Discrete Mathematics 28 (1979) 247-254

GENERALIZED RAMSEY THEORY FOR GRAPHS, X: DOUBLE STARS

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Received 8 May 1978
Revised 22 May 1979

Dedicated to Paul Erdös and Ronald Graham, double stars!

The double star $S(n, m)$, where $n \geq m \geq 0$, is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with a line joining their centers. Its ramsey number $r(S(n, m))$ is the least number p such that there is a monochromatic copy of $S(n, m)$ in any 2-coloring of the edges of K_p. It is shown that $r(S(n, m)) = \max(2n+1, n+2m+2)$ if n is odd and $m \leq 2$; and $r(S(n, m)) = \max(2n+2, n+2m+2)$ otherwise, for $n \leq \sqrt{2}m$ or $n \geq 3m$.

1. Introduction

It is by now a well-known definition that the *ramsey number* of a graph G is the least integer p such that if the lines of the complete graph K_p are 2-colored red and blue, then either the red subgraph or the blue subgraph of K_p contains a copy of G. Ramsey numbers (and various generalizations) have been computed for many classes of graphs, including stars, paths, and cycles; see [1] for a compilation of the results known in 1973 and [6] for a listing of open questions as of 1975.

Our object is to investigate the ramsey numbers of the double stars. We define a *double star* as the union of two stars with a line joining the centers. When the ratio of the number of spikes on the two stars is either greater than or equal to 3, or between 1 and $\sqrt{2}$ inclusive, we determine this ramsey number exactly. Although we have not been able to extend the proof techniques used here, we conjecture that the results obtained will also hold for the remaining cases.

More precisely, for $n \geq m \geq 0$ the *double star* $S(n, m)$ is the graph on the points $\{v_0, v_1, \ldots, v_n, w_0, w_1, \ldots, w_m\}$ with lines

$$\{(v_0, w_0), (v_0, v_i), (w_0, w_j): 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Note that $S(n, m)$ is not defined if $n < m$. For convenience the line (v_0, w_0) is called *the bridge* of $S(n, m)$ and the subgraphs $\langle v_1, \ldots, v_n \rangle$ and $\langle w_0, \ldots, w_m \rangle$ are called the *n-star at* v_0 and the *m*-star *at* w_0 respectively. We denote the ramsey number of $S(n, m)$ by $r(S(n, m))$ in the usual way. Notation and terminology not specified here can be found in [5].

Our principal results are that the ramsey numbers of the double stars satisfy

(1) $r(S(n, m)) = \max(2n+1, n+2m+2)$ if n is odd and $m \leq 2$; and

(2) $r(S(n, m)) = \max(2n+2, n+2m+2)$ if n is even or $m \geq 3$, provided that $n \leq \sqrt{2}m$ or $n \geq 3m$.

In Section 2 we show that these numbers are lower bounds for all double stars. Section 3 contains the proofs that these numbers are upper bounds for the specified cases. We also obtain a weaker upper bound that holds in general. We conclude with a list of several related unsolved problems.

2. Lower bounds

In this section we establish lower bounds for the ramsey numbers of all double stars. We begin by presenting these lower bounds in Theorem 2.1, and then provide their proofs via a series of lemmas.

Theorem 2.1. *The ramsey numbers of the double stars satisfy*

$$r(S(n, m)) \geq \begin{cases} \max(2n+1, n+2m+2) & \text{if } n \text{ is odd and } m \leq 2, \\ \max(2n+2, n+2m+2) & \text{otherwise.} \end{cases}$$

Lemma 2.2. $r(S(n, m)) \geq n+2m+2$.

Proof. Consider a coloring of K_{n+2m+1} where the red subgraph consists of $K_{n+m+1} \cup K_m$, so that the blue one is the complete bipartite graph $K(n+m+1, m)$. It is easy to see that there is no red $S(n, m)$ since $S(n, m)$ is connected and has $n+m+2$ points. As the blue subgraph is $K(n+m+1, m)$, there is no blue $S(n, m)$, since $S(n, m)$ contains two adjacent points with degrees $n+1$ and $m+1$ respectively.

Lemma 2.3

$$r(S(n, m)) \geq \begin{cases} 2n+1 & \text{if } n \text{ is odd,} \\ 2n+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. This follows immediately from the ramsey numbers of stars given in [3] since $S(n, m)$ contains $K_{1,n+1}$, the star with $n+1$ spikes.

