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Some Properties of Two Analytic Functions Associated with Complex Polynomials

ZALMAN RUBINSTEIN

University of Haifa, Haifa, Israel;
and
University of Michigan, Ann Arbor, Michigan 48109

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Two basic analytic functions $\alpha(z)$ and $\beta(z)$ defined in domains depending on the location of the zeros of a complex polynomial $P(z)$ are given by $P'/P = n/(z - \alpha)$ and $P = (z - \beta)^n$. These functions are studied with respect to their growth and their Laurent expansion coefficients. Applications to the location of zeros of complex polynomials are indicated.

1. INTRODUCTION

Associated with an nth-degree monic polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = \prod_{k=1}^{n} (z - z_k) \tag{1}$$

are two functions, referred to as the coincidence functions $\alpha(z)$ and $\beta(z)$ which satisfy the relations

$$\frac{P'(z)}{P(z)} = \frac{n}{z - \alpha(z)} \tag{2}$$

and

$$P(z) = (z - \beta(z))^n. \tag{3}$$

These functions are quite basic in the theory of the location of zeros of various functions of $P(z)$ and its derivative and have been introduced by Walsh [3, 4] and studied further and applied, for example, in [1, 2, 5, 6].

In these studies a relatively simple case was considered, namely, the case where a gap appears in the expansion of $\alpha(z)$ and $\beta(z)$. No general formula for the coefficients of these functions was obtained and no relations that exist between the coefficients of $\alpha(z)$, $\beta(z)$ and those of $P(z)$ were applied. It is the purpose of this note to close somewhat this gap. In Section 2 a general formula for the coefficients of the expansion of $\alpha(z)$ is obtained in terms of the power

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sums of the zeros of $P(z)$. Also a recursive relation between these coefficients and the coefficients of $P(z)$ is indicated. These basic relations are applied to obtain growth estimates for the function $\alpha(z)$ in its domain of analyticity and its Laurent expansion coefficients. Also one typical zero location result is indicated.

In Section 3 a similar study is made of the function $\beta(z)$ defined by (3) and corresponding results are obtained.

In Section 4 an application of the previous results is made to the case of polynomials with one fixed zero.

In Section 5 the particular case of a trinomial is considered. As an application of the general theory a necessary and sufficient condition is obtained for a trinomial to have all its zeros in the closed interior or in the closed exterior of the unit disk.

This condition is expressed as an analytical inequality and differs from the classical conditions involving determinantal inequalities or iterative calculations.

Throughout this note it will be assumed for simplicity that all the z_k, $k = 1, 2, \ldots, n$, lie in the closed unit disc and the coefficients a_k in (1) will be defined also for $k > n$ as $a_k = 0$. In this case it is known that the functions $\alpha(z)$ and $\beta(z)$ can be defined as analytic functions in z for all z satisfying $|z| > 1$, and such that $|\alpha(z)| \leqslant 1$ and $|\beta(z)| \leqslant 1$ there. We shall also denote by t_p, $p = 1, 2, \ldots$, the sums

$$t_p = \sum_{i=1}^{n} z_i^{p}. \tag{4}$$

2. Some Properties of the Function $\alpha(z)$

Lemma 1. *Let the expansion of the function $\alpha(z)$ as defined by* (2) *for z, $|z| > 1$ be*

$$\alpha(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots. \tag{5}$$

then

$$\begin{aligned} c_k = \frac{1}{n} t_{k+1} - \frac{1}{n^2} \sum_{i_1+i_2=k+1} t_{i_1} t_{i_2} + \frac{1}{n^3} \sum_{i_1+i_2+i_3=k+1} t_{i_1} t_{i_2} t_{i_3} \\ - \cdots + (-1)^k \frac{1}{n^{k+1}} t_1^{k+1}, \qquad k = 0, 1, \ldots, \end{aligned} \tag{6}$$

where t_p is defined by (4) *and where the sums are taken over all permutations $(i_1, i_2, \ldots)$, of positive integers subject to the conditions indicated.*

Proof. Relation (2) can be expressed as

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i} = \frac{n}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots = \frac{n}{z - \alpha(z)}.$$

