fact that $A^{-1}T$ and TA^{-1} are compact. If $\lambda \neq 0$ then $A^{-1}\psi_m - \lambda^{-1}\psi_m \to 0$, so that $\lambda^{-1} \in \sigma_c(A^{-1})$. But $\sigma_c(A^{-1}) = \{0\}$. This contradiction proves that $\lambda \notin \sigma_c(B)$. If $\lambda = 0$ then $\psi_m \to 0$. This is impossible as ψ_m is not compact. So $\lambda \notin \sigma_c(B)$. Let $\lambda \in \sigma_r(B)$, $(B^* - \bar{\lambda}I)f = 0$. Since $B = A(I + A^{-1}T)$, it follows that $B^* = (I + A^{-1}T)^* A^* = A^* + T^*$. Suppose now that $\lambda \notin \sigma(A)$. Then $f + (A^* - \bar{\lambda}I)^{-1} T^*f = 0$. As $T(A - \lambda I)^{-1}$ is compact the equation $g + T(A - \lambda I)^{-1} g = 0$ has a solution $g \neq 0$. Hence $(A + T - \lambda I) h = 0$, $h = (A - \lambda I)^{-1} g \neq 0$, $\lambda \in \sigma_p(B)$. It implies that $\lambda \notin \sigma_r(B)$. If $\lambda \in \sigma(A)$ then $\lambda + \epsilon \notin \sigma(A)$, where $\epsilon > 0$ is sufficiently small, as $\sigma(A)$ is discrete. Therefore the equality $f + [A^* - (\bar{\lambda} + \epsilon) I]^{-1} \times (T^* + \epsilon I) f = 0$ holds. Since $[A - (\lambda + \epsilon) I]^{-1}$ and $[A - (\lambda + \epsilon) I]^{-1} T$ are compact (only here use is made of the compactness of A^{-1}), equation

$g + (T + \epsilon I) \cdot [A - (\lambda + \epsilon) I]^{-1} g = 0$ has a solution $g \neq 0$. As above it follows that $\lambda \in \sigma_p(B)$ and therefore that $\lambda \notin \sigma_r(B)$. If $0 \notin \sigma(B)$, then $\sigma_p(B) = \sigma_d(B)$.[1]

Proof of Theorem 2. As $A^{-1}T$ is compact in H_A it follows that $H_B = H_A$, and $\sigma(B)$ is discrete [see 3, pp. 38, 42], and by Lemma 3

$$a_n \equiv \sup_{f \perp L_n(A)} \frac{(Tf, f)}{A[f, f]} = \sup_{f \perp\!\!\!\perp L_n(A)} \frac{[A^{-1}Tf, f]}{[f, f]} \to 0, \qquad n \to \infty.$$

Here $L_n(A)$ is a linear span of n first eigenelements of operator A, and sign $\perp\!\!\!\perp$ denotes orthogonality in H_A. Using the inequality $\inf a(1 + b) \geqslant \inf a(1 - \sup b)$, $a \geqslant 0$, $-1 < b < 1$ one has

$$\lambda_{n+1}(B) = \sup_{L_n} \inf_{f \perp L_n, f \in D(B)} \frac{(Bf, f)}{(f, f)}$$

$$\geqslant \inf_{f \perp L_n(A), f \in D(A)} \left\{ \frac{(Af, f)}{(f, f)} \left(1 + \frac{(Tf, f)}{(Af, f)}\right)\right\} > \lambda_{n+1}(A)(1 - a_n)$$

where $a_n \to 0$. By symmetry $\lambda_{n+1}(A) \geqslant \lambda_{n+1}(B)(1 - b_n)$, $b_n \to 0$. Thus $\lambda_n(B)\lambda_n^{-1}(A) \to 1$, $n \to \infty$.

Remark 2. The proof implies that only a finite number of eigenvalues $\lambda_n(B)$ can be negative.

