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Perturbation Theory for Eigenvalues and Resonances of Schrödinger Hamiltonians

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Suppose that $e^{2\epsilon|x|}V \in \mathrm{Re}\, L^p(\mathbb{R}^3)$ for some $p > 2$ and for $g \in \mathbb{R}$, $H(g) = -\Delta + gV$. The main result, Theorem 3, uses Puiseaux expansions of the eigenvalues and resonances of $H(g)$ to study the behavior of eigenvalues $\lambda(g)$ as they are absorbed by the continuous spectrum, that is $\lambda(g) \nearrow 0$ as $g \searrow g_0 > 0$. We find a series expansion in powers of $(g - g_0)^{1/2}$, $\lambda(g) = \sum_{n=2}^{\infty} a_n(g - g_0)^{n/2}$ whose values for $g < g_0$ correspond to resonances near the origin. These resonances can be viewed as the traces left by the just absorbed eigenvalues.

Suppose that for some $\epsilon > 0$, $e^{2\epsilon|x|}V \in \mathrm{Re}\, L^p(\mathbb{R}^3)$ for some $p > 2$.[1] We study the dependence on g of the eigenvalues and resonances of

$$H(g) = -\sum_{L=1}^{3} \frac{\partial^2}{\partial x_i^2} + gV,$$

the self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain the Sobolev space $W^{2,2}(\mathbb{R}^3)$. An important use of the general results is to study the behavior of eigenvalues as they approach $[0, \infty)$, the continuous spectrum of H. For example, suppose that $g_1 > 0$ and $\lambda(g_1) < 0$ is an eigenvalue of $H(g)$. Rellich [11] has proved that for g near g_1, there is a holomorphic function $\lambda(g)$ whose values are negative eigenvalues of $H(g)$. In addition, λ is a decreasing function of g. In this way one obtains an analytic function $\lambda(g)$ on $(g_0, g_1]$ such that $\lambda(g) \nearrow 0$ as $g \searrow g_0 > 0$. At g_0 the eigenvalue arrives at the continuous spectrum and standard perturbation theory fails to describe the behavior of $\lambda(g)$ for g near g_0. For short range V, as above, we will obtain a convergent Puiseux expansion for $\lambda(g)$ in powers of $(g - g_0)^{1/2}$ which is valid on a neighborhood of g_0. The series for $g < g_0$ gives

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[1] Some of the results, in particular Theorems 1 and 2, are valid without the reality assumption. In this case $H(g)$ with domain $W^{2,2}(\mathbb{R}^3)$ is not self-adjoint but $D(H^*) = D(H)$ and $H - H^*$ is compact relative to $H + H^*$.

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the location of resonances (poles of an analytic continuation of the resolvent). Thus we have the pleasing result that $\lambda(g)$ does not disappear at $g = g_0$ but merely becomes a resonance rather than a bound state. Turning the picture around, as $g \nearrow g_0$ a resonance converges toward 0 and for $g > g_0$ emerges as a bound state. The framework of this paper is the same as the earlier work [10] except that we have altered several sign conventions to agree with standard notation.

To describe the results precisely we must recall some facts about $H(g)$ and its resolvent. Let $H_0 = H(0)$. The spectrum, $\sigma(H_0)$, is the positive real axis $[0, \infty)$ and the resolvent $R^0(z) = (z - H_0)^{-1}$ of H_0 is an integral operator with

$$\text{kernel } R^0(-\zeta^2) = -\frac{e^{-\zeta|y-x|}}{4\pi\,|y-x|}, \qquad \operatorname{Re}\zeta > 0. \tag{1}$$

Let $E_\epsilon \in \operatorname{Hom}(L^2(\mathbb{R}^3))$ be the operator $E_\epsilon\phi = e^{-\epsilon|x|}\phi$ and let $R_\epsilon^0 = E_\epsilon R^0 E_\epsilon$.[2] Then, $R_\epsilon^0(-\zeta^2)$ is holomorphic for $\operatorname{Re}\zeta > -\epsilon$ with values in the Hilbert–Schmidt operators on $L^2(\mathbb{R}^3)$. The surprising thing is that as ζ crosses the imaginary axis, $-\zeta^2$ crosses $\sigma(H_0) = [0, \infty)$ so that even though $R^0(-\zeta^2)$ is singular the localized resolvent has an analytic continuation (see Fig. 1).

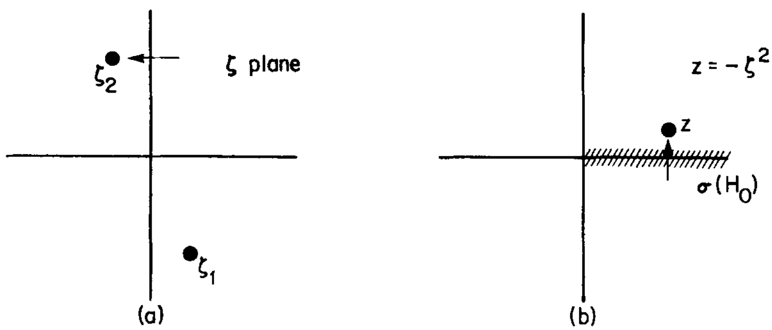

FIGURE 1

Note that for a fixed point $z \in \mathbb{C}\backslash[0, \infty)$ there may be two values ζ_1, ζ_2 with $\operatorname{Re}\zeta_j > -\epsilon$ and $z = -\zeta_j^2$ $j = 1, 2$. Such a situation is depicted in Fig. 1. Then if $\operatorname{Re}\zeta_1 > 0$ we have $R_\epsilon^0(-\zeta_1^2) = E_\epsilon(z - H_0)^{-1}E_\epsilon$ while $R_\epsilon^0(-\zeta_2^2)$ is not related immediately to the resolvent of H but arise by analytic continuation of $E_\epsilon(z - H_0)^{-1}E_\epsilon$ across the spectrum as indicated in (b) of Fig. 1. In the physics

