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RESIDUAL POTENTIAL METHOD IN SPHERICAL COORDINATES AND RELATED APPROXIMATIONS

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Introduction

The Residual Potential Method, introduced by Geers [1], appears to be a very promising technique in the solution of solid-fluid interaction problems. The theoretical development of the method in cylindrical coordinates was presented in [1,2]. In [3], Geers presents some numerical results obtained using the Residual Potential Method in spherical coordinates; however, the theoretical treatment is not provided. The purpose of the present work is to express the classical spherical wave equation in terms of a residual potential and to discuss the accuracy of the various forms of the related acoustic approximations.

Theory

The axisymmetric (two-dimensional) wave equation in spherical coordinates is given by

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2\Phi}{\partial\phi^2} + \cot\phi\,\frac{\partial\Phi}{\partial\phi}\right) = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{1}$$

in which Φ is the velocity potential; r and ϕ denote, respectively, the radial and meridional coordinates. The speed of sound in the acoustic medium is c and time is t. The analysis to follow can be extended to the three-dimensional wave equation without any difficulty. Expand $\Phi\langle r,\phi,t\rangle$ in terms of Legendre functions:

$$\Phi\,\langle r,\phi,t\rangle = \sum_{n=o}^{\infty} \Phi_n\langle r,t\rangle\, P_n\,\langle\cos\phi\rangle \tag{2}$$

Abbreviated Communication

Substitution of (2) into (1) yields

$$\frac{\partial^2 \Phi_n}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi_n}{\partial r} - \frac{\lambda_n}{r^2} \Phi_n = \frac{1}{c^2} \frac{\partial^2 \Phi_n}{\partial t^2} \tag{3}$$

in which $\lambda_n = n(n+1)$. For outgoing waves, the solution of (3) is given by the following [4]:

$$\Phi_n \langle r,t \rangle = r^n \left(\frac{\partial}{r\partial r}\right)^n \left[\frac{1}{r} h_n \langle r-ct \rangle\right] \tag{4}$$

in which h_n is yet unknown. Noting that

$$\frac{\partial h_n}{\partial r} = - \frac{1}{c} \frac{\partial h_n}{\partial t} \tag{5}$$

Φ_n can be expressed in the following form:

$$\Phi_n = \frac{(-1)^n}{c^n \, r} \sum_{k=o}^{n} \Gamma_k^n \left(\frac{c}{r}\right)^k \frac{\partial^{n-k} h_n}{\partial t^{n-k}} \tag{6}$$

in which

$$\Gamma_k^n = \frac{(n+k)!}{2^k k! \, (n-k)!} \tag{7}$$

Using (6) and (5), it can be shown that

$$\frac{\partial \Phi_n}{\partial r} + \frac{1}{c} \frac{\partial \Phi_n}{\partial t} + \frac{1}{r} \Phi_n = \frac{1}{r} \Phi_{Rn} \tag{8}$$

in which the residual velocity potential Φ_{Rn} is given by

$$\Phi_{Rn} = - \frac{(-1)^n}{c^n \, r} \sum_{k=o}^{n} k \, \Gamma_k^n \left(\frac{c}{r}\right)^k \frac{\partial^{n-k} h_n}{\partial t^{n-k}} . \tag{9}$$

Employing the Laplace and, then, inverse transformations on (6) and (9), one obtains the following reduced spherical wave equation:

$$r \frac{\partial \Phi_n}{\partial r} + \frac{r}{c} \frac{\partial \Phi_n}{\partial t} + (n+1) \, \Phi_n = \int_o^t f_n \langle t-\bar{t} \rangle \frac{\partial \Phi_n}{\partial \bar{t}} \, d\bar{t} \tag{10}$$

in which $f_o\langle t \rangle = 0$,

$$f_n\langle t \rangle = \sum_{k=1}^{n} \exp \left(a_{nk} \frac{ct}{r}\right), \; (n=1,2,\ldots) \tag{11}$$

where a_{nk} are the n roots of the following polynomial:

$$\sum_{k=o}^{n} \Gamma_k^n \, s^{n-k} = 0 \, . \tag{12}$$

For a submerged spherical shell, if one is interested in the transient response of the shell only, the reduced spherical wave equation (10) proves to be very practical. The $f_n\langle\tau\rangle$ versus nondimensional time $\tau = ct/a$ curves are given in Figure 1 for a few values of n. Here, a denotes the radius of the spherical fluid cavity. In this form, $f_n\langle\tau\rangle$ is independent of all physical parameters. The curves presented in Figure 1 are useful for all response computations. A similar conclusion was obtained by Geers [1] for his reduced cylindrical wave equation. The MacLaurin series representation of $f_n\langle\tau\rangle$ is given by

$$f_n\langle\tau\rangle = n - \frac{\lambda_n}{2}\tau + \frac{\lambda_n}{2}\frac{\tau^2}{2!} + \frac{\lambda_n}{4}\left(\frac{\lambda_n}{2} - 3\right)\frac{\tau^3}{3!} - \frac{\lambda_n}{2}(\lambda_n - 3)\frac{\tau^4}{4!} + \ldots\ldots \tag{13}$$

The following additional properties of f_n can be obtained easily:

$$\int_0^\infty f_n\langle\tau\rangle\, d\tau = 1 \quad , \quad \int_0^\infty \frac{df_n}{d\tau}\, d\tau = -n\ . \tag{14}$$

In [3], Geers presents three doubly asymptotic approximations; namely, DAA_1, DAA_2 with $\omega_n = nc/a$ and DAA_2 with $\omega_n = (n+1)c/a$. Refer to [3] for a discussion of these approximations. In our analysis, the doubly asymptotic approximations correspond to the following expressions for $f_n\langle\tau\rangle$:

$$\begin{aligned}
&\text{for } DAA_1, \quad f_n\langle\tau\rangle = 0,\\
&\text{for } DAA_2 \text{ with } \omega_n = nc/a, \quad f_n\langle\tau\rangle = ne^{-n\tau}\ ,\\
&\text{for } DAA_2 \text{ with } \omega_n = (n+1)c/a, \quad f_n\langle\tau\rangle = (n+1)e^{-(n+1)\tau}\ .
\end{aligned} \tag{15}$$

Note that both of the DAA_2 approximations satisfy the first of equation (14). Moreover, they reasonably approximate the actual $f_n\langle\tau\rangle$ curves given in Figure 1.

Numerical Example and Discussion

A spherical shell submerged in an infinite medium is subjected to a concentrated load at the apex in the form of a Heaviside function in time. This is the same problem that was studied by Lou and Klosner [5]. The shell and fluid parameters used are the same as those used in

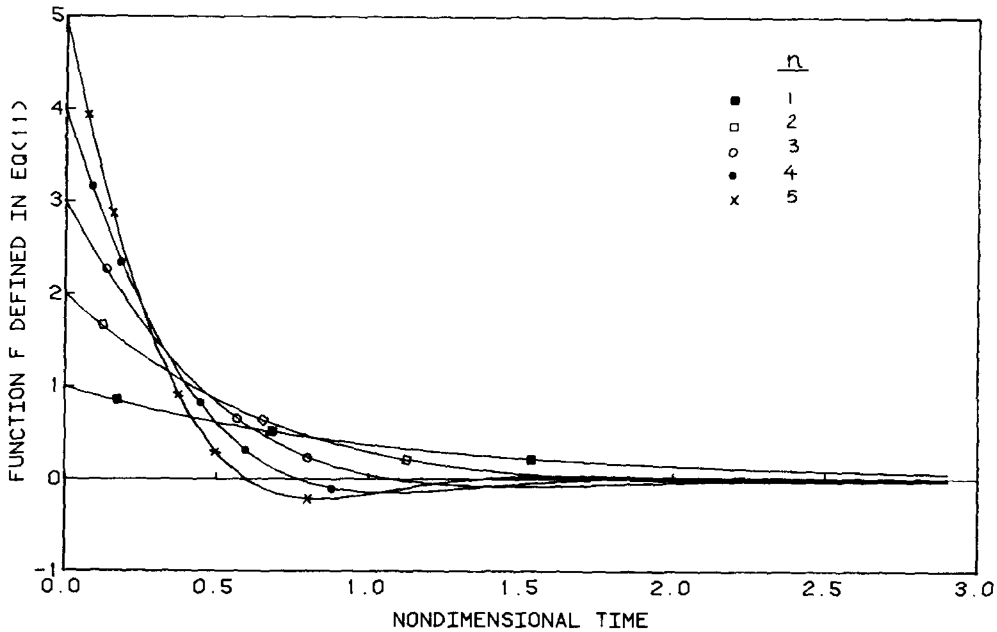

FIG. 1

Function $f_n<\tau>$ defined in equation (11).