Proof of Theorem 3. As $s_n^2(Q + QS) = \lambda_n\{(I + S^*) U(I + S)\}$, $U = Q^*Q$, $\lambda_n(U) = s_n^2(Q)$. Since S is compact and $N(I + S) = \{0\}$, one has $(I + S)^{-1} = I + \Gamma$, and $(I + S^*)^{-1} = I + \Gamma^*$, where Γ is compact. Let $V \equiv (I + S^*) U(I + S) \geqslant 0$. If it can be shown that $(*)$ $\lambda_n(V) \leqslant \lambda_n(U)(1 + a_n)$, $a_n \to 0$, $n \to \infty$, then by symmetry $\lambda_n(U) \leqslant \lambda_n(V)(1 + b_n)$, $b_n \to 0$, $n \to \infty$, and $\lambda_n(V) \cdot \lambda_n^{-1}(U) \to 1$, $n \to \infty$. This is equivalent to the first statement of Theorem 3.

The second statement of Theorem 3 can be proved similarly. Next it will be shown that

$$\lambda_{n+1}(V) = \inf_{L_n} \sup_{f \perp L_n} \frac{(Vf, f)}{(f, f)} \leqslant \sup_{f \perp M_n} \left\{\frac{(Ug, g)}{(g, g)} \cdot \frac{(g, g)}{(f, f)}\right\}$$

$$\leqslant \sup_{f \perp M_n} \frac{(Ug, g)}{(g, g)} \cdot \sup_{f \perp M_n} \frac{(g, g)}{(f, f)} \leqslant \lambda_{n+1}(U)(1 + a_n), \qquad (*)$$

$a_n \to 0$, $n \to \infty$. Here $g = (I + S)f$, M_n is the linear n-dimensional subspace so chosen that the condition $f \perp M_n$ is equivalent to the condition $g \perp L_n(U)$,

[1] See Appendix.

$L_n(U)$, being the linear span of the first eigenelements $\phi_1, ..., \phi_n$ of the operator U, M_n is the span of the elements $\psi_j = (I + S^*)\phi_j$. If $(g, \phi_j) = 0$ then $0 = (f, (I + S^*)\phi_j) = (f, \psi_j)$. As $I + S^*$ is invertible, the system $\psi_1, ..., \psi_n$ is linearly independent, $\dim M = n$ and $\rho(f, M_n) \to 0$, $n \to \infty$, $\forall f \in H$. (*) involves the use of the equality

$$\sup_{f \perp M_n} \frac{(g, g)}{(f, f)} = 1 + \sup_{f \perp M_n} \frac{|(Sf, f)| + |(f, Sf)| + (Sf, Sf)}{(f, f)} = 1 + a_n,$$

$a_n \to 0$, $n \to \infty$, which in turn follows from Lemma 3 and the compactness of S.

Proof of Theorem 4. By Theorem 1, $\sigma(B)$ is discrete. As $0 \notin \sigma(B)$, $B = A(I + C)$, $C = A^{-1}T$, thus $N(I + C) = \{0\}$, $B^{-1} = (I + C)^{-1} A^{-1}$, $s_n^2(B) = s_n^{-2}(B^{-1}) = s_n^{-2}\{(I + S)Q\}$. Here $Q = A^{-1}$, $I + S = (I + C)^{-1}$, Q, and S are compact. By Theorem 3, $s_n^{-1}(Q)\, s_n\{(I + S)Q\} \to 1$, $n \to \infty$. Thus $s_n(B)\, s_n^{-1}(A) \to 1, n \to \infty$. To prove the second statement of Theorem 4, recall that if $A = A^*$ and if $\sigma(A)$ is discrete then A^{-1} is compact. As B is normal $\sigma_r(B) = \varnothing$. If TA^{-1} is compact then it follows from the argument of Theorem 1 that $\sigma_c(B) = 0$. As $0 \notin \sigma(B)$ operator B^{-1} is compact, $B^{-1} = A^{-1}(I + TA^{-1})^{-1}$, $N(I + TA^{-1}) = \{0\}$, $(I + TA^{-1})^{-1} = I + \Gamma$, where Γ is compact. Consequently it follows that $s_n(B)\, s_n^{-1}(A) \to 1$, $n \to \infty$. As $A = A^*$ and B is normal, it is known that $|\lambda_n(A)| = s_n(A)$ and that $s_n(B) = |\lambda_n(B)|$. To prove that $\lambda_n(B)\, \lambda_n^{-1}(A) \to 1$, $n \to \infty$ it is sufficient to prove that $|\lambda_n(B)|\, \lambda_n^{-1}(B) \to 1$, $n \to \infty$. Let $Af_n + Tf_n = \lambda_n f_n$, $\lambda_n = \lambda_n(B)$, $(f_n, f_m) = \delta_{nm}$. Then $f_n + Cf_n = \lambda_n A^{-1} f_n$, and $C = A^{-1}T$ is compact, so $\lambda_n = (A^{-1}f_n, f_n)^{-1} [1 + (Cf_n, f_n)]$. Hence

$$|\operatorname{Im} \lambda_n| \cdot |\operatorname{Re} \lambda_n|^{-1} \leqslant \frac{|(Cf_n, f_n)|}{1 - |(Cf_n, f_n)|} \to 0, \qquad n \to \infty.$$

