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# Analytic Functions Satisfying Hölder Conditions on the Boundary

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## 1. INTRODUCTION

Let $G$ be an open set in the finite $z$ plane and suppose that $f(z)$ is regular in $G$ and continuous on its closure $\bar{G}$. We denote by $\partial G$ the frontier of $G$ and suppose that $\partial G$ has at least two finite points. We then prove the following.

THEOREM 1. *Suppose, with the above assumptions, that there exist constants* $\alpha$, $0 \leqslant \alpha \leqslant 1$, *and* $M > 0$ *such that*

$$|f(z_1) - f(z_2)| \leqslant M |z_1 - z_2|^\alpha \tag{1.1}$$

*whenever* $z_1$, $z_2$ *belong to* $\partial G$ *and, further, that*

$$f(z) = o(|z|) \tag{1.2}$$

*if* $\alpha < 1$ *and*

$$f(z) = o(|z|^2) \tag{1.3}$$

*if* $\alpha = 1$, as $z \to \infty$ *in any unbounded component of* $\bar{G}$. *Then* (1.1) *holds for every pair of points* $z_1$, $z_2$ *in* $\bar{G}$.

*Further, if* (1.1) *holds for a fixed* $z_1 \in \partial G$ *and a variable* $z_2 \in \partial G$, *then* (1.1) *also holds for this* $z_1$ *and any* $z_2 \in \bar{G}$.

The functions $z$ and $z^2$, respectively, show that $o$ cannot be replaced by 0 in (1.2) and (1.3), when $G$ is $|z| > 1$.

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Hardy and Littlewood proved in [4, p. 427] that if $G$ is the unit disk, then (1.1) on the boundary implies the same on the closed disk if $M$ is replaced by $CM$ for some $C > 1$. Walsh and Sewell [9, Theorem 1.2.7, p. 17; see also 11] extended the result to Jordan domains with $C = 1$. Pointwise results of the same kind were obtained by Warschawski [12]. Two other proofs for $C = 1$ $(0 < \alpha \leqslant 1)$ in the unit disk were given by Rubel *et al.* [8, p. 27], based on $H^p$-theory and the theory of two complex variables. Tamrazov [10] proved this result for bounded functions defined on an open set $G$ such that $\partial G$ has positive capacity and either $\bar{C} \backslash G$ is connected or for every $z_0 \in \partial G$,

$$\liminf_{r \to 0} r^{-1} \operatorname{cap}(\{z \,|\, |z - z_0| \leqslant r\} \backslash G) > 0,$$

where $\operatorname{cap} A$ denotes the capacity of the set $A$.

If $\omega(\delta)(\tilde{\omega}(\delta))$ denotes the modulus of continuity of $f$ on $\bar{G}$ (on $\partial G$), results of the form $\tilde{\omega}(\delta) \leqslant \phi(\delta) \Rightarrow \omega(\delta) \leqslant C\phi(\delta)$ for an absolute constant $C$ have also been obtained for functions $\phi(\delta)$ other than $\phi(\delta) = \delta^\alpha$, $\alpha > 0$. Assuming that $G$ is simply connected and that the conformal mappings from $G$ to $D$ and $D$ to $G$, where $D$ is the unit disk, satisfy Hölder conditions on the boundaries, M. B. Gagua obtained this result for $\phi(\delta) = |\log \delta|^{-p}$, $p > 0$ [2, 3]. Similar, but less general, results were proved earlier by Magnaradze [7]. Finally, Tamrazov proved in [10] that $\tilde{\omega} \leqslant \phi$ implies $\omega \leqslant C\phi$ $(C = 108)$ for more general functions $\phi$ in open sets satisfying certain capacity conditions on the boundary.

## 2. A Preliminary Result

To prove Theorem 1 we need the following generalisation of a result of Fuchs [1, Theorem 1].

THEOREM 2. *Suppose that $u(z)$ is subharmonic and positive in an open set $G$, whose complement contains at least one finite point, and that*

$$\overline{\lim}\, u(z) \leqslant 0 \tag{2.1}$$

*as $z$ approaches any boundary point of $G$ from inside $G$ except the boundary point $\zeta = \infty$. Write*

$$B(r) = \sup_{G \cap (|z| = r)} u(z), \tag{2.2}$$

$$I(r) = \frac{1}{2\pi r} \int_{G \cap (|z| = r)} u(z) |dz|. \tag{2.3}$$

*Then there exists* $\beta$, *such that* $0 < \beta \leqslant \infty$ *and*

$$\lim_{r\to\infty} \frac{B(r)}{\log r} = \lim_{r\to\infty} \frac{I(r)}{\log r} = \beta. \tag{2.4}$$

*Suppose further that* $\beta < \infty$, *and that* $u(z)$ *is harmonic in* $G$ *and possesses there a local conjugate* $v$, *such that for some* $\alpha$, *where* $0 < \alpha \leqslant 1$, *and some positive* $R_0$

$$F(z) = z^{1-\alpha} \exp(u + iv) \tag{2.5}$$

*remains one valued in* $G \cap (|z| > R_0)$. *Then* $F(z)$ *has a pole of order* $p$, *say, at* $\zeta = \infty$, $\zeta$ *is an isolated boundary point of* $G$ *and* $\beta = \alpha + p - 1$.