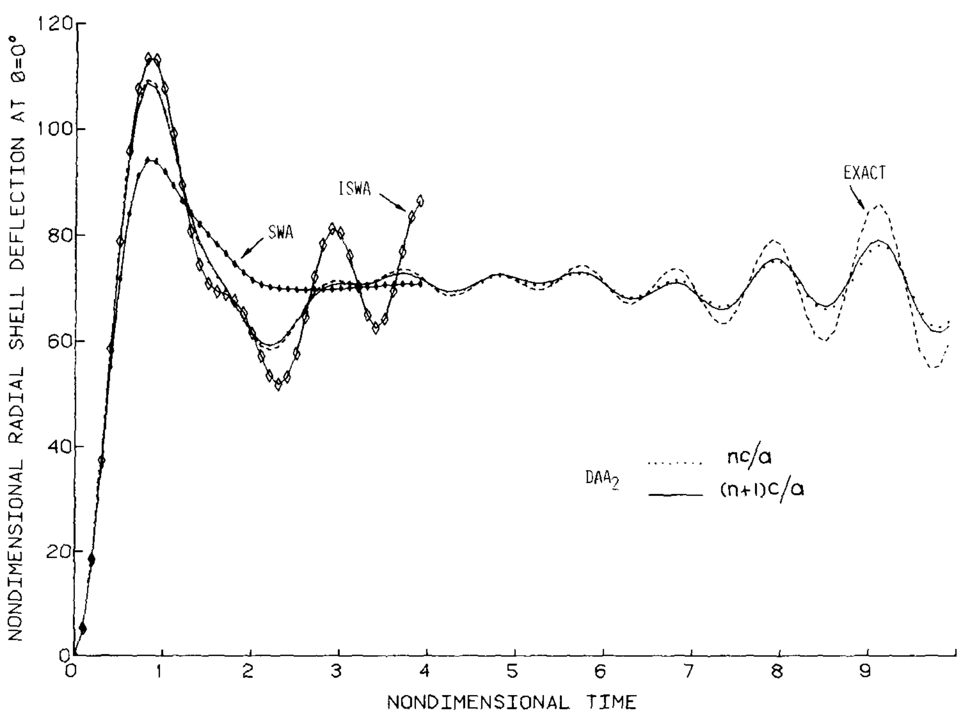

FIG. 2

Results for a submerged spherical shell.

[5]. The radial deflection at the shell apex is plotted as a function of τ in Figure 2. Only the first ten modes were superimposed, excluding the translational mode n = 1. We realize that ten modes are not sufficient to determine the response of the shell. Lou and Klosner [5] use 80 modes for this problem. The superposition of the first ten modes only is believed to be sufficient for our purpose which is to show the applicability of the method presented and to discuss the validity of the approximations. Figure 2 contains, in addition to the exact solution, four approximate solutions. Two of the approximate solutions are Geers' DAA_2 defined in (15). The other two approximations are

$$f_n\langle\tau\rangle = n \tag{16}$$

and

$$f_n\langle\tau\rangle = n - \frac{\lambda_n}{2} \; (1-e^{-\tau}) \; . \tag{17}$$

As seen in Figure 2, the doubly asymptotic approximations are much better than the other two. The reason for plotting the two approximations described by (16) and (17) is as follows: Equation (16) corresponds to the classical spherical wave approximation (SWA):

$$\frac{\partial\Phi}{\partial r} + \frac{1}{c} \; \frac{\partial\Phi}{\partial t} + \frac{1}{a} \; \Phi = 0 \tag{18}$$

which is discussed in detail in [5]. Equation (17) corresponds to the following approximate wave equation

