MECHANICS RESEARCH COMMUNICATIONS Vol. 6(5),309-316,1979. Printed in the USA.
0093-6413/79/050309-08$02.00/0

GREEN'S FUNCTIONS FOR PLANAR THERMOELASTIC CONTACT PROBLEMS - EXTERIOR CONTACT

J. Dundurs
Department of Civil Engineering, Northwestern University, Evanston, Illinois 60201, U.S.A.
Maria Comninou
Department of Civil Engineering, University of Michigan, Ann Arbor, Michigan 48109, U.S.A.

(Received 1 August 1979; accepted for print 31 August 1979)

Introduction

The purpose of this note is to record the Green's function that is suitable for formulating planar exterior thermoelastic contact problems. Exterior contact problems arise when the contacting bodies locally separate as heat is conducted through the interface. It is convenient to write the governing integral equations for such problems on the separation rather than the contact zones. The Green's function for the exterior contact consists of a thermoelastic field (heat vortex) that allows one to construct an arbitrary temperature discontinuity across the interface, while maintaining continuity of heat flux, tractions and normal displacements, and a mechanical field (edge dislocation at a frictionless interface) which is required to introduce separation between the solids. No derivations are given because it is readily confirmed that the results satisfy the field equations of thermoelasticity and the appropriate boundary conditions at the interface. The simplifying assumption used is that the contact is frictionless.

Heat Vortex

The coordinate system is placed in relation to the contacting solids as shown in Fig. 1. The two bodies are distinguished by the subscripts 1 and 2. The thermal conductivity is denoted by k, the coefficient of thermal expansion by α, the shear modulus by μ and Poisson's ratio by ν.

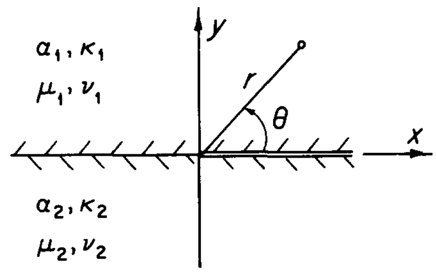

FIG. 1

Consider the temperature distributions

$$T_1 = \omega \frac{k_2}{k_1+k_2}\left(\frac{\theta}{\pi} - 1\right) \tag{1}$$

$$T_2 = \omega \frac{k_1}{k_1+k_2}\left(\frac{\theta}{\pi} - 1\right) \tag{2}$$

$$0 \leq \theta = \tan^{-1} \frac{y}{x} \leq 2\pi \tag{3}$$

which satisfy the differential equation of steady state heat conduction and lead to the following components of heat flux:

$$q_x^{(1)} = q_x^{(2)} = \frac{\omega}{\pi} \frac{k_1 k_2}{k_1+k_2} \frac{y}{r^2} \tag{4}$$

$$q_y^{(1)} = q_y^{(2)} = -\frac{\omega}{\pi} \frac{k_1 k_2}{k_1+k_2} \frac{x}{r^2} \tag{5}$$

Herein ω is a constant. The conditions satisfied on the interface by the thermal fields are

$$T_1(x,0) + \omega H(x) = T_2(x,0) \tag{6}$$

$$q_y^{(1)}(x,0) = q_y^{(2)}(x,0) \tag{7}$$

where H( ) denotes the Heaviside step function. It is seen that the temperature distributions contain a jump across the interface for $x > 0$. The heat flow lines are circles centered on the origin (hence the name heat vortex). The thermal strains corresponding to the harmonic temperature distributions (1) and (2) are compatible and can be integrated for displacements (free expansion displacements) in each of the bodies. However, the resulting expressions show that the normal displacements at the interface do not match, and either a variable gap or overlapping of material develop between the two solids.

