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DIGRAPHS WITH REAL AND GAUSSIAN SPECTRA

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The conventional binary operations of cartesian product, conjunction, and composition of two digraphs D_1 and D_2 are observed to give the sum, the product, and a more complicated combination of the spectra of D_1 and D_2 as the resulting spectrum. These formulas for analyzing the spectrum of a digraph are utilized to construct for any positive integer n, a collection of n nonisomorphic strong regular nonsymmetric digraphs with real spectra. Further, an infinite collection of strong nonsymmetric digraphs with nonzero gaussian integer values is found. Finally, for any n, it is shown that there are n cospectral strong nonsymmetric digraphs with integral spectra.

0. Introduction

In order that this presentation be self-contained we include definitions of the fundamental concepts, most of which can be found in [5] and [7]. A *digraph* D consists of a finite set V of points $v_1, \ldots, v_p$ and a set of ordered pairs of distinct points, written (u, v) or briefly uv, called *arcs*. A *dipath* $u \to v$ is an alternating sequence of distinct points and arcs beginning at u and ending at v. A *dicycle* is obtained from a $u \to v$ dipath by adding the arc vu. We say v is *reachable* from u if there exists a dipath $u \to v$. A digraph is *strongly connected* or more briefly *strong* if every two points are mutually reachable. If arcs uv and vu are both in D, they form a *symmetric pair* of arcs.

The *underlying graph* $\mathscr{G}(D)$ is obtained when we replace its arcs by undirected lines so that either a single arc or a symmetric pair of arcs in D just becomes a single line. Then the *chromatic number* $\chi(D)$ is defined as $\chi(\mathscr{G}(D))$. Thus D is *bipartite* if $\mathscr{G}(D)$ is.

The *adjacency matrix* $A = A(D)$ of a labelled digraph D is the $p \times p$ matrix $[a_{ij}]$ with $a_{ij} = 1$ if v_iv_j is an arc of D, and 0 otherwise. The *characteristic polynomial* (or *spectral polynomial*) of D is written

$$\Phi(D) = \Phi(D; x) = \det(xI - A) = \sum_{i=0}^{p} a_i x^{p-i}. \tag{1}$$

The sequence $\lambda_1, \ldots, \lambda_p$ of the roots of $\Phi(D)$ is called the *spectrum* $S(D)$.

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A digraph D is *nonsymmetric* if not every arc lies in a symmetric pair. Thus matrix $A(D)$ is nonsymmetric if and only if D is. In general, this leads to a spectrum $S(D)$ containing both real and complex eigenvalues. A matrix A is called *irreducible* if there exists no permutation matrix P such that

$$P^{T}AP = \begin{bmatrix} A_{11} & \theta \\ A_{21} & A_{22} \end{bmatrix} \tag{2}$$

where A_{11}, A_{22} are square submatrices and θ indicates a zero submatrix.

1. Primitive and imprimitive digraphs

Every strong digraph D has an irreducible adjacency matrix A which possesses a simple positive eigenvalue of greatest modulus called its *spectral radius* λ_1. If there are exactly h eigenvalues of modulus λ_1, then A or D is called *primitive* if $h = 1$, otherwise *imprimitive* with *index of imprimitivity* h. This topic was developed in detail by Dulmage and Mendelsohn [3]. The following observation depends on a relation between the length of dicycles in D and the index of imprimitivity of $A(D)$.

Theorem 1. *For an imprimitive strong digraph D the chromatic number $\chi(D)$ does not exceed three.*

Proof. For a strong digraph D, let the index of imprimitivity be $h \geq 2$. Then the point set $V(D)$ can be partitioned into h independent subsets $V(D) = V_1 \cup \cdots \cup V_h$ so that uv is an arc of D only if $u \in V_i$, $v \in V_{i+1}$ with $1 \leq i \leq h$ and $V_{h+1} = V_1$ (as illustrated in Fig. 1). If we replace the arcs by lines it is evident that $\chi(\mathscr{G}(D)) = \chi(C_h)$, so $\mathscr{G}(D)$ can be colored using at most three colors, with the points of each set V_i having the same color. □