[2] In [10] it was convenient to modify $|x|$ near 0 to be a smooth function of x. The associated operators E_ϵ are related by a similarity transformation as are the operators R_ϵ^0, R_ϵ defined with their help. Thus the assertions proved in [10] with $|x|$ modified imply parallel results with E_ϵ as defined here.

literature these are called the values of the resolvent on the "unphysical sheet." In our notation we see that the values of $R^{\epsilon 0}(-\zeta^2)$ for Re $\zeta > 0$ correspond to the "physical sheet" and for Re $\zeta < 0$ to the "unphysical" sheet.

For any real g the spectrum of $H(g)$ consists of $[0, \infty)$ plus a finite number of negative eigenvalues of finite multiplicity. Thus for Re $z \ll -1$, $(z - H(g))^{-1} = R(z)$ exists[3] and we have the resolvent identity

$$R(z) - R^0(z) = gR(z)\, VR^0(z) \tag{2}$$

relating R^0 and R. In addition, $R(z)$ is meromorphic in the slit plane $\mathbb{C}\backslash[0, \infty)$ with poles precisely at the negative eigenvalues of $H(g)$. Multiplying (2) on the left and right by E_ϵ yields the localized equation for Re $\zeta > 0$, $-\zeta^2 \notin \sigma(H)$,

$$R_\epsilon(-\zeta^2)\,[I + K(g, \zeta)] = R_\epsilon{}^0(-\zeta^2), \tag{3}$$

where $K(g, \zeta)$ is the integral operator with

$$\text{kernel } K(g, \zeta) = + \frac{g e^{\epsilon|x|}\, V(x)\, e^{-\zeta|y-x|} e^{-\epsilon|y|}}{4\pi\, |\, y - x\,|}. \tag{4}$$

Assuming that $e^{-2\epsilon|x|} V \in L^2(\mathbb{R}^3)$, it follows that the map $(g, \zeta) \mapsto K(g, \zeta)$ is holomorphic on $\mathbb{C} \times \{\text{Re}\, \zeta > -\epsilon\}$ with values in the Hilbert–Schmidt operators on $L^2(\mathbb{R}^3)$. For g fixed, $\|\, K(g, \zeta)\| \leqslant \|\, K(g, \zeta)\|_{\text{Hilbert–Schmidt}} \to 0$ as Re $\zeta \to \infty$, so the analytic Fredholm theorem implies that the function

$$\zeta \mapsto R_\epsilon(-\zeta^2) = R_\epsilon{}^0(-\zeta^2)\,[I + K(g, \zeta)]^{-1} \tag{5}$$

is meromorphic on Re $\zeta > -\epsilon$ with values in Hom$(L^2(\mathbb{R}^3))$ and has no poles for Re $\zeta \gg 1$. As remarked above, the poles of $R_\epsilon(-\zeta^2)$ in Re $\zeta > 0$ correspond to negative eigenvalues of H. A generalization of this result to positive eigenvalues is proved in [10] assuming that $e^{2\epsilon|x|} V \in L^p(\mathbb{R}^3)$ for some $p > 2$. Precisely, a point $\zeta_0 \in i\mathbb{R}\backslash 0$ is a pole of $R_\epsilon(-\zeta^2)$ if and only if $-\zeta_0{}^2 \in (0, \infty)$ is an eigenvalue of H. It is generally believed that for potentials this small at infinity there can be no such eigenvalues embedded in the continuous spectrum but the proofs in the literature require additional regularity (see, for example, the proposition following Theorem 4). The singular behavior of $R_\epsilon(-\zeta^2)$ for Re $\zeta \geqslant 0$, $\zeta \neq 0$ is summarized in the following statements.

(i) There are no poles in Re $\zeta > 0$ except on the axis $(0, \infty)$

(ii) $\zeta_0 \in (0, \infty) \cup (i\mathbb{R}\backslash 0)$ is a pole of $R_\epsilon(-\zeta^2)$ if and only if $-\zeta_0{}^2$ is an eigenvalue of H. (6)

[3] Notice that the g dependence of $R(z)$ is not apparent in this notation.

Since nullspace $R_\epsilon^0(-\zeta^2) = \{0\}$[4] formula (5) implies

(iii) ζ is a pole of $R_\epsilon(-\zeta^2)$ if and only if $I + K(g, \zeta)$ is not invertible.