Lemma 2.4. *If* $m \geq 3$ *and* n *is odd, then* $r(S(n, m)) \geq 2n+2$.

Proof. Since $n+2m+2 \geq 2n+2$ if $n \leq 2m$, we may assume $n > 2m \geq 6$ in view of Lemma 2.2. We will show that the following coloring of K_{2n+1} does not contain a monochromatic $S(n, m)$.

We first construct a graph G with $2n+1$ points. Let $V(G) = W \cup X \cup Y$ where $|W| = 3$ and $|X| = |Y| = n-1$. Also let $\langle W \rangle$ be P_3, let $\langle X \rangle$ be regular with degree

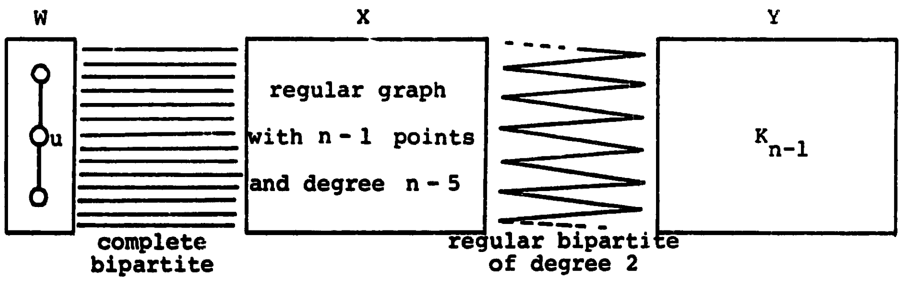

Fig. 1. The red subgraph of a coloring of K_{2n+1} which does not contain a monochromatic $S(n, m)$, where n is odd and $n > 2m \geq 6$.

$n-5$ (which is possible since n is odd), and let $\langle Y \rangle$ be K_{n-1}. Finally the lines between W and X form the complete bipartite graph on W and X, those joining X and Y form a regular bipartite graph of degree 2 on X and Y, and there are no $W-Y$ lines. This graph is illustrated in Fig. 1.

Now color K_{2n+1} so that its red subgraph is G. It is easy to check that the only point in either monochromatic subgraph with degree at least $n+1$ is the center point u of the red path in W, whose degree is $n+1$ in the red subgraph. Thus the only possible monochromatic $S(n, m)$ is red and has bridge (u, v) for some $v \in W \cup X - \{u\}$. However, there are at most two red lines from v to points not in the red n-star at u. Therefore if $m \geq 3$, there is no monochromatic $S(n, m)$.

Combining the results of Lemmas 2.2, 2.3, and 2.4, we see that the proof of Theorem 2.1 is completed.

3. Upper bounds

This section establishes upper bounds for the ramsey numbers of double stars. Theorem 3.1 provides a weak upper bound that holds for all double stars. Theorems 3.2 and 3.3 show that the lower bounds given in Section 2 are also upper bounds for each of the cases $m \leq 2$ and n odd, $n \geq 3m$, and $n \leq \sqrt{2}m$.

We begin by stating these theorems and then again present the proofs via a series of lemmas, thus completing the proof of the results stated in the introduction.

Theorem 3.1. *The ramsey numbers of the double stars satisfy*

$$r(S(n, m)) \leq 2n + m + 2.$$

Theorem 3.2. *The ramsey numbers of the double stars satisfy*

$$r(S(n, m)) \leq n + 2m + 2 \quad \text{if } n \leq \sqrt{2}m.$$

Theorem 3.3. *The ramsey numbers of the double stars satisfy*

$$r(S(n,m)) \leqslant \begin{cases} 2n+1 & \text{if } n \text{ is odd and } m \leqslant 2, \\ 2n+2 & \text{otherwise, if } n \geqslant 3m. \end{cases}$$

We begin by introducing some notation which will be used throughout this section. Let a red-blue coloring of the lines of a complete graph K be given. For a point v and a subset W of points in K, let red-$d(v)$ be the number of points joined to v by red lines; let red-$d_W(v)$ be the number of these points in W; blue-$d(v)$ and blue-$d_W(v)$ are defined similarly. Our proofs focus on a fixed point u in K with maximum monochromatic degree. Thus without loss of generality, we may assume that red-$d(u) \geqslant$ red-$d(v)$ and red-$d(u) \geqslant$ blue-$d(v)$ for all v. Write red-$d(u) = m+n-k$, where the integer k is not necessarily positive. Finally let

$$A = \{v : \text{the line } (u, v) \text{ is red}\}$$

and let

$$B = \{v : \text{the line } (u, v) \text{ is blue}\}.$$

Note that $|A| = n+m-k$.