For a graph G let $D(G)$ denote a digraph obtained from an *orientation* of G, i.e., to each line is assigned either of the two possible directions. Then as Robbins

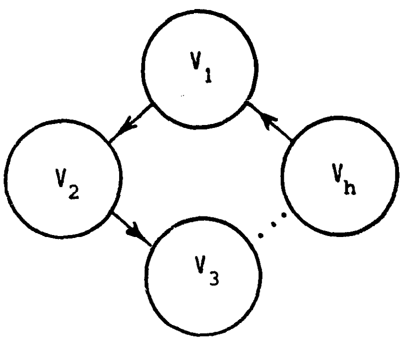

Fig. 1. A strong digraph D which is imprimitive.

[12] showed, there exists a strong orientation of G if and only if G is connected and bridgeless.

Corollary 1a. *For a graph G with chromatic number $\chi(G) \geq 4$, every strong orientation $D(G)$ has a primitive adjacency matrix.*

This result is best possible as clearly the graph $G = C_5 + e$ has $\chi(G) = 3$ and no strong orientation of G is imprimitive. But of course if the length of every cycle in a connected graph G is divisible by an odd integer, then each block of G has an imprimitive orientation which again implies that $\chi(G) \leq 3$.

2. The spectrum of a digraph

Schwenk [14] showed that the spectral polynomial of a graph G can be expressed in terms of the polynomials of subgraphs obtained from G by deleting a single point or a set of points. Our object is to generalize some of these results to digraphs. For this purpose it is very useful to have the theorem of Sachs [13] which gives the coefficients of the spectral polynomial of a digraph in terms of subgraphs whose components are dicycles.

Two nonisomorphic digraphs D_1 and D_2 are called *cospectral* if $S(D_1) = S(D_2)$. If in addition $S(D_1 - u) = S(D_2 - v)$ for some points u, v they are called *cospectrally rooted*. The *coalescence* $(D_1, u) \cdot (D_2, v)$ of the two rooted digraphs (D_1, u) and (D_2, v) is the digraph obtained by identifying their roots. As the proofs of the next theorem and its corollary are essentially the same as given by Schwenk [14] for graphs, we omit them.

Theorem 2. *Let v be a point of a digraph D and let $\mathscr{C}(v)$ be the set of all dicycles $\vec{C}$ containing v. Then the characteristic polynomial $\Phi(D)$ satisfies the equation,*

$$\Phi(D) = x\Phi(D - v) - \sum_{\vec{C} \in \mathscr{C}(V)} \Phi(D - v(\vec{C})). \tag{3}$$

Corollary 2a. *Let $D = (D_1, u) \cdot (D_2, v)$ be the coalescence of rooted digraphs (D_1, u) and (D_2, v). Then for all points w in D, $\Phi(D)$ satisfies*

$$\Phi(D) = \Phi(D_1)\Phi(D_2 - v) + \Phi(D_1 - u)\Phi(D_2) - x\Phi(D_1 - u)\Phi(D_2 - v). \tag{4}$$

Sketch of proofs. By Sach's theorem [13] the coefficients a_i of $\Phi(D)$ for a digraph D are given by

$$a_i = \sum_{D(i) \subset D} (-1)^{c(D(i))} \tag{5}$$

where the summation extends over all subdigraphs $D(i)$ with i points, whose components are dicycles and where the exponent $c(D(i))$ is the number of dicycles in $D(i)$.

We now derive (3) from (5). There are two possibilities for the point v:

(i) $v \notin D(i)$. Then $D(i)$ corresponds to $D'(i)$ defined as $D(i)$ in $D-v$.

(ii) $v \notin \vec{C}_n - D(i)$. Then $D(i)$ corresponds to $D'(i)$ defined as $D(i) - V(\vec{C}_n)$ in $D - V(\vec{C}_n)$.

