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THE SMALLEST GRAPHS WITH CERTAIN ADJACENCY PROPERTIES

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A graph is said to have property $P_{1,n}$ if for every sequence of $n+1$ points, there is another point adjacent only to the first point. It has previously been shown that almost all graphs have property $P_{1,n}$. It is easy to verify that for each n, there is a cube with this property. A more delicate question asks for the construction of the smallest graphs having property $P_{1,n}$. We find that this problem is intimately related with the discovery of the highly symmetric graphs known as cages, and are thereby enabled to resolve this question for $1 \leq n \leq 6$.

1. The notation

For two points u and v of a graph G we write uAv if u and v are adjacent, and $u\bar{A}v$ if they are not. The set of points of G adjacent to a given point v is the *neighborhood* $N(v)$. The subgraph induced by $N(v)$ is the *link* of v in G, written *link* (v). The *subgraph of G induced by all points neither equal to nor adjacent to v is denoted* G_v. Thus this is the subgraph of G obtained by removing the closed neighborhood $N^*(v)$. As usual, the minimum degree of G is denoted by $\delta(G)$, the maximum degree by $\Delta(G)$.

We say of two graphs G_1 and G_2 that G_1 is *smaller than* G_2 if $p_1 < p_2$ or if $p_1 = p_2$ and $q_1 < q_2$. For other graph theoretic notation and terminology we follow [3].

2. The problem

Some fascinating adjacency properties of graphs have been found in [2] to hold for almost all graphs. However, to our consternation, almost no graphs have been constructed which enjoy these properties. We generalize and then investigate a special case, not only to discover graphs with these properties but also to find the smallest such graphs.

Axiom n in Blass and Harary [2] states that for every sequence of $2n$ points $(u_1, \ldots, u_n; v_1, \ldots, v_n)$, there is another point w such that wAu_i and $w\bar{A}v_j$, for $i, j = 1, \ldots, n$. We generalize this to property $P_{m,n}$. A graph G has *property* $P_{m,n}$ (written $G \in \boldsymbol{P}_{m,n}$) if for any sequence of points $(u_1, \ldots, u_n; v_1, \ldots, v_n)$ there is another point w such that wAu_i and $w\bar{A}u_j$ for all $i=1,\ldots,m$ and $j=1,\ldots,n$. Obviously $G \in \boldsymbol{P}_{m,n}$ implies $\bar{G} \in \boldsymbol{P}_{n,m}$.

It was shown in [2] that for each n, almost all graphs satisfy *Axiom n*, i.e., have property $P_{n,n}$. It follows at once that for each m and n with say $m \leq n$, almost all graphs are in $\boldsymbol{P}_{m,n}$. In other words, if we let $\boldsymbol{G}_p$ be the family of all graphs with p points then

$$\lim_{p\to\infty} \frac{|\boldsymbol{G}_p \cap \boldsymbol{P}_{m,n}|}{|\boldsymbol{G}_p|} = 1. \tag{1}$$

But in spite of (1), it appears to be very difficult to construct graphs in $\boldsymbol{P}_{m,n}$ for general m and n.

As a special case of this problem we concentrate on graphs with property $P_{1,n}$. We note that for sufficiently large k, Q_k (the k-cube) is in $\boldsymbol{P}_{1,n}$. This observation is made more precise in Lemma 1. Our object is to find the smallest graphs in $\boldsymbol{P}_{1,n}$. Cubes are not the answer. We have succeeded for $n=1,\ldots,6$ and propose a conjecture related to values of $n>6$ by linking the determination of such smallest graphs to the discovery of certain cages [3]. We include here proofs only for $n=2$ and 3, as the other arguments are long, complicated, and analogous.

In what follows we use $(u; v_1,\ldots,v_k)$ to denote *the set of points of G adjacent to u, but not adjacent to any* v_i. If $k \leq n$ and $G \in \boldsymbol{P}_{1,n}$, then $(u; v_1,\ldots,v_k) \neq \emptyset$, for all $(k+1)$-sequences. And if wAu and $w\bar{A}v_j$, for $i=1,\ldots,k$, we write $w \in (u; v_1,\ldots,v_k)$, i.e., w is a point in the set $(u; v_1,\ldots,v_k)$.