Proof of Lemma 5. If $T[f, f]$ is compact relative to $A[f, f]$, then $T[f, f]$ is bounded in H_A. Recall that $T[f, f] = [tf, f]$, t being the self-adjoint bounded operator in H_A. As $T[f, f]$ is compact relative to $A[f, f]$, t is compact in H_A, $t = t_n + t_\epsilon$, where t_n has finite rank, $|t_\epsilon| < \epsilon$, $|t_\epsilon|$ being the norm of operator in H_A. Hence $T[f, f] = [t_n f, f] + [t_\epsilon f, f] = T_n[f, f] + T_\epsilon[f, f]$, $|T_\epsilon[f, f]| \leqslant \epsilon[f, f]$. Let $\phi_1, ..., \phi_n$ be the basis of the diagonal representation of t_n, $[t_n f, f] = \sum_{j=1}^{n} \lambda_j |[f, \phi_j]|^2$, $\phi_j \in H_A$. As $D(A)$ is dense in H_A one can find $\{\psi_j\} \in D(A)$, $\| \psi_j - \phi_j \|_{H_A} < \delta$ so that $|[f, \phi_j]|^2 \leqslant |[f, \psi_j] + [f, \phi_j - \psi_j]|^2 \leqslant 2\, |[f, \psi_j]|^2 + 2\, \| f \|^2 \delta^2$, $|[f, \psi_j]|^2 = |(f, A\psi_j)|^2 \leqslant c\, | f |^2$. Hence $|[t_n f, f]| \leqslant c\delta^2 \| f \|^2 + C\, | f |^2$, $C = C(n, \lambda_j)$. From here we get $B[f, f] = A[f, f] + T[f, f] \geqslant A[f, f]\, (1 - C\delta^2 - \epsilon) - C\, | f |^2$. Setting $C\delta^2 + \epsilon < 1$ one can see that $B[f, f]$ is semibounded in H from below, and that $H_B = H_A$.

Proof of Theorem 5. As $B[f, f]$ is semibounded in H from below there exists an $m > 0$ such that $B_m[f, f] = B[f, f] + m(f, f)$ is positive definite in H. Let

$B_m = B + m$ and B be the self-adjoint operators generated by the forms $B_m[f, f]$ and $B[f, f]$ respectively. It is clear that $\lambda_n(B_m) = \lambda_n(B) + m$. As $\lambda_n(A) \to +\infty$ the equality $\lambda_n(B)\,\lambda_n^{-1}(A) \to 1$, $n \to \infty$ is equivalent to the equality $\lambda_n(B_m)\,\lambda_n^{-1}(A) \to 1$, $n \to \infty$. Thus it can be supposed that operator B is positive definite. The spectrum $\sigma(B)$ is discrete iff $W(H_B \to H)$ is compact (Rellich theorem). But $W(H_B \to H)$ is compact iff $W(H_A \to H)$ is compact because $H_A = H_B$. As $\sigma(A)$ is discrete the operator $W(H_A \to H)$ is compact. And thus $\sigma(B)$ is discrete. Further, the following holds:

$$\lambda_{n+1}(B) = \sup_{L_n} \inf_{f \perp L_n} \frac{B[f,f]}{(f,f)} \geqslant \inf_{f \perp L_n(A)} \frac{A[f,f]}{(f,f)} \left(1 + \frac{T[f,f]}{A[f,f]}\right)$$

$$\geqslant \lambda_{n+1}(A) \left(1 - \sup_{f \perp L_n(A)} \frac{T[f,f]}{A[f,f]}\right) = \lambda_{n+1}(A)\,(1 - a_n),$$

where $a_n \to 0$, $n \to \infty$. The relation $a_n \to 0$ follows from Lemma 3 and the compactness of $T[f, f]$ relative to $A[f, f]$, $L_n(A)$ denotes the linear span of the first n eigenelements of the operator A. By symmetry $\lambda_{n+1}(A) \geqslant \lambda_{n+1}(B)\,(1 - b_n)$, $b_n \to 0$, $n \to \infty$. Thus $\lambda_n(B)\,\lambda_n^{-1}(A) \to 1$, $n \to \infty$.