This establishes a one-to-one correspondence between the subdigraphs $D(i)$ and $D'(i)$ so that if a subdigraph of D on the left side of (3) adds an amount to a coefficient a_i of $\Phi(D)$, then $D'(i)$ adds the same amount to a_i on the right, proving the theorem. □

If we now apply (3) to $(D_1, u) \cdot (D_2, v)$ so that their identified roots are chosen as the point v in (3), we obtain an equation which can be transformed to (4), proving the corollary. □

Obviously if (D, w), (D_1, u) and (D_2, v) are three rooted graphs and D_1 and D_2 are cospectrally rooted, then

$$\Phi((D, w) \cdot (D_1, u)) = \Phi((D, w) \cdot (D_2, v)). \tag{6}$$

The four cospectral strong digraphs with 4 points illustrated in Fig. 2 were listed in [6]. We now note that the first pair and the last pair are cospectrally rooted (with roots marked by circles). All four digraphs have the characteristic polynomial $\Phi(x) = (x^3 - x^2 - 1)(x + 1)$.

Fig. 2. Two smallest pairs of cospectrally rooted digraphs.

3. Binary operations on digraphs

It is very useful to construct new classes of graphs by binary operations on smaller graphs. The *conjection* $G = F_1 \wedge G_2$ of two graphs (or digraphs [11]) can be defined by taking as the adjacency matrix of G the tensor product of those of G_1 and G_2. It was observed in [9] that the cartesian product and composition of two graphs are also expressible in terms of matrix operations. Both Schwenk [14] and Cvetković [2] give the spectra of graphs formed by three different abelian operations of two graphs G and H in terms of the spectra of G and H. Their proofs are straightforward but complicated. However, the results can be obtained more naturally by applying spectral properties of polynomials of matrix tensor products to it. Our object is to show this more generally for three well known binary operations on digraphs: cartesian product, conjunction, and composition.

Let E and F be two digraphs with point sets $U=\{u_i\}$ and $V=\{v_i\}$. The next three operations define digraphs having $U\times V$ as its point set:

Conjunction $D=E\wedge F$. Here $((u_1, v_1), (u_2, v_2))$ is an arc of D whenever (u_1, u_2) and (v_1, v_2) are arcs in E and F.

Cartesian product $D=E\times F$. Now $((u_1, v_1), (u_2, v_2))$ is an arc of D whenever $u_1=u_2$ and (v_1, v_2) is an arc of F or $v_1=v_2$ and (u_1, u_2) is an arc of E.

Composition $D=E[F]$. Define $((u_1, v_1), (u_2, v_2))$ as an arc of D whenever (u_1, u_2) is an arc of E or $u_1=u_2$ and (v_1, v_2) is an arc of F.

Let A and B be matrices of order p_1 and p_2 with complex elements. Then the *tensor product* of $A=[a_{ij}]$ and B written $A\otimes B$, is defined as the partitioned matrix,

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p_1}B \\ & & \vdots & \\ a_{p_11}B & a_{p_12}B & \cdots & a_{p_1p_1}B \end{bmatrix}. \tag{7}$$

We now extend all matrix equations known for graphs [9] to digraphs (noting that this was already done by McAndrew [11] for the conjunction). As the proofs are easy and analogous to these for graphs, they are omitted. It is customary to denote by J_m the $m\times m$ *unit matrix* with every entry 1.

Lemma 3a. *If the adjacency matrices of the digraphs* D_1 *and* D_2 *are* A_1 *and* A_2, *then*

$$A(D_1\wedge D_2)=A_1\otimes A_2, \tag{8}$$

$$A(D_1\times D_2)=A_I\otimes I_{p_2}+I_{p_1}\otimes A_2, \tag{9}$$

$$A(D_1[D_2])=A_1\otimes J_{p_2}+I_{p_1}\otimes A_2. \tag{10}$$

We note that $D_1[D_2]$ is not abelian and

$D_1\wedge D_2$ is strong if and only if D_1 and D_2 are strong and D_1 or D_2 has an odd cycle (McAndrew [11]). (11)

$D_1\times D_2$ is strong if and only if D_1 and D_2 are both strong. (12)

$D_1[D_2]$ is strong if and only if D_1 is strong. (13)

Consider a polynomial $\psi(x; y)$ in two complex variables,

$$\psi(x; y)=\sum_{i,j=0}^{n} c_{ij}x^iy^j. \tag{14}$$

Then for two matrices A, B defined as above we mean by $\psi(A;B)$ the *tensor polynomial*,

$$\psi(A;B)=\sum_{i,j=0}^{n} c_{ij}A^{i}\otimes B^{j}. \tag{15}$$

We call a digraph *regular* if all row and column sums of its adjacency matrix are equal. The superscript in $\alpha^{(k)}$ will be used to designate the multiplicity of the number α, thus $\alpha^{(k)}$ means $\alpha,\ldots,\alpha$ with k terms.

