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On the Quasi-static Boundary Value Problem of Electrodynamics*

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1. INTRODUCTION

Let Ω be the exterior of a bounded domain D with a smooth boundary Γ and let n denote its outer normal. The quasi-static problem of the title consists in solving the equations (cf. [1, 3]):

$$\begin{aligned} \operatorname{Curl} E &= 0, \\ \operatorname{Div} E &= 0, \end{aligned} \tag{1}$$

subject to the boundary conditions on Γ that

$$n \times E = -n \times E_0, \tag{2}$$

where E_0 is a given initial static field

$$E_0 = -\nabla\phi, \qquad \text{where} \qquad \Delta\phi = 0.$$

In Section 2 this problem is reduced to one for a scalar potential while in Section 3 the main result is obtained by an iterative process for the above boundary value problem.

2. REDUCTION TO A DIRICHLET PROBLEM

Since Curl $E = 0$ it follows that there exists a u such that $E = -\nabla u$ with $\Delta u = 0$ in Ω and where the boundary condition becomes $n \times \nabla(u + \phi)|_\Gamma = 0$. This boundary condition is equivalent to requiring that an arbitrary unit tangent vector t of Γ satisfies

$$t \cdot \nabla(u + \phi)|_\Gamma = 0 \tag{3}$$

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and this in turn can be satisfied by requiring that

$$u + \phi_\Gamma = C = \text{constant}. \tag{4}$$

The constant C can be evaluated from the condition

$$\int_\Gamma \frac{\partial u}{\partial n}\, ds = 0. \tag{5}$$

The last condition is merely the physical statement that the total charge on Γ must be zero.

Thus the boundary problem for u has been reduced to finding a u such that

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega, \\ u &= -\phi + C \quad \text{on} \quad \Gamma, \end{aligned} \tag{6}$$

and

$$\int_\Gamma \frac{\partial u}{\partial n}\, ds = 0. \tag{7}$$

To solve this problem we first find a solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega, \\ u\,|_\Gamma &= \psi, \end{aligned} \tag{8}$$

and then determine C so that (7) is satisfied. In fact this is the necessary and sufficient condition for the solution to problem (6) to be obtained in the form of a double layer potential.

Let $r_{ts} = |\, t - s \,|$ and let A denote the operator

$$A\sigma = \int_\Gamma \frac{\partial}{\partial n_t} \frac{1}{4\pi r_{ts}}\, \sigma(s)\, ds$$

and A^* the operator

$$A^*\tau = \int_\Gamma \frac{\partial}{\partial n_s} \frac{1}{4\pi r_{st}}\, \tau(s)\, ds.$$

Now the solution to (6) is of the form

$$u = \frac{a}{|\, x \,|} + \int_\Gamma \frac{\partial}{\partial n_s} \frac{1}{4\pi r_{sx}}\, \sigma(s)\, ds$$

for an appropriately chosen a, but it is easy to see that (7) implies that $a = 0$.

From the representation

$$u = \int_\Gamma \frac{\partial}{\partial n_s} \frac{1}{4\pi r_{sx}} \tau(s)\, ds \tag{9}$$

and the boundary condition (8) one gets

$$\tau + A^*\tau = 2\psi. \tag{10}$$

This equation will have a solution, by the Fredholm alternative, if and only if

$$\int_\Gamma \psi\sigma\, ds = 0, \tag{11}$$

where σ is any nontrivial solution of the homogeneous equation

$$\sigma + A\sigma = 0.$$

However, this last equation admits only the solution

$$\sigma = K\sigma_0(t),$$

where $K =$ constant $\neq 0$ and $\sigma_0(t)$ is the equilibrium charge distribution on the surface Γ of the perfect conductor. In a previous work [2] an iterative process for calculating $\sigma_0(t)$ was established. It is easy to see that (7) and (11) imply that

$$C = \left(\int_\Gamma \phi\sigma_0(t)\, dt\right)\left(\int_\Gamma \sigma_0(t)\, dt\right)^{-1}. \tag{12}$$

In order to complete the solution an iterative process which solves problem (6) when C is defined by (12) will be constructed.

3. The Iterative Process and Main Theorem

If

$$M = -A^*\nu + \int_\Gamma \nu\, dt \tag{13}$$

and

$$\nu = M\nu + 2\psi, \qquad \psi = -\phi + C, \tag{14}$$

where C is given by (12), the iterative process is defined by

$$\nu_{n+1} = M\nu_n + 2\psi, \qquad \nu_0 = 2\psi, \qquad \nu = \lim_{n\to\infty} \nu_n . \tag{15}$$

In terms of this the following is the main result:

THEOREM. *Equation* (14) *implies* (10) *and the solution of Eq.* (10) *is given by the iterative process* (15).