Our approach to studying the eigenvalues of $H(g)$ is to describe, with precision, the singular set $\mathscr{S}$ defined by

$$\mathscr{S} \equiv \{(g, \zeta) \in \mathbb{C} \times \{\operatorname{Re} \zeta > -\epsilon\}: I + K(g, \zeta) \text{ is not invertible}\}. \tag{7}$$

For spherically symmetric potentials V a related idea was proposed by Ciafaloni and Menotti [1]. Their analysis began with the observation that *for fixed* ζ (g, ζ) is in the set $\mathscr{S}$ if and only if g^{-1} is an eigenvalue of $K(1, \zeta)$. They use results about the eigenvalues of integral operators and perturbation theory to study the *dependence of* g *on* ζ. This is to be contrasted with the usual perturbation theory which leads one to study the dependence of the eigenvalue $-\zeta^2$ on g. Geometrically, Ciafaloni and Menotti study the dependence of the constant ζ sections of $\mathscr{S}$ on ζ while in the usual theory with g as the parameter one studies the dependence on g of the constant g sections of $\mathscr{S}$. From this point of view it is clear that one stands to improve both pictures by studying the structure of the set $\mathscr{S}$ as a subset of $\mathbb{C}^2$ rather than just its sections.

Our analysis of $\mathscr{S}$ falls into two parts. The first result asserts that $\mathscr{S}$ is a variety.

THEOREM 1. *If* $K(g, \zeta)$ *and* $\mathscr{S}$ *are defined by* (4) *and* (7), *respectively, then there is a nonzero holomorphic function* $f: \mathbb{C} \times \{\operatorname{Re} \zeta > -\epsilon\} \to \mathbb{C}$ *such that* $\mathscr{S} = \{(g, \zeta): f(g, \zeta) = 0\}$.

Proof 1. Both B. Simon [13] and A. Jensen [5, 6] have observed that using the renormalized determinant $\det_2$ the proof is immediate with $f(z, \zeta) = \det_2(I = K(g, \zeta))$.

Proof 2. Since $\mathscr{S}$ is a closed subset of $\mathbb{C} \times \{\operatorname{Re} \zeta > -\epsilon\}$ and, the cohomology group $H^2(\mathbb{C} \times \{\operatorname{Re} \zeta > -\epsilon\}, \mathbb{Z}) = 0$ the solution of the Cousin II problem in several complex variables (see [3, Lemma 12, p. 251) implies that we need only prove that $\mathscr{S}$ is locally the zero set of a holomorphic function. That is, we must show that for each (g_0, ζ_0) there is a neighborhood N of (g_0, ζ_0) in $\mathbb{C}^2$ and a holomorphic function f on N such that $\mathscr{S} \cap N$ coincides with the zero set of f. This, in turn, is an immediate consequence of the following lemma.

[4] For completeness we sketch a proof. If $u \in L^2$ and $R_\epsilon^0(-\zeta^2)u = 0$ then the convolution $(e^{-\zeta|x|}/4\pi \mid x \mid) * (E_\epsilon u)$ vanishes. Take Fourier transform to find $(\mid \xi \mid^2 + \zeta^2)^{-1} \widehat{E_\epsilon u}(\xi) = 0$ for all $\xi \in \mathbb{R}^3$. Thus $\widehat{E_\epsilon u}$ vanishes on the dense open set $\{\xi: \mid \xi \mid^2 \neq -\zeta^2\}$. Since $\widehat{E_\epsilon u}$ is continuous it follows that $\widehat{E_\epsilon u} = 0$ and therefore that $u = 0$.

LEMMA. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{U}$ an open set in $\mathbb{C}^n$. Suppose the map $z = (z_1, ..., z_n) \mapsto K(z)$ is holomorphic on $\mathscr{U}$ with values in the compact operators on $\mathscr{H}$. Then for any $z^0 \in \mathscr{U}$ there is an open $\mathscr{U}_1 \subset \mathscr{U}$ with $z^0 \in \mathscr{U}_1$ and a holomorphic function $f: \mathscr{U}_1 \to \mathbb{C}$ such that for $z \in \mathscr{U}_1$, $I + K(z)$ is not invertible if and only if $f(z) = 0$.*

The proof is a straightforward generalization of the standard analytic Fredholm theorem. Choose K_0 an operator of finite rank such that $\| K(z^0) - K_0 \| < 1/2$. Then $\| K(z) - K_0 \| < 1/2$ on an open neighborhood $\mathscr{U}_1$ of $z^0 \in \mathbb{C}^n$. For $z \in \mathscr{U}_1$ let $(I + K - K_0)^{-1} = 1 + S(z)$; then $S: \mathscr{U}_1 \to \mathrm{Hom}(\mathscr{H})$ is a holomorphic compact operator-valued function and if $C \equiv K_0(I + S)$ then

$$I + K = (I + C)(I + K - K_0). \tag{8}$$

Thus for $z \in \mathscr{U}_1$, $I + K$ is invertible if and only if $I + C$ is invertible. Note that range $C \subset$ range K_0. Let $b_1, ..., b_l$ be a basis for range K_0; then there exist uniquely determined vectors $c_i(z) \in \mathscr{H}$ such that

$$C(z)h = \sum_i (h, c_i(z))\, b_i .$$

A straightforward calculation shows that $I + C(z)$ is invertible if and only if $\det[\delta_{ij} + (b_j, c_i(z))] \neq 0$. This proves the lemma with $f(z) = \det[\delta_{ij} + (b_j, c_i)]$.

