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On Some Distribution Problems in Manova and Discriminant Analysis*

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Asymptotic expansions, valid for large error degrees of freedom, are given for the multivariate noncentral F distribution and for the distribution of latent roots in MANOVA and discriminant analysis. The asymptotic results are expressed in terms of elementary functions which are easy to compute and the results of some numerical work are included. The Bartlett test of the null hypothesis that some of the noncentrality parameters in discriminant analysis are zero is also briefly discussed.

1. INTRODUCTION AND SUMMARY

In multivariate analysis of variance situations it is usually of interest to test whether a matrix of noncentrality parameters is zero, at least as a first step in the analysis. If such a test is rejected, questions arise as to the rank of the noncentrality matrix. To fix the ideas and motivate the problems, consider a typical one-way analysis of variance with independent samples from r groups; in the ith group there are m_i observations drawn from a p-variate normal distribution with mean $\boldsymbol{\mu}_i$ and covariance matrix $\Sigma (i = 1,..., r; r > p)$. Let W and B denote respectively the "within-groups" and "between-groups" matrices of sums of squares and sums of products, constructed in the usual way. These matrices are independently distributed; W has the (central) Wishart distribution $W_p(n_2, \Sigma)$, where $n_2 = M - r$ with $M = \sum_{i=1}^{r} m_i$, and B has a noncentral Wishart distribution $W_p(n_1, \Sigma; \Delta)$, where $n_1 = r - 1$ and $\Delta = \Sigma^{-1}\Lambda$ is the noncentrality matrix with

$$\Lambda = \sum_{i=1}^{r} m_i(\boldsymbol{\mu}_i - \boldsymbol{\mu})(\boldsymbol{\mu}_i - \boldsymbol{\mu})', \qquad \boldsymbol{\mu} = M^{-1} \sum_{i=1}^{r} m_i \boldsymbol{\mu}_i .$$

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The null hypothesis that the mean vectors are all equal is equivalent to $\Lambda = 0$. If this is rejected it is reasonable to look for linear functions which best discriminate between the groups. The number of meaningful discriminant functions is equal to the dimension of the subspace spanned by $\mu_1, \ldots, \mu_r$ or, equivalently, to the rank of the noncentrality matrix Δ (see e.g. Kshirsagar [15, Ch. 9]). Hence, in discriminant analysis, it is of interest to test the null hypothesis

$$H_k\colon \omega_1 \geqslant \cdots \geqslant \omega_k > \omega_{k+1} = \cdots = \omega_p = 0, \tag{1.1}$$

where $\omega_1 \geqslant \cdots \geqslant \omega_p$ ($\geqslant 0$) denote the latent roots of $\mathbf{Q}$. Statistics used for testing H_k are functions of the latent roots $l_1, \ldots, l_p$ of the matrix $B(B+W)^{-1}$. The joint density function of these roots depends only on $\omega_1, \ldots, \omega_p$ and is (Constantine [7])

$$k_1 \prod_{i=1}^{p} \left[l_i^{(1/2)(n_1-p-1)} (1-l_i)^{(1/2)(n_2-p-1)} \right] \prod_{i<j} (l_i - l_j)$$
$$\times \exp\left(-\tfrac{1}{2}\sum_{i=1}^{p} \omega_i\right) {}_1F_1^{(p)}(\tfrac{1}{2}(n_1+n_2); \tfrac{1}{2}n_1; \tfrac{1}{2}\Omega, L) \tag{1.2}$$
$$(1 > l_1 > \cdots > l_p > 0),$$

where

$$k_1 = \pi^{(1/2)p^2} \Gamma_p(\tfrac{1}{2}(n_1+n_2)) / [\Gamma_p(\tfrac{1}{2}n_1)\, \Gamma_p(\tfrac{1}{2}n_2)\, \Gamma_p(\tfrac{1}{2}p)] \tag{1.3}$$

with

$$\Gamma_p(a) = \pi^{(1/4)p(p-1)} \prod_{i=1}^{p} \Gamma(a - \tfrac{1}{2}(i-1)),$$