The case $\alpha = 1$ of this result is a slight extension of Fuchs' Theorem. To prove Theorem 2, we define $u(z) = 0$ in the complement of $G$ and deduce that $u(z)$ is subharmonic and not constant in the plane. It follows from standard convexity theorems [5, p. 67] that the limits

$$\beta_1 = \lim_{r\to\infty} \frac{B(r)}{\log r} \qquad \text{and} \qquad \beta_2 = \lim_{r\to\infty} \frac{I(r)}{\log r}$$

exist and $0 \leqslant \beta_2 \leqslant \beta_1$ clearly. Also $\beta_2 > 0$ unless $u$ is harmonic in the plane, and this is impossible since $u$ attains its minimum 0 at a finite boundary point of $G$ and $u$ is not constant. Again we have, for $0 < r < R$ [5, p. 127],

$$B(r) \leqslant \frac{R+r}{R-r} I(R)$$

so that for each fixed $K > 1$ we obtain

$$\beta_1 = \lim_{r\to\infty} \frac{B(r)}{\log r} \leqslant \frac{K+1}{K-1} \lim_{r\to\infty} \frac{I(Kr)}{\log(Kr)} = \frac{K+1}{K-1}\beta_2,$$

i.e., $\beta_1 \leqslant \beta_2$. Thus $\beta_1 = \beta_2 = \beta$ and this proves (2.4).

Next, if $\beta < \infty$, $u(z)$ has order zero and is finite at the origin so that [5, p. 155] $u(z)$ has the representation

$$u(z) = u(0) + \int \log|1 - z/\zeta| \, d\mu(\zeta)$$

in terms of the Riesz mass $\mu$ of $u(z)$. Also if $n(r)$ denotes the total mass in $|z| < r$ then Jensen's formula [5, p. 127] shows that

$$I(r) = \int_0^r n(t)\, dt/t + u(0) \tag{2.6}$$

so that

$$\beta = \lim_{r \to \infty} n(r) \tag{2.7}$$

is the Riesz mass of the whole plane. Also since $u(z)$ has order zero it follows from Heins' extension of Wiman's theorem [6] that

$$A(r) = \inf_{|z| = r} u(z)$$

is unbounded as $r \to \infty$. In particular $G$ contains a sequence of circles

$$|z| = r_\nu, \quad \text{where } R_0 < r_1 < r_2 < ,..., r_\nu \to \infty \text{ as } \nu \to \infty.$$

By hypothesis these circles belong to $G$, since $u = 0$ outside $G$ and so $G$ has only one unbounded component. In view of the maximum principle and (2.1) $G$ cannot have any bounded components, so that $G$ is connected. Next, (2.6) shows that for $r = r_\nu$,

$$\begin{aligned} n(r) = r \frac{d}{dr} I(r) &= \frac{1}{2\pi} \int_0^{2\pi} r \frac{\partial}{\partial r} u(re^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \theta} v(re^{i\theta})\, d\theta \\ &= n_\nu + \alpha, \end{aligned}$$

where $n_\nu$ is an integer, since $F(z)$, given by (2.5), is one valued.

Thus since $n(r)$ is increasing and bounded, $n_\nu$ is constant for large $\nu$ and so $n(r)$ is constant and equal to $\beta$ for $r > R_1$, say. Thus there is no Riesz mass in $R_1 < |z| < \infty$ and so $u(z)$ is harmonic there. Hence $F(z)$ has an isolated singularity at $\infty$ and since when $|z| = r$

$$|F(z)| \geqslant r^{1-\alpha},$$

then $F(z)$ has a pole at $\infty$ if $\alpha < 1$. If $\alpha = 1$ and $F(z)$ is finite at $\infty$, then $u(z)$ is bounded as $z \to \infty$ and so $\beta = 0$ in (2.4), which gives a contradiction. Thus $F(\infty) = \infty$ in all cases. If $p$ is the order of the pole of $F(z)$ at $\infty$ then

$$u(z) = (\alpha + p - 1) \log |z| + 0(1) \qquad \text{as} \quad z \to \infty,$$

so that $\beta = \alpha + p - 1$. In particular,

$$u(z) \to \infty \qquad \text{as} \quad z \to \infty,$$

so that the complement of $G$ in the open plane is bounded. This completes the proof of Theorem 2.

We note that Theorem 2 has a converse. If $u$ is harmonic and positive near $\infty$ then there exists $\alpha$ such that $0 < \alpha \leqslant 1$ and $F(z)$ given by (2.5) has a pole at $\infty$.

We state for future reference a form of Theorem 2 when the exceptional boundary point $\zeta$ is finite.