3. The lemmas

It is convenient to develop six preliminary results before proving the main theorems.

Lemma 1. *The cube* Q_{2n+1} *is in* $\boldsymbol{P}_{1,n}$.

Proof. Each point v of Q_{2n+1} has degree $2n+1$. Let $u_1,\ldots,u_{2n+1}$ be the points adjacent to v. It follows at once from the definition of a cube that $u_i\bar{A}u_j$, for $i,j=1,\ldots,2n+1$. Also no point of Q_{2n+1}, except u, is adjacent to more than two of the u_i. Thus for each set X of n points of this cube, there is a point u such that uAv and for all $x \in X$, $u\bar{A}x$, hence $Q_{2n+1} \in \boldsymbol{P}_{1,n}$. □

We note in passing that $Q_{2n} \notin \boldsymbol{P}_{1,n}$.

Lemma 2. *If* $G \in \boldsymbol{P}_{1,n}$, *then* $\delta(G) \geq n+1$.

Proof. Assume the contrary and suppose u is a point of G with $\deg u = k \leq n$. Let $v_1, \ldots, v_k$ be the points of *link* (u). Then there is no point w such that $w \in (u; v_1, \ldots, v_k)$. This contradiction proves the lemma. □

Lemma 3. *If* $G \in P_{1,n}$ *and* $\deg u = n+1$, *then* u *is on no* 3- *or* 4-*cycles.*

Proof. Let $v_1, \ldots, v_{n+1}$ be the points of *link* (u). If u is on a 3-cycle then the other two points on this 3-cycle are two of the v_i, say v_1 and v_2. But then there is no point w in $(u; v_2, \ldots, v_{n+1})$. So u is not on a 3-cycle. If u is on a 4-cycle, we can with no loss of generality suppose that v_1 and v_2 are on the 4-cycle as well. Let x be the point opposite u on the 4-cycle. Then there is no point w in $(u; x, v_3, \ldots, v_{n+1})$. Hence u is not on a 4-cycle. □

Lemma 4. *For any sequence* $u, v_1, \ldots, v_k$ *in* $V(G)$, $|(u; v_1, \ldots, v_k)| \geq n-k+1$.

Proof. If $|(u; v_1, \ldots, v_k)| \leq n-k$, let $w_1, \ldots, w_m$ be all the points in $(u; v_1, \ldots, v_k)$, where $m \leq n-k$. Then there is no point x in $(u; v_1, \ldots, v_k, w_1, \ldots, w_m)$, a contradiction. □

Lemma 5. *If* $G \in \boldsymbol{P}_{1,n}$ *and* w *is any point of* G, *then* $G_w \in \boldsymbol{P}_{1,n-1}$.

Proof. Let $u, v_1, \ldots, v_{n-1}$ be any sequence of n points of G. Then some point x of G is in $(u; w, v_1, \ldots, v_{n-1})$. But $x \neq w$ and $x\bar{A}w$, so $x \in G_w$; hence $G_w \in \boldsymbol{P}_{1,n-1}$. □

Lemma 6. *If every graph in* $\boldsymbol{P}_{1,n-1}$ *has at least* r *points and if* $G \in \boldsymbol{P}_{1,n}$ *has order* p *and maximum degree* Δ, *then* $p \geq 1 + \Delta + r$.

Proof. It is sufficient to show that for all points v of G, $p \geq 1 + \deg v + r$. But this follows at once from the facts that $|V| = |\{v\}| + |N(v)| + |V(G_v)|$, and that $G_v \in \boldsymbol{P}_{1,m-1}$ by Lemma 5. □

The *girth* g of a graph G (which is not a forest) is the smallest cycle length in G.

Lemma 7. *If* $G \in \boldsymbol{P}_{1,n}$ *and* $g \geq 5$ *and if* G *is not an* $(n+1)$-*regular graph of girth* 5, *then* $p \geq n^2 + 3n + 2$.