$\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_p)$, $L = \operatorname{diag}(l_1, \ldots, l_p)$, and ${}_1F_1^{(p)}$ is a hypergeomemetric function having the matrices Ω, L as arguments (see James [13]). That part of the distribution involving only the noncentrality parameters $\omega_1, \ldots, \omega_p$, namely

$$\exp\left(-\tfrac{1}{2}\sum_{i=1}^{p} \omega_i\right) {}_1F_1^{(p)}(\tfrac{1}{2}(n_1+n_2); \tfrac{1}{2}n_1; \tfrac{1}{2}\Omega, L) \tag{1.4}$$

can be regarded as a marginal likelihood. There are substantial difficulties involved in calculating the ${}_1F_1^{(p)}$ function exactly using the zonal polynomial expansion of James and Constantine, especially in cases which are of particular statistical interest, for example, large error degrees of freedom n_2, large noncentrality matrix $\mathbf{Q}$ (i.e. large Ω). These difficulties stem primarily from problems involved in the calculation of zonal polynomials and in the extremely slow convergence of the series. For these reasons it makes sense to ask how the ${}_1F_1^{(p)}$ function behaves asymptotically, thus giving rise to asymptotic forms for the density (1.2) and likelihood (1.4). There are a number of asymptotic approaches of statistical interest, some of which have been studied previously (see Constantine and Muirhead [8], Glynn [9] and Srivastava and Carter [19]). This paper is concerned primarily with a situation which has not yet been studied, namely the asymptotic behavior of ${}_1F_1^{(p)}$ as the error degrees of freedom n_2 become large, with the noncentrality matrix remaining fixed. This is essentially the case when the differences between the means are assumed to be small. An asymptotic representation is given in Section 4 for the density (1.2) when the $p-k$ smallest noncentrality parameters are zero. Bartlett's [3] statistic for testing the null hypothesis H_k is also briefly discussed. In Section 3 the asymptotic behavior is obtained for the density function of the matrix $F = B^{1/2}W^{-1}B^{1/2}$ (the noncentral multivariate F); this distribution involves a ${}_1F_1$ function of one matrix argument.

2. Preliminaries

It will be shown later that the asymptotic behavior of both the one and two matrix ${}_1F_1$ functions can be expressed in terms of ${}_0F_1$ functions (Bessel functions) of large matrix argument. The latter functions occur in the noncentral Wishart density function and in the density function of latent roots in the case of noncentral means with known covariance. For definitions, etc., see [11], [7], [13]. The asymptotic behaviors of these ${}_0F_1$ functions have been studied by Anderson [1], Leach [16] and Muirhead [18], and can be expressed in terms of elementary functions which are easy to compute. For convenience the relevant results are stated here in the following two theorems

THEOREM 2.1. *Let $R = \operatorname{diag}(r_1, \ldots, r_p)$ where each r_i is positive. As $n \to \infty$*

$$ {}_0F_1(c; nR) \sim k_2(n) \exp\left(2n^{1/2} \sum_{i=1}^{p} r_i^{1/2}\right) \prod_{i=1}^{p} r_i^{(1/4)(p-2c)} \prod_{i<j}^{p} (r_i^{1/2} + r_j^{1/2})^{-1/2} $$

$$ \times \{1 + n^{-1/2}P_1 + O(n^{-1})\}, \tag{2.1} $$

where

$$ k_2(n) = \Gamma_p(c)\, 2^{-p}\pi^{-(1/4)p(p+1)} n^{-(1/8)p(4c-p-1)} $$

and

$$ P_1 = \frac{1}{16} \sum_{i<j}^{p} (r_i^{1/2} + r_j^{1/2})^{-1} - \frac{1}{16}(2c-p)(2c-p-2) \sum_{i=1}^{p} r_i^{-1/2}. $$