We now prove two lemmas which will be used repeatedly in this section.

Lemma 3.4. *If $k<0$, then every 2-coloring of K_p contains a monochromatic $S(n, m)$ for $p \geqslant n+2m+2$.*

Proof. Clearly it is enough to prove the lemma for $p = n+2m+2$, and we may ignore the trivial case $n=0$. Note that $|A| \geqslant n+m+1$ since $k<0$. Now if red-$d(v) \geqslant m+1$ for some $v \in A$, then K_{n+2m+2} contains a red $S(n, m)$ with bridge (u, v). To see this, note that after we form a red m-star at v, there are at least n points left to form a red n-star at u. Hence we may assume that for every $v \in A$ we have red-$d(v) \leqslant m$ and therefore

$$\text{blue-}d(v) \geqslant n+2m+1-m = n+m+1.$$

Combining this fact with $|A| \geqslant n+m+1$ and $n+m+1 > \frac{1}{2}(n+2m+2)$, we see that there must be a blue line between some pair of points of A. Clearly any such blue line forms the bridge of a blue $S(n, m)$.

Lemma 3.5. *If $k \geqslant 0$ and there are more than $k(n+m-k)$ red lines between A and B, then K_p contains a monochromatic $S(n, m)$ for $p \geqslant n+2m+2$.*

Proof. If there are more than $k(n+m-k)$ red lines between A and B, then red-$d_B(v) \geqslant k+1$ for some $v \in A$. Furthermore red-$d(v) \geqslant$ blue-$d(u)$ since red-$d(u) \geqslant$ blue-$d(v)$, so

$$\text{red-}d(v) \geqslant (p-1)-(n+m-k) \geqslant m+k+1 \geqslant m+1.$$

We can thus construct a red $S(n, m)$ with bridge (u, v) by using at least $k+1$ points from B and at most $m-(k+1)$ points from $A-\{v\}$ to form the red m-star at v. This leaves at least

$$m+n-k-1-(m-(k+1))=n$$

points in $A-\{v\}$ to form the red n-star at u.

We are now ready to give the proof of Theorem 3.1, which gives a weak upper bound.

Proof of Theorem 3.1. We must show that (every 2-coloring of) K_{2n+m+2} contains a monochromatic $S(n, m)$. Note that $|A|=n+m-k$, $|B|=n+k+1$ and necessarily $k \leqslant \frac{1}{2}(m-1)$. By Lemma 3.4 we may assume $k \geqslant 0$, so $|B| \geqslant n+1$. If there is a point $w \in B$ such that blue-$d_A(w) \geqslant m$, then there is clearly a blue $S(n, m)$ with bridge (u, w). Hence we may assume that

$$\text{red-}d_A(w) \geqslant (n+m-k)-(m-1)=n-k+1$$

for each $w \in B$, so there are at least $(n-k+1)(n+k+1)$ red lines between A and B. If K_{2n+m+2} does not contain a monochromatic $S(n, m)$, then by Lemma 3.5 there are at most $k(m+n-k)$ red lines between A and B. Thus we have

$$\begin{aligned}(n+1)^2-k^2 &= (n-k+1)(n+k+1)\\ &\leqslant k(m+n-k)=k(m+n)-k^2\\ &\leqslant \tfrac{1}{2}(m-1)(2n)-k^2<mn-k^2 \leqslant n^2-k^2\end{aligned}$$

which is clearly impossible.

Next we consider the case in which $n \leqslant 2m$.

Lemma 3.6. *If $n \leqslant 2m$ and $k \geqslant 0$ and there are fewer than $(m-k+1)(m+k+1)$ red lines between A and B, then K_{n+2m+2} contains a monochromatic $S(n, m)$.*

Proof. Recall that $|A|=m+n-k$, so now $|B|=m+k+1$. Thus if there are fewer than $(m-k+1)(m+k+1)$ red lines between A and B, then red-$d_A(w) \leqslant m-k$ for some $w \in B$, and so

$$\text{blue-}d_A(w) \geqslant |A|-(m-k)=n.$$

Since $|B| \geqslant m+1$, this implies that K_{n+2m+2} contains a blue $S(n, m)$ with bridge (w, u).