Proof. If G is regular of degree $n+1$, the result follows from a theorem in Biggs [1, p. 153], which states that if a k-regular graph has odd girth g, then

$$p \geq 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} \tag{1}$$

and if it has even girth,

$$p \geq 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{(g-2)/2}. \tag{2}$$

If G is not $(n+1)$-regular, then by Lemma 2, G contains a point of degree $d \geq n+2$. If $v_1, \ldots, v_d$ are the points of G adjacent to u, then

$$p \geq 1+d+\sum_{i=1}^{d} |(u; v_i)| \geq 1+(n+2)+n(n+2) > n^2+3n+2,$$

as required. □

Lemma 8. *If G has girth $g \geq 5$ and $\delta(G) \geq n+1$, then $G \in \boldsymbol{P}_{1,n}$.*

Proof. For the purposes of this proof we shall write uBv if $u \in N^*(v)$, i.e., uAv or $u = v$. Let u be any point of G and let $v_1, \ldots, v_{n+1}$ be the points of *link* u. Then for any point $w \neq u$, wBv_i for at most one value of i. Thus for any $w_1, \ldots, w_n$ we have $(u; w_1, \ldots, w_n) \neq \emptyset$. □

4. Some solutions

We now proceed to indicate the smallest graphs in $\boldsymbol{P}_{1,n}$, $n = 1, \ldots, 6$. One easily verifies that C_5 is the smallest graph in $\boldsymbol{P}_{1,1}$. An (m, n)-*cage* is defined as a smallest m-regular graph of girth n. Note that C_5 is the $(2, 5)$-cage and in general C_p is the $(2, p)$-cage.

P:

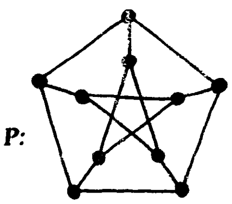

Fig. 1. The Petersen graph.

Theorem 1. *The smallest graph in $\boldsymbol{P}_{1,2}$ is the Petersen graph P which has order* 10 *and is the* $(3, 5)$*-cage. Every other graph in $\boldsymbol{P}_{1,2}$ has at least* 12 *points.*

Proof. If $G \in \boldsymbol{P}_{1,2}$ then $\delta(G) \geq 3$. Since C_5 is the smallest graph in $\boldsymbol{P}_{1,1}$, Lemma 6 implies that $p \geq 1+3+5 = 9$. But if $p = 9$, then G must be 3-regular, which is impossible. So $p \geq 10$ and $\delta(G) \geq 3$, which with Lemma 8 means that P is the smallest graph in $\boldsymbol{P}_{1,2}$.

To show that no graph in $\boldsymbol{P}_{1,2}$ has 11 points, we first observe that if $G \in \boldsymbol{P}_{1,2}$ and $p = 11$, then the degree set of G is a subset of $\{3, 4, 5\}$, since for any point v, $\deg v \geq 3$ by Lemma 2 and $\deg v \leq 5$ by Lemma 6. Since not all points of an 11-point graph can have odd degree, there is a point v of G having degree 4. Let u_1 to u_4 be the points of *link* v. Now observe that the induced subgraph, *link* v, contains at most one line, and that no point outside *link* v other than v is adjacent to more than two points of *link* v, or $P_{1,2}$ will be violated. Let A be the set of points, not in the link, adjacent to just one point of *link* v; let B contain the points adjacent to exactly two of them.

If *link* v contains no lines, then either some point of *link* v is adjacent to every point of B or some point of *link* v is adjacent to no points of B. To show that one of these must hold, suppose that no point of *link* v is adjacent to every point of B. If B is empty, the second situation prevails, so let us say that $x_0 \in B$ is adjacent to both u_1 and u_2. Then there exist points x_1 and x_2 in B with $x_1 \bar{A} u_1$ and $x_2 \bar{A} u_2$, or we would have a point in *link* v adjacent to every point of B. If $x_1 = x_2$, then $x_1 A u_3$ and $x_1 A u_4$, which means $P_{1,2}$ is violated. So we have $x_1 \neq x_2$ and neither is adjacent to both u_3 and u_4, or else one of the sets $(v; x_0, x_1)$ and $(v; x_0, x_2)$ is empty. So with no loss in generality we have $x_0 A u_1, u_2$; $x_1 A u_1, u_3$; and $x_2 A u_2, u_3$. But then no point of B can be adjacent to u_4, and we have the second case. So one of the two cases must hold.

