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VISCOMETRIC FLOWS OF THIRD GRADE FLUIDS

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1. Introduction

The fluid of second grade has a remarkable property in common with the Classical linearly viscous fluid in that viscometric measurements suffice to determine the form of the constitutive relation for all flows (cf. Truesdell [1], [2]). This property does not carry over in general to arbitrary third grade fluids. However, if the third grade fluids are required to be thermodynamically compatible in the sense that all arbitrary motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid be a minimum in equilibrium, it is found that the class of thermodynamically compatible third grade fluids can indeed be characterized completely by means of viscometric data.

2. Fluids of Grade Three

The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous third grade fluid is assumed to be related to the fluid motion in the following manner [3]:

$$\mathbf{T} = -p\mathbf{1}+\mu(\theta)\mathbf{A}_1+ {}_1(\theta)\mathbf{A}_2+\alpha_2(\theta)\mathbf{A}_1^2+\beta_1(\theta)\mathbf{A}_3$$
$$+\beta_2(\theta)[\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1]+\beta_3(\theta)(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1 \quad , \qquad (1.1)_1$$

where $\mathbf{v}$ is the velocity, and $\mathbf{A}_n$ are the Rivlin-Ericksen tensors defined recursively through

$$\underset{\sim}{A}_1 = \text{grad}\underset{\sim}{v} + (\text{grad}\underset{\sim}{v})^T \ , \qquad (1.1)_2$$

and

$$\underset{\sim}{A}_n = \dot{\underset{\sim}{A}}_{n-1} + \underset{\sim}{A}_{n-1}(\text{grad}\underset{\sim}{v}) + (\text{grad}\underset{\sim}{v})^T\underset{\sim}{A}_{n-1} \ . \qquad (1.1)_3$$

In equation $(1.1)_1$, $-p\underset{\sim}{1}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, μ the viscosity, α_1 and α_2 the normal stress moduli, and β_1, β_2 and β_3 material moduli which resemble shear dependent viscosity: all the material moduli being functions of temperature. The dot in equation $(1.1)_3$ denotes material time differentiation. If the coefficients β_1, β_2 and β_3 are set to be zero in equation $(1.1)_1$, we obtain the special subclass of fluids of second grade. If in addition the normal stress moduli α_1 and α_2 are set to be zero, we obtain the classical Navier-Stokes fluid.

While the constitutive relation (1.1) may be considered to be a third order approximation to the response of a simple fluid of the fading memory type in the sense of retardation (cf. Coleman and Noll [4], Truesdall and Noll [3]), since it is properly invariant it may also be considered as an exact model for some fluid, as is done, for example, for fluids wherein all the material moduli except the coefficient of viscosity μ are set equal to zero - the classical Navier-Stokes fluid. In this analysis we shall investigate the viscometric flows of fluids which are considered to be modeled exactly by (1.1).

As mentioned in the introduction, the fluid of second grade has a property in common with the classical linearly viscous fluid in that viscometric measurements suffice to determine the form of the constitutive equation for all flows. This property, however, does not carry over in general to a fluid modeled by $(1.1)_1$, because, in viscometric flows the tensor $\underset{\sim}{A}_3 \equiv 0$, and hence no information whatsoever can be provided about the nature of β_1.

Recently, a detailed study regarding the thermodynamics of fluids which are modeled exactly by a constitutive relation of the form $(1.1)_1$ has been carried out by Fosdick and Rajagopal [5]. They find that the material coefficients have to meet certain restrictions if the fluid is to undergo motions which are compatible with thermodynamics in the sense that all arbitrary motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum when the fluid is locally at rest.* It will be shown in the next section that these restrictions imply that the viscometric measurements are indeed sufficient to determine the form of the constitutive equation for all flows of thermodynamically compatible third grade fluids.

3. Viscometric Flows

Coleman and Noll [6] exhibited that the solution of any viscometric flow can be expressed in terms of three functions $\tau(\cdot)$, $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$; where the first function is odd and the last two even. These functions are generally referred to as viscometric functions. In the case of simple shearing by an amount κ,

$$T_{xy} = \tau(\kappa) \quad , \tag{2.1$_1$}$$

$$T_{xx} - T_{zz} = \sigma_2(\kappa) \quad , \tag{2.1$_2$}$$

and

$$T_{yy} - T_{zz} = \sigma_1(\kappa) \quad . \tag{2.1$_3$}$$

One can easily show (cf. Truesdell [3]) that for a fluid of grade three, the viscometric functions $\tau(\kappa)$, $\sigma_1(\kappa)$ and $\sigma_2(\kappa)$ are of the form

$$\tau(\kappa) = \kappa[\mu + 2(\beta_2 + \beta_3)\kappa^2] \quad , \tag{2.2$_1$}$$