THEOREM 3. *Suppose that $u(z)$ is harmonic and positive in an open set $G$ in the closed plane, whose complement contains at least two points and that $u(z)$ satisfies* (2.1) *as $z$ approaches any boundary point of $G$ excluding one finite boundary point $\zeta$. Suppose further that $u$ possesses a local conjugate $v$, such that*

$$F(z) = (z - \zeta)^{\alpha - 1} \exp(u + iv)$$

*remains regular, i.e., one valued in $G \cap (|z - \zeta| < \delta)$, where $\delta > 0$ and $0 < \alpha \leqslant 1$. Then either*

$$\overline{\lim}\, |z - \zeta|^m \, |F(z)| = \infty \tag{2.8}$$

*as $z \to \zeta$ for every positive integer $m$, or else $F(z)$ has a pole at $\zeta$ and $\zeta$ is an isolated boundary point of $G$.*

We apply Theorem 2 to $U(z) = u(\zeta + z^{-1})$ and deduce Theorem 3.

## 3. PROOF OF THEOREM 1

Suppose that $f(z)$ satisfies the hypotheses of Theorem 1. We write for any $z_1 \in \partial G$

$$u(z) = \log |f(z) - f(z_1)| - \alpha \log |z - z_1| - \log M \tag{3.1}$$

and proceed to show that

$$u(z) \leqslant 0 \text{ in } G. \tag{3.2}$$

Suppose first that $G$ is bounded. If $\alpha = 0$ it follows from (1.1) that

$$\overline{\lim}\, u(z) \leqslant 0 \tag{3.3}$$

as $z$ approaches any boundary point $z_2$ of $G$ other than $z_1$, and since $f(z)$ is continuous at $z_1$, (3.3) holds also as $z$ approaches $z_1$. Thus in this case (3.2) follows at once from the maximum principle, since $u(z)$ is subharmonic in $G$.

Assume next that $\alpha > 0$ and that (3.2) is false. Let $G_0$ be the subset of $G$ in which $u(z) > 0$ and define

$$u_0(z) = u(z), \qquad z \in G_0, \tag{3.4}$$

$$u_0(z) = 0, \qquad \text{elsewhere.} \tag{3.5}$$

Then it follows from (3.3) that $u_0(z)$ is subharmonic in the open plane, except possibly at $z_1$, and also at $\infty$, since $G$ is bounded. Also $u_0(z)$ is not constant. Thus $u_0(z)$ satisfies the hypotheses for $u(z)$ of Theorem 3, with $\zeta = z_1$, $G = G_0$ and

$$F(z) = (f(z) - f(z_1))/M(z - z_1).$$

We deduce that $F(z)$ has a pole at $z_1$, which contradicts our assumption that $f(z)$ is continuous at $z_1$ as a function in $\bar{G}$. Thus (3.2) holds in all cases if $G$ is bounded.

Suppose next that $G$ is unbounded. We first apply the result we have just proved with the domain

$$G_1 = G \cap (|z - z_1| < 1)$$

instead of $G$. Then $u(z)$ is bounded above by some positive constant $M'$ on $G \cap (|z - z_1| = 1)$, since $f(z)$ is continuous in $\bar{G}$ and so in $\bar{G}_1$. Thus the argument we have just given when applied to $u(z) - M'$ in $G_1$ shows that

$$u(z) \leqslant M' \quad \text{in} \quad G_1. \tag{3.6}$$

Suppose now again that (3.2) is false. Let $G_0$ be the subset of $G$ where $u(z) > 0$ and define $u_0(z)$ by (3.4) and (3.5). Then $u_0(z)$ is subharmonic in the closed plane except possibly at $z = z_1$ and $z = \infty$. However, by (3.6) $u_0(z)$ is bounded above near $z_1$. It now follows [5, p. 237] that $u_0(z)$ can be extended as a subharmonic function to the whole open plane. We now apply Theorem 2. If $0 \leqslant \alpha < 1$ we deduce from Theorem 2, applied with $1 - \alpha$ instead of $\alpha$, that $f(z) - f(z_1)$ has a pole at $\infty$, which contradicts (1.2). If $\alpha = 1$ we deduce from Theorem 2, applied with $\alpha = 1$, that $(f(z) - f(z_1))/(z - z_1)$ has a pole at $\infty$, which contradicts (1.3). Thus (3.2) holds in all cases. This proves the last sentence of Theorem 1.

We now take a fixed point $z_2 \in G$ and consider

$$u(z) = \log|f(z) - f(z_2)| - \alpha \log|z - z_2| - \log M.$$

Then $u(z)$ is subharmonic in $G$ if we define

$$u(z_2) = -\infty \qquad \text{when} \quad \alpha < 1,$$

$$u(z_2) = \log|f'(z_2)/M| \qquad \text{when} \quad \alpha = 1.$$

Also by what we have just proved, if $f(z)$ satisfies the hypotheses of Theorem 1, then (3.3) holds as $z$ approaches any finite boundary point of $G$. If (3.2) is false we again define $u_0(z)$ by (3.4) and (3.5) and apply Theorem 2. Once again (1.2) or (1.3) leads to a contradiction so that (3.2) holds in $\bar{G}$. Thus (1.1) is proved in all cases.

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