THEOREM 2.2. *Let $R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$, where R is $p \times p$, $R_1 = \operatorname{diag}(r_1, \ldots, r_k)$ with $r_1 > \cdots > r_k > 0$, and let $S = \operatorname{diag}(s_1, \ldots, s_p)$ with $s_1 > \cdots > s_p > 0$. As $n \to \infty$*

$$
{}_0F_1^{(p)}(c; nR, S) \sim k_3(n) \exp\left(2n^{1/2} \sum_{i=1}^{k} (r_i s_i)^{1/2}\right) \prod_{i=1}^{k} (r_i s_i)^{(1/4)(p-2c)}
$$

$$
\times \prod_{\substack{i=1 \\ i<j}}^{k} \prod_{j=1}^{p} c_{ij}^{-1/2} \{1 + n^{-1/2} Q_1 + O(n^{-1})\}, \tag{2.2}
$$

where

$$
\begin{aligned}
c_{ij} &= (r_i - r_j)(s_i - s_j) \qquad (i, j = 1, \ldots, k) \\
&= r_i(s_i - s_j) \qquad (i = 1, \ldots, k; j = k+1, \ldots, p), \\
k_3(n) &= \Gamma_k(c)\, \Gamma_k(\tfrac{1}{2} p)\, 2^{-k} \pi^{-(1/2)k(k+1)} n^{(1/4)k(k-p-2c+1)}
\end{aligned} \tag{2.3}
$$

and

$$
Q_1 = \tfrac{1}{4} \sum_{i<j}^{k} c_{ij}^{-1}[(r_i s_i)^{1/2} + (r_j s_j)^{1/2}] + \tfrac{1}{4} \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{-1}(r_i s_i)^{1/2}
$$

$$
- \frac{1}{16}(2c - p)(2c - p - 2) \sum_{i=1}^{k} (r_i s_i)^{-1/2}.
$$

The term Q_1 of order $n^{-1/2}$ in (2.2) has not been given previously, except when $k = p$ (Muirhead [18]). Q_1 was found using a partial differential equation for the ${}_0F_1^{(p)}$ function.

3. The Noncentral Multivariate F Distribution

Let the matrices B and W be distributed as in Section 1, i.e. B is $W_p(n_1, \Sigma; \Delta)$ and W is independently $W_p(n_2, \Sigma)$. The density function of the matrix $F = B^{1/2} W^{-1} B^{1/2}$ is (see [13])

$$
k_4 \det F^{(1/2)(n_1-p-1)} \det(I + F)^{-n}
$$

$$
\times \exp(-\tfrac{1}{2} \operatorname{tr} \Delta)\, {}_1F_1(n; \tfrac{1}{2}n_1; \tfrac{1}{2}\Delta(I + F^{-1})^{-1}) \tag{3.1}
$$

where $n = \frac{1}{2}(n_1 + n_2)$ and

$$
k_4 = \Gamma_p(n) / (\Gamma_p(\tfrac{1}{2}n_1)\, \Gamma_p(\tfrac{1}{2}n_2)).
$$

Since the ${}_1F_1$ function in (3.1) depends on the matrix $R = \frac{1}{2}\Delta(I + F^{-1})^{-1}$ only through its latent roots we can assume, without loss of generality, that $R = \operatorname{diag}(r_1, \ldots, r_p)$ where each r_i is positive.

THEOREM 3.1. *As* $n \to \infty$

$$ {}_1F_1(n; c; R) \sim \exp(\tfrac{1}{2} \operatorname{tr} R)\, {}_0F_1(c; nR). \tag{3.2} $$

Proof. To avoid continually writing out long expressions we merely sketch the proof. We can write

$$ \begin{aligned} {}_1F_1(n; c; R) = {} & \frac{n^{(1/4)p(4n-p-1)}}{\Gamma_p(n)}\, e^{-pn}\, {}_0F_1(c; nR) \\ & \times \int_{S+n^{1/2}I>0} \exp(-n^{1/2} \operatorname{tr} S) \det(I + n^{-1/2}S)^{n-(1/2)(p+1)} \\ & \times \frac{{}_0F_1(c; nR(I + n^{-1/2}S))}{{}_0F_1(c; nR)}\, dS; \end{aligned} $$

this follows by expressing ${}_1F_1$ as a Laplace transform of ${}_0F_1$ (see Herz [11]) and rearranging slightly. Now let $n \to \infty$; Theorem 2.1 can be used to show that the ratio of the two ${}_0F_1$ functions in the integrand tends to $\exp(\operatorname{tr} R^{1/2}S)$, while

$$ \exp(-n^{1/2} \operatorname{tr} S) \det(I + n^{-1/2}S)^{n-(1/2)(p+1)} \to \exp(-\tfrac{1}{2} \operatorname{tr} S^2). $$