A question that arises is how is one to interpret points $(g, \zeta) \in \mathscr{S}$ with $g \in \mathbb{R}$ and $\mathrm{Re}\, \zeta < 0$. There are several reasons why this question is natural. From the point of view adopted here these points stand on an equal footing with the points $(g, \sqrt{-\lambda})$, λ an eigenvalue of $H(g)$, so they seem to be generalized eigenvectors of some sort. Second, we will see that if g_0 is a threshold coupling constant, that is, if there is an eigenvalue $\lambda(g)$ with $\lambda(g) \nearrow 0$ as $g \searrow g_0 > 0$ then the associated $\zeta(g) = \sqrt{-\lambda(g)}$ will continue to $g < g_0$ with $(g, \zeta(g)) \in \mathscr{S}$. Thus, after the eigenvalue, $\lambda(g)$, is absorbed by the continuous spectrum it persists as a point of $\mathscr{S}$.

Fortunately, there are three distinct, though related, interpretations, each of which is informative. One begins with the identity (derived by solving Schrödinger's equation by Laplace transform)

$$E_\epsilon e^{-itH} E_\epsilon = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} R_\epsilon(i\tau)\, e^{\tau t}\, d\tau \tag{9}$$

valid for $a \gg 1$.[5] The meromorphic continuation of $R(-\zeta^2)$ to $\mathrm{Re}\, \zeta > -\epsilon$

[5] See Rauch [10] for the details of this first interpretation. For dilation–analytic potentials an analysis following similar lines has been carried out by Jensen [6].

provides a meromorphic continuation of $R_\epsilon(i\tau)$ across the line $\operatorname{Re}\tau = 0$ into the slit plane $\mathbb{C}\backslash\mathbb{R}_-$. One can deform the contour in (9) to show that

$$E_\epsilon e^{-itH} E_\epsilon = \sum \operatorname{Res}\,(e^{\tau t} R_\epsilon(i\tau) \colon \tau = \tau_j) + \frac{1}{2\pi i}\int \Gamma R_\epsilon(i\tau)\, e^{\tau t}\, d\tau + O(e^{-\epsilon/2t}), \quad (10)$$

where the contour Γ starts at $\epsilon/2 - iO$, proceeds along the "bottom of the real axis" to a point close to the origin, then encircles the origin and returns to $\epsilon/2 + iO$. The poles in the sum are finite in number and lie in the region $\operatorname{Re}\tau \geqslant -\epsilon/2$. Notice that under the map $i\tau = -\zeta^2$, points $(g, \zeta) \in \mathscr{S}$ with $\operatorname{Re}\zeta \geqslant 0$ correspond to poles of $R_\epsilon(i\tau)$ on the axis $\operatorname{Re}\tau = 0$, and, these yield the yield the contribution $\sum_{\lambda_j} E_\epsilon e^{-i\lambda_j t}\pi_{\lambda_j} E_\epsilon$ of the point spectrum of H. The poles with $\operatorname{Re}\tau < 0$ correspond to points $(g, \zeta) \in \mathscr{S}$ with $\operatorname{Re}\zeta < 0$ and $-\zeta^2 = i\tau$. Their contribution to (10) has time dependence of the form (polynomial in t)$e^{\tau_j t}$. When $\operatorname{Re}\tau_j$ is close to zero these are long-lived but decaying modes and this is one reason why the poles are called resonances.

A second interpretation is that $(g, \zeta) \in \mathscr{S}$ if and only if there is a solution of the reduced wave equation

$$-\Delta u + gVu + \zeta^2 u = 0 \text{ in } \mathbb{R}^3 \quad (11)$$

which satisfies an appropriate radiation condition at infinity.[6] The numbers $-\zeta^2$ are sometimes referred to as scattering eigenvalues because of this interpretation. Finally, if $S\colon \mathbb{R} \to \operatorname{Hom}(L^2(S^2))$ is the scattering matrix for $H(g)$ then $S(\sigma^2)$ is the restriction to $\mathbb{R}$ of a function meromorphic in $\operatorname{Im}\sigma \geqslant -\epsilon$ with poles precisely the points $i\zeta$, where $(g, \zeta) \in \mathscr{S}$, $\operatorname{Re}\zeta < 0$. If ζ is close to the imaginary axis and $S(\sigma^2)$ has a pole at ζ then one expects $S(\sigma^2)$ to be large for $\sigma \approx \operatorname{Im}\zeta$, $\sigma^2 \approx (\operatorname{Im}\zeta)^2 \approx -\operatorname{Re}(\zeta^2)$. Physically this corresponds to enhanced scattering, i.e., large total cross section, at energies close to $-\operatorname{Re}(\zeta^2)$, a second reason for the name resonance.

Recall that $\mathscr{S}$ is the zero set of a holomorphic function $f(g, \zeta)$ of two variables. This allows us to obtain Puiseux expansions for $\zeta(g)$.

THEOREM 2 (perturbation theory for resonances). *If $(g_0, \zeta_0) \in \mathscr{S}$, then there is a disk $D = \{g \in \mathbb{C} \colon |g - g_0| < r\}$, integers, $k_1, k_2, \ldots, k_l$, a positive number ρ, and holomorphic functions $h_j : \{z \in \mathbb{C} \colon |z| < r^{1/k_j}\} \to \mathbb{C}$ for $j = 1, 2, \ldots, l$, so that for $g \in D$ the point $\zeta \in \mathscr{S} \cap \{\zeta \colon |\zeta - \zeta_0| < \rho\}$ if and only if $\zeta = h_j(w)$ for some j and w such that $w^{k_j} = g - g_0$. That is, the points of $\mathscr{S}$ near (g_0, ζ_0) are given by the values of the Puiseux series $h_j((g - g_0)^{1/k_j})$, $j = 1, 2, \ldots, l$.*

[6] For V more general than that considered here this result is proved by Jensen [5]. Less general results were obtained by Lax and Phillips [9] and Shenk and Thoe [12], the latter authors building on the earlier work of Dolph *et al.* [2].