*In the case of an incompressible homogeneous fluid of grade three it has been shown [5], that the specific Helmholtz free energy ψ has the specific form $\psi = \hat{\psi}(\theta, \underset{\sim}{A}_1, \underset{\sim}{A}_2)$. By the specific Helmholtz free energy be a minimum when the fluid is locally at rest, we mean $\hat{\psi}(\theta,0,0) \leq \hat{\psi}(\theta,\underset{\sim}{A}_1,\underset{\sim}{A}_2)$ for all traceless $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$.

$$\sigma_1(\kappa) = (2\alpha_1 + \alpha_2)\kappa^2 \quad , \qquad (2.2)_2$$

and

$$\sigma_2(\kappa) = \alpha_2\kappa^2 \quad . \qquad (2.2)_3$$

It can be observed from equation $(2.2)_{1,2,3}$ that the material moduli β_1 does not appear in the viscometric functions at all, thus, leading one to the conclusion that a third grade fluid cannot be characterized by viscometric data alone.

However, if the third grade fluid is to be thermodynamically compatible then its material moduli have to meet the following restrictions (cf. Theroem 2, [5]):

$$\begin{gathered} \mu \geq 0 \quad , \quad \alpha_1 \geq 0 \quad , \\ -\sqrt{24\mu\beta_3} \leq (\alpha_1 + \alpha_2) \leq \sqrt{24\mu\beta_3} \quad , \\ \beta_1 = 0 \ , \quad \beta_2 = 0 \text{ and } \beta_3 \geq 0 \quad . \end{gathered} \qquad (2.3)$$

Thus, we see that in the case of third grade fluids which are compatible with thermodynamics, since $\beta_1 = 0$, the problem of the viscometric functions not being able to characterize the fluid completely do not arise. However such a problem might arise for a fluid of higher grade, say four, where the Cauchy stress $\underset{\sim}{T}$ is given by [1],

$$\begin{aligned} \underset{\sim}{T} = & -p\underset{\sim}{1}+\mu\underset{\sim}{A}_1+\alpha_1\underset{\sim}{A}_2+\alpha_2\underset{\sim}{A}_1^2+\beta_1\underset{\sim}{A}_3+\beta_2(\underset{\sim}{A}_1\underset{\sim}{A}_2+\underset{\sim}{A}_2\underset{\sim}{A}_1) \\ & +\beta_3(\mathrm{tr}\underset{\sim}{A}_2)\underset{\sim}{A}_1+\gamma_1\underset{\sim}{A}_4+\gamma_2(\underset{\sim}{A}_3\underset{\sim}{A}_1+\underset{\sim}{A}_1\underset{\sim}{A}_3)+\gamma_3\underset{\sim}{A}_2^2 \\ & +\gamma_4(\underset{\sim}{A}_2\underset{\sim}{A}_1^2+\underset{\sim}{A}_1^2\underset{\sim}{A}_2)+\gamma_5(\mathrm{tr}\underset{\sim}{A}_2)\underset{\sim}{A}_2+\gamma_6(\mathrm{tr}\underset{\sim}{A}_2)\underset{\sim}{A}_1^2 \\ & +\gamma_7\mathrm{tr}(\underset{\sim}{A}_3)\underset{\sim}{A}_1+\gamma_8\mathrm{tr}(\underset{\sim}{A}_2\underset{\sim}{A}_1)\underset{\sim}{A}_1 \quad , \end{aligned} \qquad (2.4)$$

and where

$$\tau(\kappa) = \kappa[\mu + 2(\beta_2 + \beta_3)\kappa^2] \quad , \tag{2.5$_1$}$$

$$\sigma_1(\kappa) = (2\alpha_1+\alpha_2)\kappa^2+[4(\gamma_3+\gamma_4+\gamma_5)+2\gamma_6]\kappa^4 \quad , \tag{2.5$_2$}$$

and

$$\sigma_2(\kappa) = \alpha_2\kappa^2 + 2\gamma_6\kappa^4 \quad . \tag{2.5$_3$}$$

The results established above for fluids of grade three, together with the similar result for fluids of grade two, leads to the following interesting question: Does the requirement that a fluid of grade n (n arbitrary) be thermodynamically compatible ensure that these fluids be characterized by means of viscometric data? It would indeed be exceedingly useful if the answer is in the affirmative.

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