It follows that

$$ \begin{aligned} {}_1F_1(n; c; R) \sim {} & \frac{n^{(1/4)p(4n-p-1)}}{\Gamma_p(n)}\, e^{-pn}\, {}_0F_1(c; nR) \\ & \times \underset{\substack{-\infty<s_{ij}<\infty \\ i<j}}{\int \cdots \int} \exp(-\tfrac{1}{2} \operatorname{tr} S^2 + \operatorname{tr} R^{1/2}S)\, dS. \quad \blacksquare \end{aligned} $$

The value of the last integral is $(2\pi)^{p/2}\, \pi^{p(p-1)/4} \exp(\frac{1}{2}\operatorname{tr} R)$ and the theorem then follows by using Stirling's formula for the asymptotic behavior of $\Gamma_p(n)$.

It can be noted that when $p = 1$, (3.2) agrees with the known asymptotic behavior of the classical confluent hypergeometric function (see Buchholz [4]).

On putting $c = \frac{1}{2}n_1$, $n = \frac{1}{2}(n_1 + n_2)$, Theorem 3.1 describes the asymptotic behavior of the ${}_1F_1$ function in the density (3.1) of the multivariate noncentral F distribution. An asymptotic result in terms of elementary functions which can be used for computational purposes follows by substituting the expansion for the ${}_0F_1$ function given by Theorem 2.1.

4. LATENT ROOTS IN MANOVA AND DISCRIMINANT ANALYSIS

The method used in the last section can also be used to derive the asymptotic behavior, for large n, of the two-matrix function ${}_1F_1^{(p)}(n; c; R, S)$ which occurs in the density function (1.2) of the latent roots of the matrix $B(B + W)^{-1}$. Alternatively, since

$$ {}_1F_1^{(p)}(n; c; R, S) = \int_{O(p)} {}_1F_1(n; c; RHSH')\, dH $$

the results of the previous sections can be used in conjunction with a multivariate extension of Laplace's method due to Hsu [12] to obtain the asymptotic behavior of this integral. The details are reasonably straightforward and are omitted. The result is summarized in the following:

THEOREM 4.1. *Let* $R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$, *where* R *is* $p \times p$, $R_1 = \operatorname{diag}(r_1, ..., r_k)$ *with* $r_1 > \cdots > r_k > 0$, *and let* $S = \operatorname{diag}(s_1, ..., s_p)$ *with* $s_1 > \cdots > s_p > 0$. *As* $n \to \infty$

$$ {}_1F_1^{(p)}(n; c; R, S) \sim \exp\left(\tfrac{1}{2} \sum_{i=1}^{k} r_i s_i\right) {}_0F_1^{(p)}(c; nR, S). \tag{4.1} $$

An asymptotic expansion for the ${}_0F_1^{(p)}$ function on the right side of (4.1) has been given by Theorem 2.2, and can be used for computational purposes. An alternative expression for the asymptotic behavior of ${}_1F_1^{(p)}$ has been given by Chattopadhyay and Pillai [5] and Chattopadhyay, Pillai and Li [6]; however the result of these authors, at least as stated, appears to be incorrect.

Some numerical work has been carried out in order to investigate the accuracy of the asymptotic approximations (4.1) and (2.2). The simplest nontrivial case is when $p = 2$ and we take $R = \operatorname{diag}(r_1, 0)$, $S = \operatorname{diag}(s_1, s_2)$; in this case there are reasonably simple expressions for ${}_1F_1^{(2)}$ and ${}_0F_1^{(2)}$ which allow exact calculation, namely (Muirhead [17])

$$ {}_1F_1^{(2)}(n; c; R, S) = \sum_{j=0}^{\infty} \frac{(n)_j}{(c)_j} \frac{[r_1(s_1s_2)^{1/2}]^j}{j!} P_j\left[\frac{s_1 + s_2}{2(s_1s_2)^{1/2}}\right] \tag{4.2} $$

and

$$ {}_0F_1^{(2)}(c; nR, S) = \sum_{j=0}^{\infty} \frac{[nr_1(s_1s_2)^{1/2}]^j}{(c)_j\, j!} P_j\left[\frac{s_1 + s_2}{2(s_1s_2)^{1/2}}\right], \tag{4.3} $$