4. Examples

1. Let $H = L^2(D)$, $D \in R^m$, be a bounded domain with the smooth boundary Γ, $A[f,f] = \int_D \{|\nabla f|^2 + |f|^2\}\,dx$, $T[f,f] = \int_\Gamma h(s)\,|f(s)|^2\,ds$, $h(s) \in C^1(\Gamma)$. Here $T[f, f]$ is assumed to be compact relative to $A[f, f]$, so $\lambda_n(B)\,\lambda^{-1}(A) \to 1$, $n \to \infty$. Here $\{\lambda_n(A)\}$ is the spectrum of the inner Neumann problem for the doomain D, $\{\lambda_n(B)\}$ is the spectrum of the following problem:

$$-\Delta f + f = \mu f \quad \text{in} \quad D, \qquad \partial f/\partial N + h(s) f = 0 \quad \text{on} \quad \Gamma.$$

2. Consider the problem $L_n u = \lambda_n u$ in D, $L = L_0 + L_1$, where L_0 is a self-adjoint elliptic differential operator of order $2r$, L_1 a differential operator of order $r_1 < 2r$ in $H = L^2(D)$. Suppose that $N(L_0) = \{0\}$. The operators $L_0^{-1}L_1$, $L_1 L_0^{-1}$ are compact in H. According to Theorem 4, $s_n(L)\,s_0^{-1}(L_0) \to 1$, $n \to \infty$. If in addition $N(L) = \{0\}$, L_0 is self-adjoint, L normal, then $\lambda_n(L)\,\lambda_0^{-1}(L_0) \to 1$, $n \to \infty$.

These examples are of illustrative nature. The results of these examples are known, but here results have been obtained without any calculations or estimates.

5. Comments

In [1, p. 35] the following theorems are proved: (1) if $Q \geqslant 0$, $\dim R(Q) = \infty$, Q compact $K = (I + S)Q$, $K = K^*$, S compact, $N(I + S) = \{0\}$ then

$\lambda_n(K)\lambda_n^{-1}(Q) \to 1$, $n \to \infty$; (2) if $Q = Q^*$, $\dim R(Q) = \infty$, S compact, $N(Q) = \{0\}$, $N(I + S) = \{0\}$, and $K = Q(I + S)$ then $s_n(K)\, s_n^{-1}(Q) \to 1$, $n \to \infty$. Both theorems are corollaries to Theorem 3, which was proven without assuming that K or Q were self-adjoint. Theorem (2) is an immediate consequence of Theorem 3, while Theorem (1) is implied by Theorem 5 once it is observed that $\lambda_n(K) > 0$ for any sufficiently large n and that $s_n(K) = \lambda_n(K)$ whenever $\lambda_n(K) > 0$.

Appendix

LEMMA. *If A^{-1} and TA^{-1} are compact and $N(B + KI) = \{0\}$ for some number $K \notin \sigma(A)$, then $\sigma(B) = \sigma_d(B)$. Here $N(B) =$* Ker B.

Proof. We have $(A + T - \lambda I)^{-1} = (A + KI)^{-1}(I + Q - \mu S)^{-1}$, where $S = (A + KI)^{-1}$, $Q = T(A + KI)^{-1}$, $\mu = \lambda + K$, S, Q are compact. If $N(B + KI) = \{0\}$ then $N\{(I + Q)(A + KI)\} = \{0\}$ and $N\{I + Q\} = \{0\}$. Thus $(I + Q)^{-1}$ exists on the whole space H. Therefore by the well-known result $(I + Q - \mu S)^{-1}$ is a finitemeromorphic operator function. It means that $(I + Q - \mu S)^{-1}$ is a meromorphic operator function in μ and its Laurent coefficients are finite rank operators. The lemma is proved. Part (b) in the proof of Theorem 1 follows from the lemma without assumption about compactness of $A^{-1}T$.

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