Theorems 1 and 2 resemble Theorems 1.1 and 1.2 of Howland's paper [4]. In fact for $\zeta \neq 0$ our theorems follow from his with the choice $A = B = E_\epsilon$. However at $\zeta = 0$, $R_\epsilon(\zeta)$ (his $Q(\zeta)$) has a branch point. Howland gives an example where a branch point of $Q(\zeta)$ can lead to an absence of Puiseux expansions for $\zeta(g)$. The critical fact is that Howland's example has a logarithmic branch point. When there is an algebraic branch point, composing with a suitable power function unwinds the singularity and one obtains Puiseux expansions. In our problem this boils down to studying the continuation of $(R_\epsilon - S^2)$ instead of $R_\epsilon(z)$. For completeness we give a self-contained proof valid at all ζ with $\operatorname{Re} \zeta > -\epsilon$.

Proof. *Step* 1. We show that with f as in Theorem 1 the function $\zeta \to f(g_0, \zeta)$ is not identically zero. For fixed g_0, $\| K(g_0, \zeta)\| \to 0$ as $\operatorname{Re} \zeta \to \infty$ so $[I + K(g_0, \zeta)]^{-1}$ exists for $\operatorname{Re} \zeta$ large. Thus $\zeta = \zeta_0$ is an isolated zero of $f(g_0, \zeta)$ so there is a smallest integer $k > 0$ so that $(\partial/\partial\zeta)^k f(g_0, \zeta_0) \neq 0$.

Step 2. Weierstrass preparation. Using the result above the preparation theorem implies that there is a neighborhood N of (g_0, ζ_0) and functions $\alpha_j(g)$ holomorphic near g_0 and $\beta(g, \zeta)$ holomorphic in N so that

$$f(g, \zeta) = \beta(g, \zeta)[(\zeta - \zeta_0)^k + \alpha_{k-1}(g)(S - S_0)^{k-1} + \cdots + \alpha_0(g)], \tag{12}$$

and $\beta \neq 0$ on N.

Step 3. Puiseux expansion. It follows that the zeros of f in N are identical to those of the Weierstrass polynomial in brackets on the right-hand side of (12). It is a classical fact[7] that the roots of such an expression have Puiseux expansions as described in Theorem 2. ▮

Remark. The integers k_j and functions h_j may be chosen so that except for a discrete set of g the k_j distinct roots $(g - g_0)^{1/k_j}$ yield k_j distinct points $(g, h_j((g - g_0)^{1/k_j})) \in \mathscr{S}$.

In some special cases the possible Puiseux series are severely restricted. The main restriction on such series arises from (6i); if g and g_0 are real then no matter what kth root $w = (g - g_0)^{1/k}$ is taken, $\sum a_n w^n$ must not lie in the set $\{\zeta: \operatorname{Re} \zeta > 0, \operatorname{Im} \zeta \neq 0\}$. That is, in the right half-plane the only permissible values are on the real axis.

As an example consider a negative eigenvalue λ_0 of $H(g_0)$. Let $\sum a_n(g - g_0)^{n/k}$ be a Puiseux series describing points of $\mathscr{S}$ near $(g_0, \sqrt{-\lambda_0})$; thus, $a_0 = +\sqrt{-\lambda_0}$ and for $|g - g_0|$ small and all roots $(g - g_0)^{1/k} = w$, $(g, \sum a_n w^n) \in \mathscr{S}$. Since for $g \approx g_0$ the point $\sum a_n w^n \approx \sqrt{-\lambda_0}$, condition (6i) implies that for g real and for all roots $(g - g_0)^{1/k}$, $\sum a_n(g - g_0)^{n/k} \in \mathbb{R}_+$. Rellich [11] observed that the only Puiseux series with this property are power series, that is, there are no fractional powers. Therefore, the points $(g, \zeta) \in \mathscr{S}$ near $(g_0, \sqrt{-\lambda_0})$ are des-

[7] See Jordan [7, Sects. 361–369].

cribed by a finite number of holomorphic functions $\zeta_j(g)$. By (6) again, the eigenvalues of $H(g)$ near λ_0 for g near g_0 are given by the convergent power series $-\zeta_j^2(g)$. In this way we recover Rellich's theorem on the behavior of isolated eigenvalues. What we gain is that Theorem 2 applies equally well to embedded eigenvalues and resonances and this allows us to describe the behavior of eigenvalues as they are absorbed into the continuous spectrum.