Theorem 3. *Let D_1 and D_2 be two digraphs of order p_1 and p_2 with spectra $S(D_1)=(\lambda_i)$ and $S(D_2)=(\mu_i)$. Then the spectrum of any of the three binary operations of* (8), (9), (10) *is a p_1p_2-sequence where for $i=1,\ldots,p_1$ and $j=1,\ldots,p_2$,*

$$S(D_1\wedge D_2)=(\lambda_i\mu_j), \tag{16}$$

$$S(D_1\times D_2)=(\lambda_i+\mu_j), \tag{17}$$

and if D_2 is a regular strong digraph, then

$$S(D_1[D_2])=((p_2\lambda_i+\mu_1)^{(p_1)},\mu_2^{(p_1)},\ldots,\mu_{p_2}^{(p_1)}). \tag{18}$$

Proof. To prove the statements we recall from [10] that the eigenvalues of the tensor polynomial (15) can be computed by substituting λ and μ for x and y in (14). Thus if A and B are two complex matrices with spectra $S(A)=(\gamma_i)$ and $S(B)=(\delta_j)$ then the eigenvalues of $\psi(A;B)$ are the p_1p_2 complex numbers $\psi(\gamma_i;\delta_j)$.

To prove (16) we choose $\psi(x;y)=xy$ which implies $\psi(A_1;A_2)=A_1\otimes A_2$; similarly (17) can be obtained by taking $\psi(x;y)=x+y$.

We are now ready to prove Eq. (18) which is considerably more difficult. The reason is that whereas (8) and (9) express conjunction and cartesian product directly in terms of matrices A_1 and A_2 and identity matrices which offer no difficulty, Eq. (10) expresses the adjacency matrix of the composition of two digraphs in terms of the unit matrix J_{p_2}, which cannot, in general, be expressed in terms of A_2. However if we restrict D_2 to be regular, it is possible to obtain J_{p_2} as a limit of powers of A_2, as we now show. The matrix theory background for the following arguments can be found in the books by Gröbner [4, 164–180] and Lancaster [10, 165–184].

Let $\mu_1,\ldots,\mu_t$ be the different eigenvalues of A_2. Then we can represent A_2 in its spectral decomposition,

$$A_2=\sum_{i=1}^{t}(\mu_iE_i+N_i), \tag{19}$$

where the projectors (or principal idempotents) E_i of A_2 have the properties

$$I=\sum E_i,\qquad \theta=E_iE_j\quad\text{for } i\neq j\quad\text{and}\quad E_i^2=E_i. \tag{20}$$

The principal nilpotents N_i of A_2 have the properties

$$N_iE_i = E_iN_i = N_i \quad \text{and} \quad N_iE_j = E_jN_i = N_iN_j = \theta \quad \text{for } i \neq j. \tag{21}$$

Since A_2 has constant row and column sum, we have at once $A_2e = \mu_1 e$ and $e^T A = \mu_1 e^T$ where e is the normed vector $e = (1, \ldots, 1)^T/\sqrt{p_2}$. The related projector E_1 is given by

$$E_1 = ee^T = J_{p_2}/p_2. \tag{22}$$

There are two possibilities depending on the primitivity of A_2.

Case 1. A_2 is imprimitive. Then its spectral radius μ_1 satisfies $\mu_1 > |\mu_i|$ for $i = 2, \ldots, t$. Combining (19), (20), and (21) it can be shown as in Gröbner [3, 180] that

$$\lim_{n\to\infty} (A_2/\mu_1)^n = E_1. \tag{23}$$

Now if we choose $\psi(x; y) = (p_2/\mu_1^n)xy^n + y$, then $\psi(A_1; A_2)$ has the eigenvalues $(p_2/\mu_1^n)\lambda_i\mu_j^n + \mu_j$.