Hence, taking into account (5) we obtain

$$n = \left(\frac{n}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots\right)\left(z - c_0 - \frac{c_1}{z} - \frac{c_2}{z^2} - \cdots\right).$$

Equating powers of z we have

$$c_{k+1} = \frac{1}{n}\,(t_{k+2} - t_{k+1}c_0 - t_k c_1 - \cdots - t_2 c_{k-1} - t_1 c_k), \qquad k = 0, 1,\ldots. \tag{7}$$

Now (6) is established by induction. One verifies directly that $c_0 = t_1/n$. Assuming (6) for all negative integers not exceeding k, and applying (7), we deduce

$$\begin{aligned} nc_{k+1} &= t_{k+2} - \sum_{j=0}^{k} t_{k+1-j}c_j \\ &= t_{k+1} - \sum_{j=0}^{k} t_{k+1-j}\left(\frac{1}{n}\,t_{j+1} - \frac{1}{n^2}\sum_{i_1+i_2=j+1} t_{i_1}t_{i_2}\right. \\ &\quad \left. + \frac{1}{n^3}\sum_{i_1+i_2+i_3=j+1} t_{i_1}t_{i_2}t_{i_3} + \cdots + (-1)^j\frac{1}{n^{j+1}}\,t_1^{j+1}\right) \\ &= t_{k+2} - \frac{1}{n}\sum_{j=0}^{k} t_{k+1-j}t_{j-1} + \frac{1}{n^2}\sum_{j=1}^{k}\sum_{i_1+i_2=j+1} t_{k+1-j}t_{i_1}t_{i_2} \\ &\quad - \frac{1}{n^3}\sum_{j=2}^{k}\sum_{\Sigma i_p=j+1} t_{k+1-j}t_{i_1}t_{i_2}t_{i_3} + \cdots + (-1)^{k+1}\frac{t_1^{k+1}}{n^{k+1}}. \end{aligned}$$

This last relation is equivalent to (6). One also notes that the number of terms multiplying $1/n^{j+1}$ in c_k is $\binom{k}{j}$. For instance,

$$c_1 = \frac{t_2}{n} - \frac{t_1^2}{n^2}, \qquad c_2 = \frac{t_3}{n} - \frac{2t_1t_2}{n^2} + \frac{t_1^3}{n^3},$$

$$c_3 = \frac{t_4}{n} - \frac{1}{n^2(2t_1t_3 + t_2^2)} + \frac{1}{n^3 3t_2t_1^2} - \frac{t_1^4}{n^4}.$$

Obviously Nowton's formulas

$$t_k + a_1t_{k-1} + \cdots + a_{k-1}t_1 + ka_k = 0, \qquad k = 1, 2,\ldots, \tag{8}$$

where $a_j = 0$ for $j > n$ allow one to express the c_k in terms of the coefficients a_k of $P(z)$. Sometimes it is advantageous to use a direct relation between the a_k and the c_k. Although no simple direct formula, similar to (6), seems to exist, it is possible to obtain a recursive relation similar to (7). This relation obtained from (2) is

$$c_k = -\frac{1}{n}\left((n-1)\,a_1c_{k-1} + (n-2)\,a_2c_{k-2} + \cdots + (n-k)\,a_kc_0 + (k+1)\,a_{k+1}\right)$$

for $k = 1, 2, \ldots,$

$$c_0 = -\frac{a_1}{n} \qquad (a_j = 0 \text{ for } j > n). \tag{9}$$

We now indicate two corollaries

COROLLARY 1. (a) *The three statements* $a_1 = a_2 = \cdots = a_{k-1} = 0$, $t_1 = t_2 = \cdots = t_{k-1} = 0$, *and* $c_0 = c_1 = \cdots = c_{k-1} = 0$ *are equivalent.*

(b) *If the coefficients of* $P(z)$ *in* (1) *satisfy* $a_1 = a_2 = \cdots = a_{k-1} = 0$, *then*

$$c_j = \frac{-(j+1)}{n}\,a_{j+1} \qquad \text{for } k-1 \leqslant j \leqslant 2k-2,$$

where the convention with regards to the coefficients a_j *indicated before applies for and throughout this note.*

COROLLARY 2. *Set* $M_p = \max_{1\leqslant k\leqslant p} |t_k|$, *then*

$$|c_j| \leqslant \frac{M}{n}\left(1+\frac{M}{n}\right)^k \qquad \text{for } j = 0, 1,\ldots, p-1.$$

The upper bound is attained by $p(z) = 1 + z + \cdots + z^n$, $t_1 = \cdots = t_n = -1$, $c_k = -(1/n)(1+1/n)^k$ *for* $k = 0, 1,\ldots, n-1$.