THEOREM 3. *Suppose that V is real valued and for some $\epsilon > 0$ and $p > 2$, $e^{2\epsilon|x|}V \in L^p(\mathbb{R}^3)$. Suppose $\lambda(g)$ is a negative eigenvalue which approaches the continuous spectrum as $g \searrow g_0 > 0$. Precisely, λ is holomorphic on a neighborhood of $(g_0, g_0 + \delta)$ $\lim_{g \searrow g_0} \lambda(g) = 0$ and for $g \in (g_0, g_0 + \delta)$, $\lambda(g)$ is a negative eigenvalue of $H(g)$. Then $\lambda(g) = -\zeta^2(g)$, where $\zeta(g)$ has a Puiseux expansion in powers of $(g - g_0)^{1/2}$. That is,*

$$\zeta(g) = h_1(g - g_0) + (g - g_0)^{1/2} h_2(g - g_0), \tag{13}$$

where h_i is holomorphic on a neighborhood of g_0 with real Taylor coefficients, $h_i(0) = 0$, and $(g - g_0)^{1/2}$ is the positive square root for $g - g_0 \in \mathbb{R}_+$.

Remarks. (1) In Theorem 4 we will show that there are additional restrictions on the expansion (13).

(2) An immediate consequence of (13) is that $\lambda = -\zeta^2 = O(|g - g_0|)$, a result obtained in much greater generality by Simon [13].

Proof. For $g_0 + \delta > g > g_0$ define $\zeta(g) > 0$ by $\lambda(g) = -\zeta^2(g)$ so that $(g, \zeta(g)) \in \mathscr{S}$ by (6ii). Theorem 2 implies that ζ is given by a Puiseux series

$$\zeta(g) = \sum a_n (g - g_0)^{n/k}. \tag{14}$$

Altering the a_n if necessary we may assume that $\zeta(g)$ is the value of this series when the positive kth root $(g - g_0)^{1/k}$ is taken. Since ζ is real valued for $g \in (g_0, g_0 + \delta)$ it follows that the coefficients a_n are real. We must show that the Puiseux series has the form (13). The basic fact that we will use is that no matter what root $(g - g_0)^{1/k}$ is taken the point $(g, \sum a_n (g - g_0)^{n/k}) \in \mathscr{S}$; so by (6i), if g is real,

$$\sum a_n (g - g_0)^{n/k} \notin \{z\colon \operatorname{Re} z > 0, \operatorname{Im} z \neq 0\} \equiv \mathscr{P} \tag{15}$$

($\mathscr{P}$ for prohibited). The analysis falls into two cases depending on whether the leading term in (14) is a fractional or integer power.

Case (i). The Puiseux series (14) begins with an integer power, that is,

$$\zeta(g) = \alpha (g - g_0)^{n_0} + o(|g - g_0|^{n_0}), \tag{16}$$

where $\alpha = a_{n_0 k} \in (0, \infty)$ and n_0 is a positive integer. Assemble the integer powers of $(g - g_0)$ in (13) into a power series and the half-integer powers into a second series so that

$$\zeta(g) = h_1(g - g_0) + (g - g_0)^{1/2} h_2(g - g_0) + \rho(g - g_0). \tag{17}$$

Precisely, if the a_j are as in (14),

$$h_1(s) = \sum_{n=1}^{\infty} a_{nk} s^n,$$

$$h_2(s) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_{nk/2} s^n \qquad \text{if } k \text{ is even, } h_2 \equiv 0 \text{ if } k \text{ is odd.}$$

We must show that $\rho \equiv 0$. Suppose on the contrary that $\rho \not\equiv 0$. Since the leading term of ζ is an integer power we have

$$\rho(g - g_0) = (g - g_0)^n (g - g_0)^{j/k} [\beta + O(1)], \tag{18}$$

where $n \geqslant n_0$, $\beta \in \mathbb{R}$ 0, $0 < j < k$, and $j/k \neq \frac{1}{2}$. For $g - g_0$ small and positive, $\operatorname{Re} \zeta(g) = \alpha(g - g_0)^{n_0}[1 + o(1)] > 0$ so (15) implies that all values of the Puiseux series (14) must be real. Since the first two terms on the right of (17) are real valued, $\rho(g - g_0)$ must be real so $[(g - g_0)^{1/k}]^j \in \mathbb{R}$ for all kth roots of $(g - g_0)$. Thus $e^{2\pi i m j/k} \in \mathbb{R}$ for $m = 0, 1, 2, 3, \ldots, k - 1$ which can only happen if $j = 0$ or $j/k = \frac{1}{2}$. Both of these possibilities were prohibited at the outset so we must have $\rho \equiv 0$.

Case (ii). The Puiseux series begins with a fractional power, that is,

$$\zeta(g) = (g - g_0)^{n_0} (g - g_0)^{j/k} [\alpha + o(1)], \tag{19}$$

where j, k, and n_0 are integers, $0 < j < k$, and $\alpha \in (0, \infty)$.

We first show that j/k must be $\frac{1}{2}$. The values of the Puiseux series (14) for $g - g_0 < 0$ are of the form

$$|g - g_0|^{n_0 + j/k} [\alpha + o(1)]\, e^{i\pi m j/k}, \qquad m = 0, 1, \ldots, k - 1, \tag{20}$$

where $|g - g_0|^{n + j/k} > 0$. If j/k is 0 or $\frac{1}{2}$ it is possible for these values to lie outside $\mathscr{P}$. However, if j/k is not equal to 0 or $\frac{1}{2}$ at least one of these values lies in each of the first and fourth quadrants, which is prohibited by (15). Since $j > 0$ we must have $j/k = \frac{1}{2}$ and $k = 2j$ is even.