The normal displacements can be made continuous by superposing a purely elastic field derived from complex potentials of the type z and zlogz. It is also possible at the same time to adjust the elastic fields so that the normal tractions are continuous and the shearing tractions vanish at the interface. After discarding rigid body terms, the total displacements obtained on this basis are

$$u_x^{(1)} = \frac{\omega}{\pi}\frac{k_1k_2}{k_1+k_2}\{\delta_1[y\log r - x(\pi-\theta)] - \frac{M(\delta_1-\delta_2)}{2\mu_1}[(\kappa_1+1)y\log r-(\kappa_1-1)x(\pi-\theta)+y]\} \tag{8}$$

$$u_x^{(2)} = \frac{\omega}{\pi}\frac{k_1k_2}{k_1+k_2}\{\delta_2[y\log r - x(\pi-\theta)] + \frac{M(\delta_1-\delta_2)}{2\mu_2}[(\kappa_2+1)y\log r-(\kappa_2-1)x(\pi-\theta)+y]\} \tag{9}$$

$$u_y^{(1)} = -\frac{\omega}{\pi}\frac{k_1k_2}{k_1+k_2}\{\delta_1[x\log r + y(\pi-\theta)] - \frac{M(\delta_1-\delta_2)}{2\mu_1}[(\kappa_1+1)x\log r+(\kappa_1-1)y(\pi-\theta)-x]\} \tag{10}$$

$$u_y^{(2)} = -\frac{\omega}{\pi}\frac{k_1k_2}{k_1+k_2}\{\delta_2[x\log r + y(\pi-\theta)] + \frac{M(\delta_1-\delta_2)}{2\mu_2}[(\kappa_2+1)x\log r+(\kappa_2-1)y(\pi-\theta)-x]\} \tag{11}$$

where

$$\delta = \frac{\alpha(1+\nu)}{k}, \quad \kappa = 3 - 4\nu \tag{12,13}$$

for plane strain, and

$$M = \frac{2\mu_1\mu_2}{\mu_1(\kappa_2+1) + \mu_2(\kappa_1+1)} \tag{14}$$

The set of first terms in (19-12) constitute the free expansion displacements, while the second terms which are mutliplied by the constant M are related to the stresses through Hooke's law. The stress components are

$$\sigma_{xx}^{(1)} = -\sigma_{xx}^{(2)} = \frac{2\omega}{\pi} M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2}(\pi-\theta- \frac{xy}{r^2}) \tag{15}$$

$$\sigma_{xy}^{(1)} = -\sigma_{xy}^{(2)} = - \frac{2\omega}{\pi} M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2} \frac{y^2}{r^2} \tag{16}$$

$$\sigma_{yy}^{(1)} = -\sigma_{yy}^{(2)} = \frac{2\omega}{\pi} M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2}(\pi-\theta+ \frac{xy}{r^2}) \tag{17}$$

The boundary conditions satisfied on the interface by (8-11) and (15-17) are

$$u_y^{(1)}(x,0) = u_y^{(2)}(x,0) \tag{18}$$

$$\sigma_{xy}^{(1)}(x,0) = \sigma_{xy}^{(2)}(x,0) = 0 \tag{19}$$

$$\sigma_{yy}^{(1)}(x,0) = \sigma_{yy}^{(2)}(x,0) \tag{20}$$

It may be noted that a term corresponding to rigid body rotation must be added to either (10) or (11) in order to enforce condition (20) in a strict sense.

Of particular interest in formulating the exterior thermoelastic contact problem are the heat flux and normal tractions transmitted by the interface. Shifting the heat vortex from the origin to an arbitrary point $(\xi,0)$ on the interface, the results are

$$q_y(x,0) = - \frac{\omega}{\pi} \frac{k_1k_2}{k_1+k_2} \frac{1}{x-\xi} \tag{21}$$

$$\sigma_{yy}(x,0) = 2\omega M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2} H(x-\xi) \tag{22}$$

Suppose that heat vortices with the density $\Omega(x)$ are distributed over the interval (a,b) on the interface. The resulting temperature discontinuity then is

$$\begin{aligned} \tau(x) = T_2(x,0) - T_1(x,0) &= 0, && x<a \\ &= \int_a^x \Omega(\xi)d\xi, && a<x<b \\ &= \int_a^b \Omega(\xi)d\xi, && b<x \end{aligned} \tag{23}$$

The corresponding heat flux and normal tractions transmitted by the interface are

$$q_y(x,0) = -\frac{1}{\pi}\frac{k_1k_2}{k_1+k_2}\int_a^b \frac{\Omega(\xi)d\xi}{x-\xi} \tag{24}$$

$$\begin{aligned} \sigma_{yy}(x,0) &= 0, && x<a \\ &= 2M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2}\int_a^x \Omega(\xi)d\xi, && a<x<b \\ &= 2M(\delta_1-\delta_2)\frac{k_1k_2}{k_1+k_2}\int_a^b \Omega(\xi)d\xi, && b<x \end{aligned} \tag{25}$$