To establish that (14) implies (10) multiply (14) by σ_0 and integrate over Γ. This yields

$$\int_\Gamma \nu\sigma_0\, dt = -(A^*\nu, \sigma_0) + \int_\Gamma \sigma_0\, dt \int_\Gamma \nu\, dt.$$

However, since

$$(A^*\nu, \sigma_0) = (\nu, A\sigma_0) = -(\nu, \sigma_0)$$

and

$$\int_\Gamma \sigma_0\, dt \neq 0,$$

it follows that

$$\int_\Gamma \nu\, dt = 0,$$

from which the desired implication follows.

In order to establish the validity of the iterative process it is sufficient to prove that M does not have any eigenvalues λ with $|\lambda| \leqslant 1$.

Suppose that λ is an eigenvalue satisfying $\nu = \lambda M\nu$, i.e.,

$$\nu = -\lambda A^*\nu + \lambda \int_\Gamma \nu\, dt, \tag{16}$$

and let

$$u = \int_\Gamma \nu(t) \frac{\partial}{\partial n_t} \frac{1}{4\pi r_{xt}}\, dt.$$

Denoting the exterior region by the subscript e and the interior by i, it follows that

$$(1 + \lambda)\, u_{\mathrm{e}} = (1 - \lambda)\, u_{\mathrm{i}} + \lambda \int_\Gamma (u_{\mathrm{e}} - u_{\mathrm{i}})\, dt. \tag{17}$$

Multiplying this equation by $\partial u/\partial n$ and taking into account that

$$\frac{\partial u}{\partial n_{\mathrm{e}}} = \frac{\partial u}{\partial n_{\mathrm{i}}}, \tag{18}$$

we get:

$$\frac{1+\lambda}{1-\lambda} \int_\Gamma u_{\mathrm{e}} \frac{\partial u}{\partial n_{\mathrm{e}}}\, dt = \int_\Gamma u_{\mathrm{i}} \frac{\partial u}{\partial n_{\mathrm{i}}}\, dt + \int_\Gamma (u_{\mathrm{e}} - u_{\mathrm{i}})\, dt \cdot \int_\Gamma \frac{\partial u}{\partial n}\, dt \cdot \frac{\lambda}{1-\lambda}.$$

Green's formula implies that

$$\int_\Gamma u_{\mathrm{e}} \frac{\partial u}{\partial n_{\mathrm{e}}}\, dt \leqslant 0, \qquad \int_\Gamma u_i \frac{\partial u}{\partial n_{\mathrm{i}}}\, dt \geqslant 0,$$

and

$$\int_\Gamma \frac{\partial u}{\partial n}\, dt = 0,$$

from which it follows that

$$\frac{1+\lambda}{1-\lambda} \leqslant 0$$

and, in turn, that λ is real with $|\lambda| \geqslant 1$.

To conclude the proof it must be shown that ± 1 are not eigenvalues of M. If $\lambda = -1$, then from (17) it follows that

$$2 \int_\Gamma u_i \frac{\partial u}{\partial n}\, dt = 0,$$

so that

$$\int_D |\nabla u|^2\, dx = 0, \qquad u = \text{const} \quad \text{in} \quad D,$$

$$\frac{\partial u}{\partial n_e} = \frac{\partial u}{\partial n_i} = 0.$$

Thus

$$\frac{\partial u}{\partial n_e} = 0, \qquad u = 0 \quad \text{in} \quad \Omega, \qquad \nu = u_e - u_i = \text{const}.$$

Without loss of generality it can be assumed that $\nu = 1$. However, for $\lambda = -1$ no solution exists for Eq. (16) as it follows from

$$1 = M1$$

that

$$(\sigma_0, 1) = -(1, \sigma_0) - S \int_\Gamma \sigma_0\, dt,$$

where $S = \text{meas}\, \Gamma$. This is a contradiction since $\int_\Gamma \sigma_0\, dt > 0$.

Finally for $\lambda = 1$, Eq. (16) reduces to

$$\nu = -A^*\nu + \int_\Gamma \nu\, dt \tag{19}$$

and by the Fredholm theorem this can have a solution if and only if

$$\int_\Gamma \nu\, dt \cdot \int_\Gamma \sigma_0\, dt = 0.$$

Since

$$\int_{\Gamma} \sigma_0 \, dt \neq 0$$

this will be possible if and only if $\int_{\Gamma} \nu \, dt = 0$, so that (19) reduces to

$$\nu = -A^* \nu,$$

which, in turn, implies that

$$\nu = \text{const} \neq 0$$

and this contradiction implies that $\lambda = 1$ is not an eigenvalue of (M) and hence the main theorem has been proved. In [3], iterative processes for solutions of the interior and exterior Dirichlet and Neumann problems are given.

References

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