Assemble the integer and half-integer powers as in (17). Again, we must show that $\rho \equiv 0$. Since the leading term in the expansion is given by (19) with $j/k = \frac{1}{2}$, if $\rho \not\equiv 0$ then it must be of the form (18) with j, n, k integers such that

$n + j/k > n_0 + \frac{1}{2}$, $0 < j < k$, $j/k \neq \frac{1}{2}$. We consider ζ for $g - g_0$ small and positive. Consider the kth roots $|g - g_0|^{1/k} e^{2\pi i m/k}$ $m = 0, 1, 2$. For the $k/2$ roots corresponding to even values of m we have $[(g - g_0)^{1/k}]^{k/2} \in \mathbb{R}_+$. Since $j = k/2$ it follows that the right-hand side of (19) has positive real part for these values of the root. By (15) the values of the Puiseux series must be real for these values of $(g - g_0)^{1/k}$. Since $h_1(g - g_0) + (g - g_0)^{1/2} h_2(g - g_0)$ is real valued, the imaginary part of $\rho(g - g_0)$ must vanish. Since ρ has the form (18), by reasoning as in Case (i) we conclude that $j/k = 0$ or $j/k = \frac{1}{2}$ and both of these possibilities were prohibited at the outset. It follows that $\rho \equiv 0$ and the proof of Theorem 3 is complete. ∎

Next we look more closely at the behavior of $\zeta(g)$ and $\lambda(g) = -\zeta^2(g)$ when ζ has an expansion as described in Theorem 3. A first remark is that for $|g - g_0|$ small $h_1(g - g_0)$ and $(g - g_0)^{1/2} h_2(g - g_0)$ have different orders of magnitude since for some integers n and m, $|h_1(s)| \sim cs^n$ and $|s^{1/2} h_2(s)| \sim cs^{m+1/2}$ (we have excluded the trivial case $h_1 \equiv 0$ or $h_2 \equiv 0$). The next result shows that not all expansions are possible. In some sense only three qualitatively different forms for ζ are possible and we describe them after the proof.

THEOREM 4. *Suppose that V, g_0, λ, ζ, h_1, and h_2 are as in Theorem 3.*

(1) *If $h_1 \not\equiv 0$ then for some $a > 0$ and odd integer n, $h_1(s) = as^n[1 + o(1)]$.*

(2) *If for some $\epsilon > 0$ and a sequence $g^n \searrow g_0$ the operators $H(g^n)$ have no eigenvalues in the interval $(0, \epsilon)$ then $h_1 \not\equiv 0$.*

Remark. The folk wisdom holds that for V which are as small at infinity as those considered here, there can be no positive eigenvalues so the hypothesis of (2) is expected to hold. The proofs require some additional regularity of V. For example, we have the following consequence of the results of Kato [8].

PROPOSITION. *Suppose $V \in L^2_{\text{loc}}(\mathbb{R}^3)$ and there exists a compact set $K \subset \mathbb{R}^3$ with Lebesgue measure zero and $\mathbb{R}^3 \backslash K$ connected such that V is continuous on $\mathbb{R}^3 \backslash K$, and $V = o(|x|^{-1})$ as $x \to \infty$. Then, for any $g \in \mathbb{R}$, $-\Delta + gV$ has no positive eigenvalues.*

Proof. If $u \in L^2(\mathbb{R}^3)$ and $(-\Delta + gV) = \lambda u$ with $\lambda > 0$ then a result of Kato [8] implies that there is an $R > 0$ so that $u \equiv 0$ for $|x| > R$. The unique continuation principle for solutions of elliptic equations then implies that $u = 0$ on the connected set $\mathbb{R}^3 \backslash K$. Thus u is an element of L^2 supported on the null set K so $u = 0$. ∎

Proof of Theorem 4, *part* (1). Since $h_1 \not\equiv 0$ there is an $a > 0$ and an integer n so that $h_1(g - g_0) = a(g - g_0)^n[1 + o(1)]$ as $g \to g_0$. We must show that n is odd. The proof divides into two cases depending on whether h_2 is identically zero or not.

Suppose first that $h_2 \not\equiv 0$. Then there is an $\eta > 0$ so that for $-\eta < g - g_0 < 0$ $(g - g_0)^{1/2} h_2(g_1 - g_0) \in i\mathbb{R}\backslash 0$. Since $\zeta(g)$ must not lie in the prohibited region $\mathscr{P}$ in (15), we must have $h_1(g - g_0) < 0$ for these values of g and it follows that n is odd.

Suppose next that $h_2 \equiv 0$ and that contrary to (1), n is even. Then, there is an $\eta > 0$ so that for $-\eta < g - g_0 < 0$, $\zeta(g)$ is a decreasing positive real-valued function of g. By (6i) it follows that $\lambda(g) = -\zeta^2(g)$ is a negative eigenvalue of $H(g)$ with $\lambda'(g) > 0$. However, it is a classical fact that

$$g\lambda'(g) = (gV\phi, \phi) = (H\phi, \phi) + (\Delta\phi, \phi)$$

$$= \int \lambda \,|\, \phi \,|^2 - |\, \nabla\phi \,|^2 \, dx < 0,$$

where ϕ is a unit eigenvector of $H(g)$ with eigenvalue $\lambda(g)$. This contradiction completes the proof. ∎

Proof of Theorem 4, *part* (2). Suppose that $h_1 \equiv 0$. Then, $h_2 \not\equiv 0$ so there is an $\eta > 0$ so that for $-\eta < g - g_0 < 0$, $\zeta(g) \in i\mathbb{R}\backslash 0$. By (6iii) $-\zeta^2(g)$ is a positive eigenvalue of $H(g)$ for each such value of g. Since $\zeta^2(g) \to 0$ as $g \to 0$ the hypothesis in (2) is violated and the proof by contraposition is complete. ∎

Assuming that the hypothesis of Theorem 4, part (2), is satisfied there are three distinct qualitative pictures of the behavior of $\zeta(g)$ and $\lambda(g)$ for $g \approx g_0$.