We now indulge in some routine manipulations. The tensor polynomial $\psi(A_1; A_2)$ contains $(A_2/\mu_1)^n$ which goes to E in the limit by (23). Then the limiting first term becomes $p_2A_1 \otimes E_1$ which equals $A_1 \otimes J_{p_2}$ by (22), which gives precisely the right side of (10), completing Case 1.

Case 2. A_2 is primitive of index h. Here we have h eigenvalues $\mu_1 = |\mu_2| = \cdots = |\mu_h|$, the roots of $\mu^h - \mu_1^h = 0$. Then it follows at once from the fact that the sum of hth roots of unity is zero that

$$\sum_{k=1}^{h} \mu_j^k \mu_1^{-k} = 0 \quad \text{for } j = 2, \ldots, h. \tag{24}$$

We now define $B = \sum_{k=1}^{h} (A_2/\mu_1)^k$ and replace A_2 by its spectral decomposition (19). Hence (24) together with (20) and (21) implies that B has a representation with a leading term hE_1 and no terms containing $E_2, \ldots, E_h$. All N_i for $i = 1, \ldots, h$ are zero matrices as $\mu_1, \ldots, \mu_h$ are simple eigenvalues. Therefore we obtain $\lim_{n\to\infty} (h^{-1}B)^n = E_1$.

Now we choose the particular polynomial

$$\psi(x; y) = p_2 x \left(h^{-1} \sum_{k=1}^{h} \mu_1^{-k} y\right)^n + y, \tag{25}$$

because it accomplishes the desired purpose. Then $\psi(A_1; A_2)$ has eigenvalues $\psi(\lambda_i, \mu_j)$. Finally, considering the limiting expression for $\psi(A_1; A_2)$, we obtain also in this case (10) and (18). □

We remark that Theorem 3 and its proof not only generalize the results of Cvetković and Schwenk from operations on graphs to those on digraphs, but also simplify their arguments for graphs by the use of the tensor polynomial.

4. Real digraphs

Just as integral graphs have been defined as graphs with an integral spectrum, we now say that a *real digraph* has a real spectrum. Although the spectrum of a digraph in general contains real and complex eigenvalues, we see that among the digraphs with four points there are a significant number of real digraphs. Of course a digraph is real if it is symmetric (as it is then a graph) and furthermore a digraph is real if and only if all its strong components are real. Hence only nonsymmetric digraphs are of interest and we restrict consideration to them.

As usual let $K(p_1, \ldots, p_n)$ be a complete n-partite graph and let $\mathscr{D}(G)$ denote the digraph obtained from a graph G on replacing each line by a symmetric pair of arcs. Then it is easy to see that every digraph obtained from $\mathscr{D}(K(p_1, \ldots, p_n))$ by removing any one arc is real.

From Section 2 we can quickly see that it is possible to construct n cospectral strong nonsymmetric digraphs. We just have to take two nonsymmetric cospectrally rooted strong digraphs D_1 and D_2 and form successive coalescence of n copies of these digraphs. But although D_1 and D_2 are real we do not know whether the resulting coalescence $D_1 \cdots D_1 \cdot D_2 \cdots D_2$ is real.

From a star $K_{1,2n}$ with $p = 2n+1$ points, we now obtain a family of real digraphs which contain for any positive number k at least k cospectral digraphs. We will construct a digraph $D_m(K_{1,2n})$ by joining the points in $\mathscr{D}(K_{1,2n})$, which are endpoints in $K_{1,2n}$, with m new arcs where $m \leq 2n$. Furthermore not more than two of these arcs are permitted to have a point u in common, and if so both must start or end in u, as illustrated in Fig. 3. (Thus, in particular, new symmetric pairs are excluded.) From Sachs' theorem we easily get the equation.

$$\Phi(D_m(K_{1,2n})) = x^{2n-2}(x^3 - 2nx - m). \tag{26}$$

Of course for $n \geq 2$ and $m \leq n-2$ we obtain nonisomorphic digraphs with the same characteristic polynomial (26). Thus for every positive integer k, we can find at least k nonisomorphic digraphs of the form $D_m(K_{1,2n})$ if we choose n large enough and $m = n$. Furthermore by (26) we see that for $n = m$, these digraphs are also real.