Lemma 3.7. *If $n \leqslant 2m$ and $k \geqslant n-m$, then K_{n+2m+2} contains a monochromatic $S(n, m)$.*

Proof. As above we have $|A|=m+n-k$ and $|B|=m+k+1$. By Lemma 3.5, if

K_{n+2m+2} does not contain a monochromatic $S(n, m)$ then there are at most $k(n+m-k)$ red lines between A and B, and hence at least

$$(|B|-k)(n+m-k)=(m+1)\,|A|$$

blue lines between A and B. Noting that $|A|\geq|B|$, we see that blue-$d_A(w)\geq m$ for some $w\in B$. Now since $k\geq n-m$, we have

$$|B|\geq m+(n-m)+1\geq n+1,$$

so there is a blue $S(n, m)$ with bridge (u, w).

We can now prove our second theorem.

Proof of Theorem 3.2. We must show that K_{n+2m+2} contains a monochromatic $S(n, m)$ if $n\leq\sqrt{2}m$. Suppose not. Then by Lemma 3.4 we have $k\geq 0$, and furthermore

$$\begin{aligned}(m+1)^2&=(m-k+1)(m+k+1)+k^2\\&\leq k(n+m-k)+k^2 \quad \text{by Lemmas 3.5 and 3.6}\\&=k(n+m)<(n-m)(n+m) \quad \text{by Lemma 3.7}\\&=n^2-m^2\\&\leq 2m^2-m^2=m^2 \quad \text{since } n\leq\sqrt{2}m\end{aligned}$$

which is clearly false.

Finally we look at the case $n\geq 2m$. For $m\geq 3$ or n even, the following lemma contains the crucial argument. Lemma 3.9 will then deal with the remaining case.

Lemma 3.8. *If $n\geq 2m$ and $k\geq 0$ and there are fewer than $(n-2m)(n-m+k)$ red lines between A and B, then K_{2n+2} contains a monochromatic $S(n, m)$.*

Proof. Note that $|A|=n+m-k$ and $|B|=n-m+k+1$, and so

$$(n-2m)(n-m+k)\leq(n-2m)\,|B|.$$

Thus if there are fewer than $(n-2m)(n-m+k)$ red lines between A and B, then

$$\text{red-}d_A(w)\leq n-2m-1$$

for some $w\in B$, and hence

$$\text{blue-}d_A(w)\geq(n+m-k)-(n-2m-1)=3m-k+1.$$

Suppose that blue-$d(w)\geq n+1$. Then we can construct a blue $S(n, m)$ with bridge (w, u) by using at least $3m-k+1$ points from A and at most $n-(3m-k+1)=n-3m+k-1$ points from B to form the n-star at w. This leaves at least

$$(n-m+k)-(n-3m+k-1)=2m+1\geq m$$

points in B to form the m-star at u. On the other hand, suppose blue-$d(w) \leq n$. Then red-$d(w) \geq n+1$, and in particular there is a point $v \in A$ joined by a red line to w. We claim that K_{2n+2} then contains a red $S(n, m)$ with bridge (w, v). Since red-$d(w) \geq n+1$, there is an n-star (excluding the line (w, v)) at w. It uses at most $n-2m-2$ points from A because red-$d_A(w) \leq n-2m-1$. By the proof of Lemma 3.5 we may assume that red-$d_B(v) \leq k$, so at most k points in B, together with the previously mentioned $n-2m-2$ points in A, are not available for forming an m-star at v. Since red-$d(v) \geq$ blue-$d(u) = n-m+k+1$, there are at least

$$(n-m+k+1)-k-(n-2m-2) \geq m$$

points available, and thus the red $S(n, m)$ exists.