We now show that in either of these cases $p \geq 12$. In the first case let u_1 be the indicated point. Then for clarity we write

$$p \geq |\{v\}| + \deg v + |(u_1; v)| + \sum_{i=2}^{4} |(u_i; u_1)| \geq 1 + 4 + 2 + 6 = 13.$$

In the second case let u_4 be the indicated point: we have

$$p \geq |\{v\}| + \deg v + |(u_4; v)| + |(u_1; v)| + |(u_2; u_1)| + |(u_3; u_2, u_1)| \geq 12.$$

If *link* v has a line, then let u_1 and u_2 be the points on the line and let u_3 and u_4 be the other two points of *link* v. Without loss of generality we can suppose that only v is adjacent to both u_1 and u_3. For if there is another point adjacent to both u_1 and u_3, and another point adjacent to both u_2 and u_4 then $P_{1,2}$ fails. So by symmetry we can suppose that no other point is adjacent to both u_1 and u_3. Also no point but v is adjacent to both u_3 and u_4. Then

$$p \geq 1 + 4 + |(u_1; v)| + |(u_2; u_1)| + |(u_3; u_2)| + |(u_4; u_1, u_2)| \geq 12.$$

And as P is the unique $(3, 5)$-cage, Lemma 3 can be used to show that no 10-point graph other than P is in $\mathbf{P}_{1,2}$. □

Corollary 1a. *The smallest graph in* $\mathbf{P}_{2,1}$ *is* $\bar{P}$, *the complement of the Petersen graph.* □

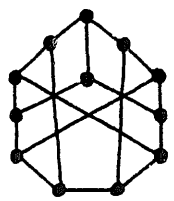

Fig. 2. A 12-point graph in $\mathbf{P}_{1,2}$.

R:

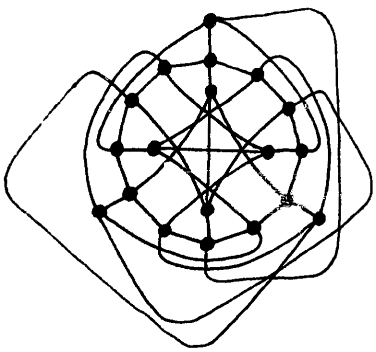

Fig. 3. Robertson's graph.

Theorem 2. *Robertson's graph, R, the* (4, 5)*-cage, is the smallest graph in* $\boldsymbol{P}_{1,3}$.

Proof. In Fig. 3 we show R, Robertson's graph, which has 19 points. We know $R \in \boldsymbol{P}_{1,3}$ by Lemma 8. We will show that any other graph in $\boldsymbol{P}_{1,3}$ has at least 20 points or has 19 points and more lines than R. (In fact the second case can be eliminated, but to do so is unnecessary here.)

From [6] we know that R is smaller than any other 4-regular graph of girth 5. By Lemma 7, it is also smaller than any graph G with girth 5 and $\Delta(G) > 4$, and is smaller than any graph G with $\delta(G) \geq 4$ and $g \geq 6$. So we proceed to tackle the case $g < 5$.

We show that if $G \in \boldsymbol{P}_{1,3}$ and has a point u of degree 5, then G is larger than R. Let

$$A = \{v : v \in V(G) \quad \text{and} \quad d(u, v) = 2\},$$

i.e., A consists of those points not in the closed neighborhood of u which are adjacent to a point of *link* u. And we let

$$B = \{v : v \in A \text{ and } v \text{ is adjacent to at least 2 points of } \mathit{link}\ u\}.$$

Observe that *link* u can contain at most one line and that no point of B is adjacent to more than two points of *link* u, lest $P_{1,3}$ be violated. Thus there are two possibilities: in Case 1, some point v_1 of *link* u is adjacent to all points of B and is a point on every line in *link* u, whilst in Case 2, there are at least two isolated points in *link* u, v_4 and v_5, with neither adjacent to a point of B.

To prove that these two cases exhaust the possibilities, suppose neither is true. We say two points of *link* u are *matched* if they are adjacent or if a point of B is adjacent to both. Since Case 2 fails to hold, at least four points of *link* u are matched. Without loss of generality, let v_1 be matched to v_2, and since Case 1 does not hold, suppose v_1 is not matched with v_3 and that v_2 is not matched with v_4. Thus since we do not have Case 2, either v_1 is matched with v_4, and v_2 with v_3, or else v_1 is matched with v_2, and v_3 with v_4. Either situation implies $G \notin \boldsymbol{P}_{1,3}$. Thus either Case 1 or Case 2 holds as claimed.