The binary operations in Section 3 allows us to form nonsymmetric strong regular digraphs which are real.

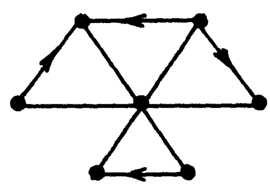

Fig. 3. A strong, real digraph obtained from a star $K_{1,6}$ by adding 4 arcs. Each undirected edge stands for a symmetric pair of arcs.

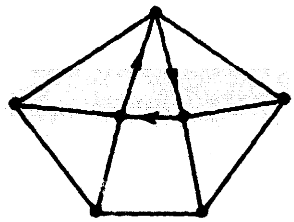

Fig. 4. A real regular nonsymmetric digraph with 7 points, with spectral polynomial $(x-3)(x^2-2)(x+1)^2(x^2+x-1)$.

Theorem 4. *For any positive integer n, we can construct n cospectral strong regular nonsymmetric digraphs which are real.*

Proof. Let G and H be two regular cospectral graphs. An example of two such cubic graphs with 14 points is given in [1]. Thus the two digraphs $\mathscr{D}(G)$ and $\mathscr{D}(H)$ are regular, of course. Furthermore we take the real regular digraph D of Fig. 4 (which was not easy to find). Then for $i=1,\ldots,n$ we define D_i by the iterated cartesian product

$$D_i = D\times\mathscr{D}(G)\times\cdots\times\mathscr{D}(G)\times\mathscr{D}(H)\times\cdots\times\mathscr{D}(H), \tag{27}$$

where we take i copies of $\mathscr{D}(G)$ and $(n-i)$ of $\mathscr{D}(H)$. Each D_i is nonsymmetric (because D is) as well as regular and strong with the same real spectrum. □

5. Gaussian digraphs

A complex number $\lambda=\alpha+i\beta$ is called a *gaussian integer* if α and β are integers. The set of all these number s is written $\mathbf{Z}[i]$.

The next lemma is useful in constructing digraphs with certain properties. The characteristic polynomial (26) of $D_m(K_{1,2n})$ can also be obtained from it. Let $(D_1, \vec{P}(u_1))\cdot(D_2, \vec{P}(u_2))$ be the *generalized coalescence*, where we not only identify u_1 and u_2, but also the dipaths $\vec{P}(u_1)$ and $\vec{P}(u_2)$ of the same length containing u_1 and u_2.

Theorem 5. *If for the rooted digraphs (D_1, u_1) and (D_2, u_2), u_1 and u_2 are contained in all dicycles of D_1 and D_2, then the generalized coalescence $D=(D_1, \vec{P}(u_1))\cdot(D_2, \vec{P}(u_2))$ has the characteristic polynomial*

$$\Phi(D)=x^{p-p_1}\Phi(D_1)+x^{p-p_2}\Phi(D_2)-x^p.$$

Proof. Let u be the identified point $u=u_1=u_2$ in D. Then the subdigraphs $D-u$ and $D-V(\vec{C}(u))$ have no strong nontrivial component. Applying Theorem 2 we

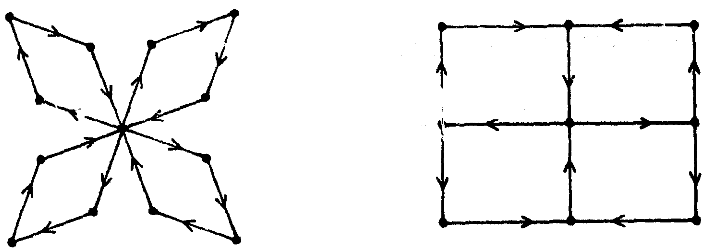

Fig. 5. Two generalized coalescences of four copies of $\vec{C}_4$ which are *gaussian*.