It is seen from (23) and (25) that $\sigma_{yy}(x,0)$ is proportional to $\tau(x)$. From (23) it also follows that

$$\Omega(x) = \frac{d\tau(x)}{dx} \tag{26}$$

If the interval (a,b) is a separation zone that is bounded by two contact zones offering no resistance to heat transfer from one body to the other, $\tau(x) = 0$ outside the separation interval, and from (25)

$$\int_a^b \Omega(\xi)d\xi = 0 \tag{27}$$

Edge Dislocation at a Slipping Interface

Consider the elastic fields

$$u_x^{(1)} = \frac{b_y M}{2\pi\mu_1}\{(\kappa_1-1)\log r - \frac{2x^2}{r^2}\} \tag{28}$$

$$u_x^{(2)} = \frac{b_y M}{2\pi\mu_2}\{(\kappa_2-1)\log r - \frac{2x^2}{r^2}\} \tag{29}$$

$$u_y^{(1)} = \frac{b_y M}{2\pi\mu_1}\{(\kappa_1+1)\theta - \frac{2xy}{r^2}\} - \frac{b_y M}{2\mu_1}(\kappa_1+1) \tag{30}$$

$$u_y^{(2)} = \frac{b_y M}{2\pi\mu_2}\{(\kappa_2+1)\theta - \frac{2xy}{r^2}\} - \frac{b_y M}{2\mu_2}(\kappa_2+1) \tag{31}$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = -\frac{2b_y M}{\pi}\frac{x}{r^2}(1 - \frac{2x^2}{r^2}) \tag{32}$$

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} = -\frac{2b_y M}{\pi}\frac{y}{r^2}(1 - \frac{2x^2}{r^2}) \tag{33}$$

$$\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} = \frac{2b_y M}{\pi}\frac{x}{r^2}(1 + \frac{2y^2}{r^2}) \tag{34}$$

where M is defined by (14), and $0 \leq \theta \leq 2\pi$. These fields satisfy the equations of equilibrium and the following boundary conditions at the interface:

$$u_y^{(1)}(x,0) + b_y H(x) = u_y^{(2)}(x,0) \tag{35}$$

$$\sigma_{xy}^{(1)}(x,0) = \sigma_{xy}^{(2)}(x,0) = 0 \tag{36}$$

$$\sigma_{yy}^{(1)}(x,0) = \sigma_{yy}^{(2)}(x,0) \tag{37}$$

It is seen from (35-37) that the given displacements and stresses correspond to an edge dislocation which has the Burgers vector

$(0,b_y)$ and is located at the freely slipping interface.

Shifting the dislocation from the origin to the point $(\xi,0)$, the gap between the solids and the normal tractions transmitted by the interface are

$$g(x) = u_x^{(1)}(x,0) - u_y^{(2)}(x,0) = -b_y H(x-\xi) \tag{38}$$

$$\sigma_{yy}(x,0) = \frac{2b_y M}{\pi}\frac{1}{x-\xi} \tag{39}$$

If edge dislocations are distributed on the interval (a,b), with $B_y(x)$ being the density,

$$\begin{aligned} g(x) &= 0, && x<a \\ &= -\int_a^x B_y(\xi)d\xi, && a<x<b \\ &= -\int_a^b B_y(\xi)d\xi, && b<x \end{aligned} \tag{40}$$

$$\sigma_{yy}(x,0) = \frac{2M}{\pi}\int_a^b \frac{B_y(\xi)d\xi}{x-\xi} \tag{41}$$

It follows from (40) that

$$B_y(x) = -\frac{dg(x)}{dx} \tag{42}$$

and also that, if the gap is to close at $x = b$, we must have

$$\int_a^b B_y(\xi)d\xi = 0 \tag{43}$$

Conclusion

The given expressions, in particular (24), (25) and (41), allow one to write at sight the governing integral equations for exterior contact problems. The unknown functions in the integral equations are the densities of heat vortices and the edge dislocations. However, the intervals of integration are generally unknown, and the equations are subject to inequality conditions expressing the requirements that the gap may not be negative and that the interface tractions may not be tensile. Once the two densities are determined, any field quantity of interest can be found by integration using the given expressions.

Acknowledgement

One of the authors (M.C.) is pleased to acknowledge the support by the National Science Foundation under the grant ENG77-25032.