Proof. By (6)

$$|c_j| \leqslant \frac{1}{n}M + \frac{1}{n^2}\binom{j}{1}M^2 + \cdots + \frac{1}{n^{j+1}}M^{j+1} = \frac{M}{n}\left(1+\frac{M}{n}\right)^j.$$

Concerning the growth of $\alpha(z)$ we have

THEOREM 1. *Set* $\alpha(z)$ *be given by* (2), *where all the* z_k, $k = 1, 2,\ldots, n$ *lie in the closed unit disc. Then for* $p = 1, 2,\ldots$

$$|z^p\alpha(z) - c_0z^p - c_1z^{p-1} - \cdots - c_{p-1}z| \leqslant 1 + |c_0| + \cdots + |c_{p-1}| \qquad \text{for } |z| > 1. \tag{10}$$

Proof. The function $a(\zeta) = \alpha(1/\zeta)$ is analytic for $|\zeta| < 1$, and $|a(\zeta)| < 1$ there. Moreover

$$a(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \cdots.$$

The function

$$b(\zeta) = \frac{a(\zeta) - c_0 - c_1\zeta - \cdots - c_{p-1}\zeta^{p-1}}{1 + |c_0| + |c_1| + \cdots + |c_{p-1}|}$$
$$= b_p\zeta^p + b_{p+1}\zeta^{p+1} + \cdots$$

satisfies $|b(\zeta)| \leqslant 1$ in $|\zeta| < 1$. The inequality (10) is deduced by applying Schwarz's lemma. Combining Theorem 1 with Corollary 1 we have

COROLLARY 3. *Let the polynomial $p(z)$ and the function $\alpha(z)$ be related by* (2). *Then the following two statements are equivalent.*

(a) *The polynomial $P(z)$ has the form $z^n + a_p z^{n-p} + \cdots + a_n$, $p \geqslant 1$ and all its zeros lie in the closed unit disc.*

(b) *The function $\alpha(z)$ is analytic in the exterior of the unit disc and satisfies there the inequality $|\alpha(z)| \leqslant |z|^{-p+1}$.*
The implication (a) $\to$ (b) *was established in* [4, 5].

COROLLARY 4. *If $P(z) = z^n + a_p z^{n-p} + \cdots + a_n$ has all its zeros in the closed unit disc, then all the zeros of the equation $P'(z)/P(z) = c$, $c \neq 0$ which do not lie in the closed unit disc, lie inside the lemniscate*

$$\left| z^{p+1} - \frac{n}{c} z^p + a_p \frac{p}{n} z \right| \leqslant 1 + \frac{p}{n} |a_p| . \tag{11}$$

Proof. If z is a zero of $P'(z)/P(z) - c$ and $|z| > 1$, then $\alpha(z) = z - n/c$. Since $c_0 = c_1 = \cdots = c_{p-2} = 0$, $c_{p-1} = -(p/n)\,a_p$, we have by (10) inequality (11). We remark that when $a_p = 0$, (11) implies the classical inequality which in our case reads

$$|z^{p-1}| \left| z - \frac{n}{c} \right| \leqslant 1.$$

It is clear that since $|z| > 1$ (11) is generally stronger than the last inequality.

3. SOME PROPERTIES OF THE FUNCTION $\beta(z)$

The function $\beta(z)$ which satisfies (3) can be defined out of the relation

$$\ln\left(1 - \frac{\beta(z)}{z}\right) = \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 - \frac{z_k}{z}\right) . \tag{12}$$

If all z_k satisfy $|z_k| \leqslant 1$ and if $|z| > 1$, it follows from the fact that the function

$\ln(1 - \zeta)$, $\ln 1 = 0$ maps the open unit disk onto a convex region univalently, that $\beta(z)$ is uniquely determined by the branch of the logarithmic function. Furthermore $\beta(z)$ is analytic in $|z| > 1$ and $|\beta(z)| \leqslant 1$ there