get

$$\Phi(D) = x^p - \sum_{\vec{C} \in \mathscr{C}(u)} \Phi(D - V(\vec{C})) = x^p - \sum_{i=1}^{2} x^{p-p_i} \sum_{\vec{C} \in \mathscr{C}(u)} \Phi(D_i - V(\vec{C}))$$
$$= x^p - \sum_{i=1}^{2} x^{p-p_i}(x^{p_i} - \Phi(D_i)),$$

which proves the lemma. □

If we take the generalized coalescence D of n copies of $(D_1, \vec{P}(u_1))$, then from Lemma 2 it follows readily that

$$\Phi(D) = nx^{p-p_1}\Phi(D_1) - (n-1)x^p. \tag{28}$$

Among the digraphs with four points, the directed cycle $\vec{C}_4$ is obviously gaussian with the spectrum $S(\vec{C}_4) = (\pm 1, \pm \mathrm{i})$.

Corollary 5a. *The generalized coalescence of* n^2 *copies of* $(\vec{C}_4, \vec{P}(u))$ *where* $\vec{P}(u)$ *is a dipath of length* 0, 1 *or* 2 *is gaussian with nonzero eigenvalues* $\pm n$, $\pm n\mathrm{i}$.

For $n^2 = 4$ two nice looking examples are given in Fig. 5.

As the adjacency matrix $A(\vec{C}_4)$ is a normal matrix with orthogonal right eigenvectors, the eigenvectors α^i of its eigenvalues -1, $\pm\mathrm{i}$ are orthogonal to the eigenvector $\beta = (1, 1, 1, 1)^T$ of 1. For the complementary digraph $\bar{\vec{C}}_4$, the adjacency matrix $A(\bar{\vec{C}}_4)$ is of course $J_4 - I_4 - A(\vec{C}_4)$. Therefore this adjacency matrix also has β and α^i as eigenvectors, belonging to the eigenvalues $2, 4, -1 \pm \mathrm{i}$. Thus $\bar{\vec{C}}_4$ is also gaussian, which follows as $\vec{C}_4$ is regular.

Finally we note that from the two gaussian digraphs $\vec{C}_4$ and $\bar{\vec{C}}_4$ we can form arbitrarily large families of gaussian digraphs with the formulas of Theorem 3 involving three binary operations.

6. Integral digraphs

Of course we define an *integral digraph* as one having a spectrum consisting only of integers; so these are all gaussian. It is rather surprising that there are two

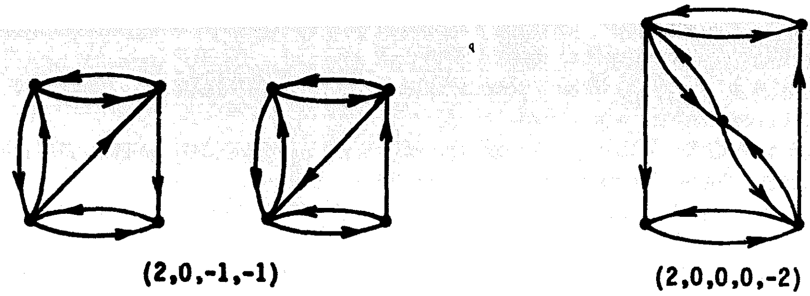

Fig. 6. Three integral digraphs with spectrum indicated.

cospectral integral digraphs with four points, which are the smallest such digraphs. They are illustrated in Fig. 6 together with an integral digraph of five points.

As for real digraphs it is possible to form cospectral integral digraphs by using the cartesian product.

Theorem 6. *For any positive integer n we can find n cospectral strong nonsymmetric digraphs which are integral.*

Proof. The proof is analogous to that of Theorem 5. For $i=1,\ldots,n$ we define D_i by the iterated cartesian product of the two cospectral digraphs D_1 and D_2 of Fig. 6,

$$D_i = D_1 \times \cdots \times D_1 \times D_2 \times \cdots \times D_2, \tag{29}$$

where we take i copies of D_1 and $(n-i)$ of D_2. □

7. Unsolved problems

(a) What are the smallest real regular nonsymmetric digraphs? Is Fig. 4 the smallest one?

(b) What is the smallest pair of cospectral regular nonsymmetric digraphs?

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