where $P_j(\cdot)$ denotes the Legendre polynomial of degree j. The values of these functions quickly become very large, as can be seen in Table 1, where the actual values of the above two functions are given for some selected values of the parameters. In Table 1, Ratio refers to the ratio of the left side of (4.1) to the right side; this ratio tends to unity as $n \to \infty$. These, and other more extensive, numerical results show that the accuracy of the approximation increases as s_1 decreases (all other values of the parameters remaining fixed), accuracy increases as c decreases, and accuracy increases as r_1 decreases. Similar results should also be true for values of p other than two, although the exact calculation of the functions involved then becomes much more difficult. The asymptotic expansion (2.2) is used in place of the exact value of ${}_0F_1^{(2)}$ in Table 2. Ratio (1) there refers to the exact value of ${}_0F_1^{(2)}$ to H, where H denotes the right side of (2.2), and Ratio (2) is the exact value of ${}_1F_1^{(2)}$ divided by $\exp(\frac{1}{2}r_1s_1) \cdot H$.

TABLE 1

Exact Values of ${}_1F_1^{(2)}$ and ${}_0F_1^{(2)}$ from (4.2) and (4.3) when $r_1 = 10$, $r_2 = 0$, $s_2 = .25$[a]

s_1	c	n	${}_1F_1^{(2)}$	${}_0F_1^{(2)}$	Ratio
.75	5	100	$.4214 \times 10^{19}$	$.1692 \times 10^{18}$	.5857
		500	$.2669 \times 10^{47}$	$.8062 \times 10^{45}$	.7786
		1000	$.5523 \times 10^{68}$	$.1553 \times 10^{67}$	.8361
.50	5	100	$.1737 \times 10^{15}$	$.2291 \times 10^{14}$	.6224
		500	$.4309 \times 10^{37}$	$.4422 \times 10^{36}$	.7999
		1000	$.8029 \times 10^{54}$	$.7736 \times 10^{53}$	.8523
.25	5	100	$.2924 \times 10^{10}$	$.1146 \times 10^{10}$	.7313
		500	$.1000 \times 10^{26}$	$.3335 \times 10^{25}$	.8593
		1000	$.1190 \times 10^{38}$	$.3805 \times 10^{37}$	.8965
.75	1	100	$.1813 \times 10^{24}$	$.4424 \times 10^{22}$	.9636
		500	$.2076 \times 10^{53}$	$.4961 \times 10^{51}$	.9842
		900	$.2305 \times 10^{71}$	$.5484 \times 10^{69}$	.9883

[a] Ratio $= {}_1F_1^{(2)}/(\exp(\frac{1}{2}r_1s_1){}_0F_1^{(2)})$.

Substitution of (4.1) and (2.2), with $R = \frac{1}{2}\Omega$, $S = L$, $c = n_1/2$, in (1.2) gives a representation for the joint density function of the latent roots $l_1, \ldots, l_p$ of $B(B + W)^{-1}$, under the assumption that the noncentrality parameters satisfy

$$\omega_1 > \cdots > \omega_k > \omega_{k+1} = \cdots = \omega_p = 0. \tag{4.4}$$

This is summarized in the following:

THEOREM 4.2. *For large error degrees of freedom n_2 an asymptotic representation of the density function* (1.2) *of $l_1, \ldots, l_p$, when the noncentrality parameters satisfy* (4.4), *is*

$$k_5 \prod_{i=1}^{p} [l_i^{(1/2)(n_1-p-1)}(1-l_i)^{(1/2)(n_2-p-1)}] \prod_{i<j}^{p} (l_i - l_j) \prod_{i=1}^{k} l_i^{(1/4)(p-n_1)}$$

$$\times \exp\left[\sum_{i=1}^{k} (\tfrac{1}{4}\omega_i l_i + (2n\omega_i l_i)^{(1/2)})\right] \prod_{i=1}^{k} \prod_{\substack{j=1 \\ j<j}}^{p} d_{ij}^{-1/2}\{1 + O(n^{-1/2})\}, \quad (4.5)$$

where $n = \frac{1}{2}(n_1 + n_2)$,

$$d_{ij} = (\omega_i - \omega_j)(l_i - l_j) \qquad (i, j = 1,..., k)$$
$$= \omega_i(1 - l_j) \qquad (i = 1,..., k; j = k + 1,..., p)$$
$$k_5 = k_1 k_3 2^{(1/4)k(n_1+p-k-1)} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{k} \omega_i\right) \prod_{i=1}^{k} \omega_i^{(1/4)(p-n_1)}$$

and k_1 *and* k_3 *are given by* (1.3) *and* (2.3).