LEMMA 2. *The function $\beta(z)$ defined by* (3) *or* (12) *to be an analytic function in the region $|z| > 1$ has the expansion*

$$\beta(z) = d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \cdots, \tag{13}$$

where the coefficients d_k, $k = 0, 1,\ldots,$ satisfy the relations $d_0 = t_1/n$ and

$$d_k = -\frac{1}{n(k+1)}(t_1 d_{k-1} + t_2 d_{k-2} + \cdots + t_k d_0 - t_{k+1}), \qquad k = 1, 2,\ldots. \tag{14}$$

Proof. For convenience let $\zeta = 1/z$. We evaluate $\beta(\zeta)$ by the formula

$$\zeta\beta(\zeta) = 1 - \exp\left(\frac{1}{n}\sum_{k=1}^{n} \ln(1 - z_k\zeta)\right). \tag{15}$$

Now, for $|\zeta| < 1$,

$$\frac{1}{n}\sum_{k=1}^{n} \ln(1 - z_k\zeta) = -t_1\zeta - \frac{t_2}{2}\zeta^2 - \cdots - \frac{t_k}{k}\zeta^k - \cdots. \tag{16}$$

To evaluate the coefficients of the exponential of a power series we make the following observation: if

$$p(z) = p_0 + p_1 z + \cdots,$$
$$q(z) = q_0 + q_1 z + \cdots,$$

and $q(z) = e^{p(z)}$, then from the differential equation $q' - p'q = 0$ one obtains

$$(k+1)\, q_{k+1} = p_1 q_k + 2p_2 q_{k-1} + \cdots + (k+1)\, p_{k+1} q_0, \tag{17}$$

$k = 0, 1,\ldots,$ and $q_0 = \exp p_0$. By (16), $jp_j = -t_j/n$, $j = 1, 2,\ldots,$ and $p_0 = 0$. The result follows now by (15) and (17).

COROLLARY 5. *If the zeros of the polynomial $P(z)$ satisfy the symmetry conditions $t_1 = t_2 = \cdots = t_k = 0$, then the coefficients d_k of the expansion of $\beta(z)$ satisfy the conditions $d_0 = d_1 = \cdots = d_{k-1} = 0$ and $d_j = t_{j+1}/nj$ for $k \leqslant j \leqslant 2k$. In particular $|d_j| \leqslant 1/j$.*

Proof. The first part follows directly by (14). For the second part consider (14) for d_j, $k \leqslant j \leqslant 2k$. It is easy to see that all the terms of the form $t_i d_{j-i}$ in (14) vanish by the first part of this corollary. As an application of Corollary 5 and (8) we mention

COROLLARY 6. *Under the hypotheses of Corollary* 5

$$a_i = -\frac{t_i}{i} \qquad \text{for } k+1 \leqslant i \leqslant 2k+1$$

and

$$d_j = \frac{-(j+1)\,a_{j+1}}{nj} \qquad \text{for } k \leqslant j \leqslant 2k.$$

A typical simple application of the above results to the locations of the zeros of various functions of $P(z)$ is

COROLLARY 7. *Let* $P(z) = z^n + a_p z^{n-p} + \cdots + a_n$, $p \geqslant 2$. *Then all the zeros of the polynomial* $P(z) - c$, *which are outside the closed unit disk, lie inside the lemniscate*

$$\left| nz^{p+1} - nc^{1/n} z^p + a_p \frac{p}{p-1} z \right| \leqslant n + | a_p | \frac{p}{p-1} \tag{18}$$

for a suitable choice of $c^{1/n}$. *When* $a_p = 0$ *the above result reduces to the known estimate in* [6, Theorem 5].