$$(1 + \frac{a}{c}\frac{\partial}{\partial t}) \;\; (\frac{\partial\Phi}{\partial r} + \frac{1}{c}\frac{\partial\Phi}{\partial t} + \frac{1}{a}\;\Phi) - \frac{1}{2a}\;(\frac{\partial^2\Phi}{\partial\phi^2} + \mathrm{Cot}\,\phi\,\frac{\partial\Phi}{\partial\phi}) \;\; = \;\; 0, \tag{19}$$

which will be called an improved spherical wave approximation (ISWA). In the solution of shell-fluid interaction problems, if an approximate wave equation similar to the SWA or the ISWA in nature is used, then it is not necessary to go through a modal analysis. As seen in Figure 2, for early times, the ISWA which is given by (19) is a much better approximation than the SWA which is given by (18). The suggestion is that it may be possible to obtain approximations which are better than any existing approximation. However, the restriction is that any improved approximation should be as practical as the SWA, ISWA, or DAA's which do not require one to use a modal analysis.

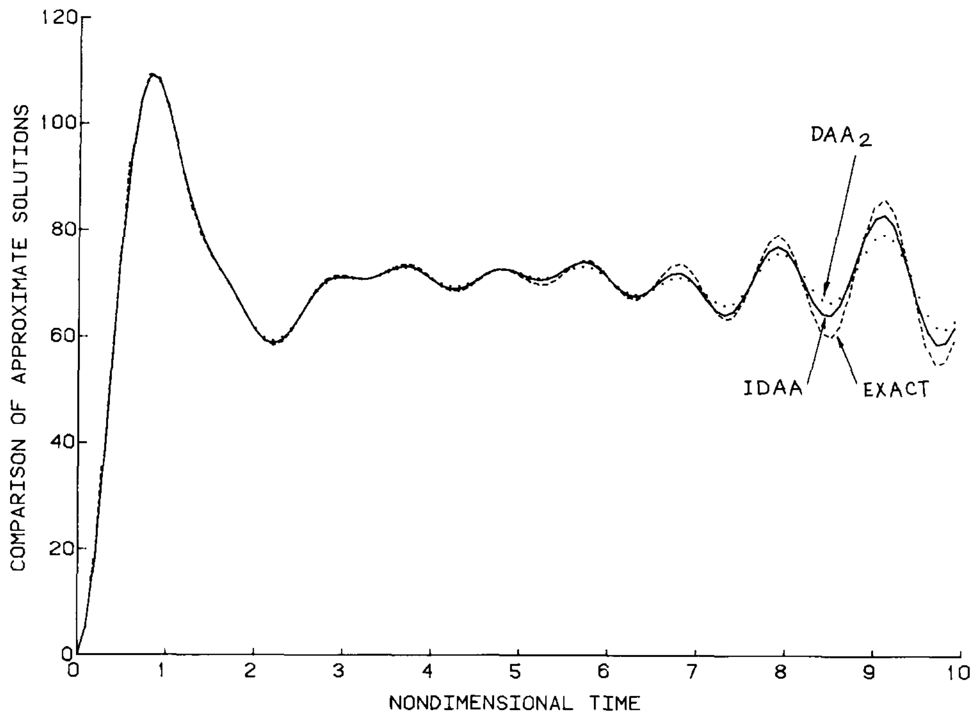

FIG. 3

Comparison of approximate solutions for the submerged shell.

For instance, an improved doubly asymptotic approximation (IDAA) is obtained using the following expression for $f_n <\tau>$:

$$f_n<\tau> = ne^{-(\pi n^2 \tau^2/4)} \quad . \tag{20}$$

The DAA_2 and the IDAA are compared with the exact solution in Figure 3. The IDAA is a better approximation, at least for the problem under consideration. However, the use of the IDAA requires one to use a modal analysis. The work is under way to obtain other IDAA's which do not present this impracticality.

References

1. T. L. Geers, J. Appl. Mech. 36, 459 (1969)
2. T. L. Geers, J. Acoust. Soc. Am. 49, 1505 (1971)
3. T. L. Geers, J. Acoust. Soc. Am. 64, 1500 (1978)
4. H. Lamb, Hydrodynamics, 6th Edn. Dover, New York (1945)
5. Y. K. Lou and J. M. Klosner, J. Appl. Mech. 40, 1078 (1973)

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Abbreviated paper - For further information, please contact the author.