TABLE 2

Ratio (1) $= {}_0F_1^{(2)}/H$ and Ratio (2) $= {}_1F_1^{(2)}/(\exp(\frac{1}{2}r_1 s_1)H)$, where H Denotes the Right Side of (2.2), when $r_1 = 10$, $r_2 = 0$, $s_2 = .25$, $c = 5$

s_1	n	Ratio (1)	Ratio (2)
.75	100	1.0057	.5890
	500	1.0011	.7794
	1000	1.0005	.8366
.50	100	1.0098	.6285
	500	1.0018	.8013
	1000	1.0009	.8530
.40	100	1.0162	.6565
	500	1.0026	.8159
	1000	1.0013	.8637

From Theorem 4.2 it is easy to obtain the following:

COROLLARY. *For large* n_2 *an asymptotic representation of the conditional density function of the* $p - k$ *smallest sample roots* $l_{k+1},..., l_p$ *given the* k *largest roots* $l_1,..., l_k$, *when the assumption* (4.4) *holds, is proportional to*

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} (l_i - l_j)^{1/2} \prod_{i=k+1}^{p} [l_i^{(1/2)(n_1-p-1)}(1-l_i)^{(1/2)(n_2-p-1)}] \prod_{\substack{k+1 \\ i<j}}^{p} (l_i - l_j). \quad (4.6)$$

It is worth remarking that although there are marked differences between the asymptotic joint density functions of $l_1, \ldots, l_p$ in the three situations discussed in Section 1, *these densities all give rise to the same asymptotic conditional density function* (4.6) *of* $l_{k+1}, \ldots, l_p$ *given* $l_1, \ldots, l_k$. If the "linkage factors"

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} (l_i - l_j)^{1/2}$$

are ignored this is just the distribution of the latent roots of $S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 have the independent Wishart distributions $W_{p-k}(n_1 - k, \Sigma)$ and $W_{p-k}(n_2 - k, \Sigma)$ respectively. Note that (4.6) does not depend on $\omega_1, \ldots, \omega_k$ (i.e., $l_1, \ldots, l_k$ are asymptotically sufficient for $\omega_1, \ldots, \omega_k$); these are nuisance parameters in a test of the null hypothesis H_k given by (1.1) and their effect can be eliminated, at least asymptotically, by using the conditional distribution (4.6). The statistic most commonly used for testing H_k is (Bartlett [2, 3])

$$T_k = -\log \prod_{i=k+1}^{p} (1 - l_i),$$

and when H_k is true, the asymptotic distribution of $n_2 T_k$, as $n_2 \to \infty$, is $\chi^2_{(n_1-k)(p-k)}$. A correction factor $\alpha(n_2)$, which improves the rate of convergence of the test statistic $\alpha(n_2)\, T_k$ to its asymptotic χ^2 distribution by improving agreement between the moments, can be found by using the conditional density (4.6) to compute moments of T_k (cf. James [14], Glynn and Muirhead [10], in other contexts). The following result, given by Glynn [9], follows directly from Theorem 4 of Glynn and Muithead [10]:

THEOREM 4.3. *When the null hypothesis H_k is true the statistic*

$$L_k = \left[n_2 - k + \tfrac{1}{2}(n_1 - p - 1) + \sum_{i=1}^{k} l_i^{-1} \right] T_k$$

has an asymptotic $\chi^2_{(n_1-k)(p-k)}$ distribution, and

$$E(L_k) = (n_1 - k)(p - k) + O(n_2^{-2}).$$

The multiplying factor suggested originally by Bartlett [3] is $n_2 + \frac{1}{2}(n_1 - p - 1)$; the multiplying factor in L_k is approximately this if the observed values of $l_1, \ldots, l_k$ are all close to one.

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