Proof. If z is a zero of $P(z) - c$ which lies outside the closed unit disc, then by (3), $\beta(z) = z - c^{1/n}$ for some choice of $c^{1/n}$. By (13), Corollaries 5 and 6

$$n\beta(z) + a_p \frac{p}{p-1} z^{-p+1} = O(z^{-p}) \qquad \text{for } | z | > 1.$$

Hence by Schwartz's lemma and the inequality $| \beta(z)| \leqslant 1$ for $| z | > 1$ we have

$$\left| n\beta(z) + a_p \frac{p}{p-1} z^{-p+1} \right| \leqslant | z |^{-p} \left(n + | a_p | \frac{p}{p-1}\right)$$

for $| z | > 1$. Substituting the value of $\beta(z)$ we obtain the desired result. We remark that (18) can be written as

$$| z | \, | f(z) + A | \leqslant 1 + | A | \tag{19}$$

with $f(z) = z^{p-1}(z - c^{1/n})$ and $A = pa_p/(n(p-1))$. This is generally better than the known result

$$| f(z)| \leqslant 1 \qquad \text{for } | z | > 1. \tag{20}$$

Indeed by Corollary 6, $|A| \leqslant 1/(p-1)$. Thus the locus (19) is contained in locus (20) at least for $|z| \geqslant p/(p-2)$, $p \geqslant 3$. In particular (19) is stronger than (20) for all $|z| \geqslant 3$ and $p \geqslant 3$.

4. Polynomials with a Known Zero

The example $p(z) = 1 + z + \cdots + z^n$, $t_k + 1 = 0$ for $k \neq (n+1)j$ and $t_k = n$ otherwise indicates that one does not have to explicitly know all the zeros of $p(z)$ to evaluate the numbers t_k. We illuminate this situation more precisely in

Lemma 3. *Let* $q(z) = (z-a)\,p(z) = (z-a)(z-z_1)\cdots(z-z_n) = z^{n+1} + a_p z^{n-p+1} + \cdots + a_{n+1}$, $p \geqslant 1$, *where* $p(z)$ *is a polynomial all of whose zeros lie in the closed unit disk and* a *a given complex constant. Then the coefficients* c_k *and* d_k *of* $\alpha(z)$ *and* $\beta(z)$ *respectively relative to* $P(z)$ *as defined by* (2) *and* (3) *satisfy the relations*

$$c_0 = d_0 = -\frac{a}{n},$$

$$c_k = -\frac{a^{k+1}}{n}\left(1-\frac{1}{n}\right)^k, \qquad k = 0, 1,\ldots, p-1 \tag{21}$$

and

$$d_k = \delta_k(n)\, a^{k+1},$$

where the constants $\delta_k(n)$ *satisfy the difference equation*

$$\delta_k(n) = \frac{1}{n(k+1)}\left(\delta_{k-1}(n) + \delta_{k-2}(n) + \cdots + \delta_0(n) - 1\right), \qquad k = 1,\ldots, p-1. \tag{22}$$

Proof. By hypothesis $t_j + a^j = 0$ for $j = 1, 2,\ldots, p-1$. By (7)

$$c_{k+1} = -\frac{1}{n}\left(a^{k+2} + a^{k+1}c_0 + a^k c_1 + \cdots + a c_k\right)$$

$k = 0, 1,\ldots$, $c_0 = t_1/n = -a/n$. This difference equation has the solution (21). Similarly applying (14) one obtains (22). We conclude this section with one simple application.

Corollary 8. *All the zeros of the derivative of the polynomial* $q(z)$ *defined in Lemma* 3 *lie in the union of the closed unit disc and the lemniscate*

$$|z|\left|(n+1)z - a\left(n-\frac{1}{n}\right)\right| \leqslant 1 + \frac{|a|}{n}.$$

Proof. Since

$$\frac{q'(z)}{q(z)} = \frac{p'(z)}{p(z)} + \frac{1}{z-a} = \frac{n}{z-\alpha(z)} + \frac{1}{z-a}$$

for $|z| > 1$, then any zero of $q'(z)$ which is not a zero of $q(z)$ and which lies in $|z| > 1$ satisfies the relation

$$\alpha(z) = (n+1)z - na. \tag{23}$$

Relation (10) with $p = 1$ combined with (21) and (23) implies Corollary 8.