Lemma 3.9. *If $m \leq 2$ and $n \geq 2m$ and n is odd, then K_{2n+1} contains a monochromatic $S(n, m)$.*

Proof. We begin with the case $m=2$. We must show that K_{2n+1} contains a monochromatic $S(n, 2)$ for $n \geq 5$. Note that $2n+1 \geq n+2m+2$. Following our previous notation we write red-$d(u) = n+2-k$. Because the number of points with odd red degree cannot be odd we must have $k \leq 1$. If $k<0$, then by Lemma 3.4, K_{2n+1} contains a monochromatic $S(n, 2)$. If $k=0$, we may assume that there are no red lines between A and B by Lemma 3.5. Hence, for any $w \in B$ we have blue-$d_A(w) = n+2$ so K_{2n+1} contains a blue $S(n, 2)$ with bridge (w, u). Finally suppose $k=1$. By Lemma 3.5 we may assume that there are at most $n+1$ red lines between A and B. Since $n+1 < 2(n-1) = 2|B|$ there is some $w \in B$ with red-$d_A(w) \leq 1$. Therefore blue-$d_A(w) \geq (n+1)-1 = n$ and thus K_{2n+1} contains a blue $S(n, 2)$.

We omit the proof for $m=1$, since it is similar. For $m=0$, $S(n, m)$ is simply the star $K_{1,n+1}$ whose ramsey number was computed in [3].

Combining Lemmas 3.8 and 3.9, we are ready to prove the last theorem.

Proof of Theorem 3.3. Since Lemma 3.9 provides the proof for n odd and $m \leq 2$ we may assume $n \geq 3m$. Supppose that K_{2n+2} does not contain a monochromatic $S(n, m)$. Then by Lemma 3.4 we may assume $k \geq 0$. Now it follows that

$$\begin{aligned} m(n-m+k) &\leq (n-2m)(n-m+k) \quad \text{since } n \geq 3m \\ &\leq k(n+m-k) \quad \text{by Lemmas 3.5 and 3.8.} \end{aligned}$$

The above inequality reduces to $(n-m-k)(m-k) \leq 0$, which is impossible since both factors are positive. This is easily seen from the inequalities

$$n+m-k = \text{red-}d(u) \geq n+1 \geq 2m+1.$$

4. Unsolved problems and further results

(1) We make the natural conjecture for the remaining cases.

Conjecture. The ramsey numbers of the double stars are

$$r(S(n, m)) = \begin{cases} \max(2n+1, n+2m+2) & \text{if } n \text{ is odd and } m \leq 2, \\ \max(2n+2, n+2m+2) & \text{otherwise.} \end{cases}$$

In addition to the results contained in this paper, we have verified the conjecture for $m \leq 4$. Thus all that remains to be proved is that $r(S(n, m)) \leq \max(2n+2, n+2m+2)$ for $\sqrt{2}m < n < 3m$, $m \geq 5$.

(2) In [1] Burr conjectured that $r(T)$ for an arbitrary tree T is equal to the lower bound determined by a simple "canonical coloring" of the type given in Lemma 2.2 and Lemma 2.3 above. The construction of Lemma 2.4 disproves this conjecture. Are there trees whose ramsey numbers are arbitrarily greater than these lower bounds?

(3) The double star $S(n, m)$ has a path P_2 joining the centers of the n-star and the m-star. We may generalize to $S(n, m; k)$, which has a path P_k joining the centers of the n-star and the m-star. Burr and Erdös [2] show that

$$r(S(n, m; 4)) = \max(2n+3, n+2m+5).$$

In general what is $r(S(n, m; k))$?

(4) In [4] Grossman studied unicyclic graphs with stars emanating from points on the cycle, while in [2] Burr and Erdös considered complete graphs with a star emanating from one point. What effect in general does adding stars emanating from points on a graph have on the ramsey number?

References

[1] S. Burr, Generalized Ramsey theory for graphs—a survey, in: R. Bari and F. Harary, eds., Graphs and Combinatorics, Lecture Notes in Math. 406 (Springer, Berlin, 1974) 52–75.

[2] S. Burr and P. Erdös, Extremal Ramsey theory for graphs, Utilitas Mathematica 9 (1976) 247–258.

[3] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs II: Small diagonal numbers, Proc. Amer. Math. Soc. 32 (1972) 389–394

[4] J. Grossman, Some ramsey numbers of unicyclic graphs, Ars Combinatoria, to appear.

[5] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

[6] F. Harary, The foremost open problems in generalized ramsey theory, in: C. Nash-Williams and J. Sheehan, eds., Proceedings of the Fifth British Combinatorial Conference (Utilitas Math., Winnipeg, 1976) 269–282.