In both cases, the points of *link* u are denoted v_1 to v_5. In Case 1,

$$p \geq |\{u\}| + \deg u + \sum_{i=2}^{5} |(v_i;\, v_1)| + |(v_1;\, u)| \geq 1 + 5 + 4 \cdot 3 + 3 = 21.$$

Note the repeated use of Lemma 4 for bounding purposes.

Case 2 gives

$$p \geq 1+5+|(v_4; u)|+|(v_5; u)|+|(v_2; v_1)|+|(v_3; v_1, v_2)| \geq 20,$$

where we label the v_i so that $v_2 \bar{A} v_3$.

If there is a point u in G of degree $d \geq 6$ then either G_u is the Petersen graph P, or $p \geq 19$ and G has more lines than R. This follows from Theorem 1 and Lemmas 2 and 5. So suppose $G_u = P$. The argument in the preceding paragraphs allows us to conclude that G has no point of degree 5. So we have $G_u = P$ and $\deg u = 6$ or 7. Let $w_1, \ldots, w_{10}$ denote the points of G_u. Since $\delta(G) \geq 4$, if G is to be smaller than R then every point of G_u is adjacent to a point of *link* u. And since for any $v_i \in N(u)$, $(v_i; u) \geq 2$, we know that every point of *link* u is adjacent to at least two points of G_u. So the fact that diam $P = 2$ implies that every point of G is on a 3- or 4-cycle. Since G has no points of degree 5, we infer from Lemma 3 that every point of G has degree 6 or 7. So every point of G_u is adjacent to at least 3 points of *link* u. Thus there are at least 30 lines between G_u and *link* u, so that two of the *link* u points, which we can call v_1 and v_2, are adjacent to a total of at least seven points of G_u, or else $\boldsymbol{P}_{1,3}$ is violated. But given any seven points of P, there is a point of P adjacent to three of the seven; let w_1 be such a point. Then $(w_1; u, v_1, v_2)$ is empty, which means that $G \notin \boldsymbol{P}_{1,3}$. This contradiction shows that we cannot have $G_u = P$ when $p < 19$, and eliminates all candidates for graphs in $\boldsymbol{P}_{1,3}$ smaller than R. Hence R is the smallest graph in $\boldsymbol{P}_{1,3}$. □

In order to obtain a general proof that an $(n+1, 5)$-cage is a smallest graph in $\boldsymbol{P}_{1,n}$, the following conjecture would be useful.

Conjecture. If $G \in \boldsymbol{P}_{1,n}$ and has girth $g < 5$, then $p \geq n^2 + 3n + 2$.

We have been unable to devise a proof of this conjecture for all n. We have, however, proved it for $n \leq 6$ by considering each of these values separately. The techniques used are very similar to those used in Theorems 1 and 2. Note that Theorem 1 establishes the conjecture for $n = 2$; verification for $n = 1$ is easy.

Of course proving that the smallest graph in $\boldsymbol{P}_{1,n}$ is a cage would probably be very difficult if no $(n+1, 5)$-cage is known. However, the conjecture is motivated by the following assertion which is verified by Table 1.

Table 1. The smallest 5-cages

Degree	p	Discovered by
3	10	Petersen [3]
4	19	Robertson [6]
5	30	Wegner [7]
6	40	O'Keefe and Wong [5]
7	50	Hoffman and Singleton [4]

Observation. For each of the known $(n+1, 5)$-cages, $p \leq n^2+3n+2$.

So the conjecture might be useful for checking whether $(n+1, 5)$-cages which will be discovered in the future are also smallest graphs in $\boldsymbol{P}_{1,n}$.

The bound in the conjecture is sharp, at least for $n=1$, in that $\bar{C}_6 \in \boldsymbol{P}_{1,1}$.

We remark that the irregularity of the numbers p in Table 1 is startling.

Unsolved Problems. What are the smallest graphs in $\boldsymbol{P}_{1,n}$ when $n \geq 7$ and more generally what are the answers for $\boldsymbol{P}_{m,n}$ with $m, n \geq 2$?

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