5. The Case of a Trinomial

The case of a trinomial is particularly interesting since we can explicitly evaluate all the coefficients. Indeed applying (9) to the irreducible trinomial $p(z) = z^n + a_p z^{n-p} + a_n$, $n \neq jp$, we obtain

$$c_{(j-1)+lp} = -\frac{j}{n}\left(\frac{p-n}{n}a_p\right)^l a_j, \qquad l = 0, 1, \ldots,$$

where $j = p, n$ and all the other c_k vanish. In particular $c_0 = c_1 = \cdots = c_{p-2} = 0$, $c_{p-1} = (p/n)a_p$, $c_p = \cdots = c_{2p-2} = 0$, $c_{2p-1} = (p(n-p)/n)a_p^2$. Moreover the function $\alpha(z)$ can be applied directly to obtain a necessary and sufficient condition different from the usual iterative or determinants criteria for a trinomial to have all its zeros in the closed interior or exterior of the unit disc.

THEOREM 2. *Let $P(z) = z^n + a_p z^{n-p} + a_n$, $a_p \neq 0$. If $2p \geqslant n$, $a_p > 0$, then a necessary and sufficient condition for the trinomial $P(z)$ to have all its zeros in the closed unit disc is that the inequality*

$$\operatorname*{Max}_{0\leqslant\theta\leqslant 2\Pi} \operatorname{Re}\left(a_n e^{i(n-p)\theta} - \frac{n-p}{p} e^{i\theta p}\right) \leqslant \frac{n(1-|a_n|^2) - (2p-n)a_p^2}{2pa_p}. \tag{24}$$

Proof. The function $\alpha(z)$ in our case satisfies by (2) and (10) the relations

$$-\alpha(z)\, z^{n-p-1} = \frac{pa_p z^{n-p} + na_n}{nz^p + (n-p)a_p}$$

and $|\alpha(z)| \leqslant |z|^{-p+1}$ for $|z| > 1$. These conditions can be expressed in the form

$$|pa_p + na_n\zeta^{n-p}| \leqslant |n + (n-p)a_p\zeta^p| \tag{25}$$

for $|\zeta| < 1$. By Corollary 6, $a_p = -t_p/p$. Thus $|a_p| \leq n/p$ and hence $|a_p| \leq n/(n-p)$ if $2p > n$ and $|a_p| = n/(n-p)$ only for $2p = n$, $a_p = 2$, and $a_n = 1$. In the latter case (24) is satisfied since both sides of Eq. (24) vanish. Assuming that $|a_p| < n/(n-p)$, inequality (25) for $|\zeta| < 1$ is equivalent by the maximum principle to the same inequality for $|\zeta| = 1$. A standard calculation shows that (24) and (25) for $|\zeta| = 1$ are also equivalent.

Various simple necessary conditions and sufficient conditions can be derived from (24). For example a simple sufficient condition is

$$4pa_p \leqslant n(1 - |a_n|^2) - (2p - n)\, a_p^2.$$

The necessary condition $|a_n| \leqslant 1$ follows from (24) since if $|a_n| > 1$, then the left-hand side of (24) exceeds $|a_n| - (n-p)/p > 0$, while the right-hand side is negative. Obviously the assumption $a_p > 0$ in Theorem 2 does not limit its generality because a preliminary transformation of the form $e^{i\alpha}P(ze^{i\gamma})$ will reduce $P(z)$ to the desired form. If $2p < n$, Theorem 2 can then be applied to the polynomial $z^nP(1/z)$. Inequality (24) will then give a necessary and sufficient condition for the zeros and lie in the closed exterior of the unit disc.

In the particular case of a quadratic equation

$$z^2 + az + b$$

$\arg a = \alpha$, one obtains the following necessary and sufficient condition for both zeros to lie in the closed unit disc.

$$|a| \, |b - e^{2i\alpha}| \leqslant 1 - |b|^2.$$

It is possible to prove this inequality also directly but not as conveniently. We conclude with a few remarks about the maximum M of the left-hand side of (24) which is attained, say for $\theta = \theta^*$. It is easy to see that θ^* satisfies the equation

$$|a_n| \sin((n-p)\,\theta^* + \arg a_n) + \sin(\theta^* p) = 0.$$

Also

$$\left(M + \frac{n-p}{p}\cos(\theta^* p)\right)^2 + \sin^2(\theta^* p) = |a_n|^2$$

and

$$\left(M + \frac{n-p}{p}\cos(\theta^* p)\right)^2 = |a_n|^2 \cos^2((n-p)\,\theta^* + \arg a_n).$$

These relations can be used for numerical calculations.

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