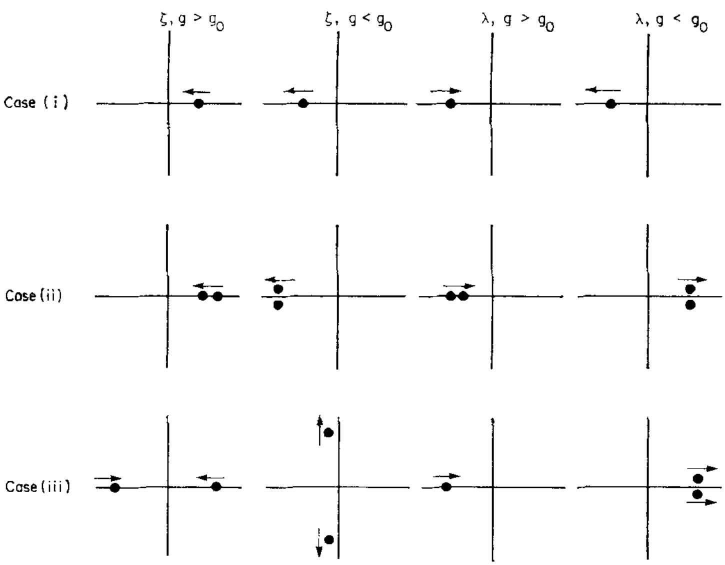

FIGURE 2

These correspond to the three situations (i) $h_2 \equiv 0$; (ii) $0 \not\equiv s^{1/2}h_2(s) = o(|h_1(s)|)$, and (iii) $h_1(x) = o(|s^{1/2}h_2(s)|)$, the last two relations holding as $s \to 0$. The behavior of ζ and λ is depicted in Figure 2, where the dots represent the values of the functions and the arrows indicate the direction of motion of the dots as *g decreases*. When interpreting the diagrams in Fig. 2 it is important to realize that values of ζ with Re $\zeta < 0$ correspond to the "unphysical sheet." For example, in case (i) the function $\lambda(g) = -\zeta^2(g)$ is holomorphic and negative; however, for $g < g_0$, Re $\zeta < 0$ so these negative values of λ correspond to resonnances, *not* eigenvalues. In cases (ii) and (iii) there are two resonances for $g > g_0$ as traces of the eigenvalue which is absorbed into the continuous spectrum when $g = g_0$.

Note that in case (ii) two eigenvalues arrive at the origin simultaneously and in case (iii) an eigenvalue and a resonance arrive simultaneously.

References

1. M. Ciafaloni and P. Menotti, Analysis of a S-matrix singularities by means of operator techniques, *Nuovo Cimento* **35**, No. 1 (1965), 160–193.
2. C. Dolph, J. McLeod, and D. Thoe, The analytic continuation of the resolvent kernel and scattering operator associated with the Schrödinger operator, *J. Math. Anal. Appl.* **16** (1966), 311–332.
3. R. Gunning and H. Rossi, "Analytic Functions of Several Complex Variables," Prentice–Hall, Englewood Cliffs, N. J., 1965.
4. J. S. Howland, Puiseux series for resonances at an embedded eigenvalue, *Pacific J. Math.* **55** (1974), 157–176.
5. A. Jensen, Local distortion technique, resonances, and poles of the S-matrix, *J. Math. Anal. Appl.* **59** (1977), 505–513.
6. A. Jensen, Local decay in time of solutions to Schrödinger's equation with a dilation–analytic interaction, *Manuscripta Math.* **25** (1978), 61–77.
7. C. Jordan, "Cours d'analyse," Vol. 1, Gauthier–Villars, Paris, 1909.
8. T. Kato, Growth properties of solutions of the reduced wave equation with variable coefficients, *Comm. Pure Appl. Math.* **12** (1959), 403–425.
9. P. D. Lax and R. S. Phillips, "Scattering Theory," Academic Press, New York, 1967.
10. J. Rauch, Local decay of scattering solutions to Schrödinger equations, *Comm. Math. Phys.* **61** (1978), 149–168.
11. F. Rellich, Störungstheorie der spektralzerlegung, I–V, *Math. Ann.* **113** (1937), 600–619, 677, 685; **116** (1939), 555–570; **117** (1940), 356–382; **118** (1942), 462–484.
12. N. Shenk and D. Thoe, Eigenfunction expansions and scattering theory for perturbations of A, *Rocky Mtn. J. Math.* **1** (1971), 89–125.
13. B. Simon, On the absorption of eigenvalues by continuous spectrum in regular perturbation problems, *J. Functional Analysis* **25** (1977), 338–344.