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On the Connection between Critical Point Theory and Leray–Schauder Degree*

E. H. ROTHE

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Submitted by C. L. Dolph

1. INTRODUCTION

Let E be a real Hilbert space with elements $x, y, \ldots$, with scalar product $\langle x, y \rangle$, with norm $\|x\| = \langle x, x \rangle^{1/2}$, and zero point θ. Let f be a map of a bounded open neighborhood Ω_1 of θ into the reals, and suppose that $g = \operatorname{grad} f$ exists in Ω_1. A point $x \in \Omega_1$ is called critical for f if it satisfies the equation

$$g(x) = \theta. \tag{1.1}$$

We assume that θ is an isolated critical point, i.e., there exists an open neighborhood $\Omega \subset \Omega_1$ of θ such that θ is the only root of (1.1) in the closure $\bar{\Omega}$ of Ω. If we assume that g is a Leray–Schauder (L–S) map, i.e.,

$$g(x) = x - G(x), \tag{1.2}$$

where G is completely continuous, then the Leray–Schauder degree $d(g, \Omega, \theta)$ is defined and independent of the specific choice of an Ω having the above properties. For such Ω we may define

$$\lambda(\theta; g) = d(g, \Omega, \theta). \tag{1.3}$$

λ is called the Leray–Schauder index of θ as the root of (1.1).

It is known that under certain additional assumptions the Morse numbers $M_0, M_1, M_2, \ldots$ of the critical point θ are zero except for a finite number and that

$$\lambda(\theta; g) = \sum_{i=0} (-1)^i M_i. \tag{1.4}$$

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(see [5, Section 8]). The proof consists of establishing (1.4) first for the finite-dimensional case and then using a passage to the limit in the dimension.

The purpose of the present paper is to establish (1.4) directly in Hilbert space for the special case that θ is a nondegenerate critical point for f. This is possible by using the "intrinsic" definition of the Leray–Schauder degree which—under proper assumptions—does not presuppose the finite-dimensional degree theory [2, 6].

In Section 2 the definition and elementary properties of a nondegenerate critical point in Hilbert space are recalled. Section 3 contains relevant background material of the intrinsic degree theory.

Finally, in Section 4 it is shown that relation (1.4) is an immediate consequence of the definitions and assertions contained in Sections 2 and 3.

2. Definition and Properties of a Nondegenerate Critical Point

The isolated critical point θ of f is called nondegenerate if the following two conditions (A) and (B) are satisfied.

(A) $f \in C''(\bar{\Omega})$, i.e., the first and second differentials $Df(x; h)$ and $D^2f(x; h, k)$ are defined and continuous for all x in some open set containing $\bar{\Omega}$. (For the definition of D and D^2 see e.g., [1, Chapter VIII].)

We note that (A) implies that $D^2(\theta; h, k)$ is a bounded symmetric bilinear form in h and k (see [1, Section 1.c]).

(B) The bounded symmetric bilinear form $D^2f(\theta; h, k)$ is nondegenerate, i.e., the relation

$$D^2f(\theta; h_0; k) = 0 \qquad \text{for all} \quad k \in E \tag{2.1}$$

implies that $h_0 = \theta$.

We recall that the index of a bounded nondegenerate quadratic form $q(h, h)$ is defined as the maximal dimension of linear subspaces L of E for which $q(h, h) < 0$ for all $h \in L - \theta$.

If the index N of the quadratic form $D^2f(\theta; h, h)$ is finite, then for $i = 0, 1, 2,...$

$$M_i = \delta_N^i \qquad (\delta_N^i \text{ is the Kronecker } \delta) \tag{2.2}$$

is called the ith Morse number of the nondegenerate critical point θ of f.

Now from our differentiability assumptions on f it follows easily that the differential $D(g; h)$ of $g = \operatorname{grad} f$ exists and that

$$D^2f(x; h, k) = \langle Dg(x; h), k \rangle, \qquad x \in \Omega. \tag{2.3}$$

(For a proof see [7, p. 363].) Since here the left member is symmetric in h and k, we see from (2.3) that for each $x \in \Omega$, the linear operator $Dg(x; h)$ is symmetric. Moreover, since g is an L–S map (cf. Eq. (1.2)) it follows by a lemma of Krasnoselskii [3, p. 135] that, for $x \in \Omega$, the linear operator $Dg(x; h)$ is L–S. In particular the operator

$$l(h) = Dg(\theta, h) = h - L(h) \tag{2.4}$$

is L–S and symmetric. Thus $L(h)$ is completely continuous and symmetric.

If now $e_1, e_2, ...$ is a full orthonormal system of eigenelements of L with corresponding nonzero eigenvalues $\lambda_1, \lambda_2, ...,$ then by (2.3), (2.4), and the classical expansion theorems (see e.g., [4, pp. 231, 232])

$$D^2 f(\theta; h, h) = \langle l(h), h \rangle = \sum_{i=1} \langle he_i \rangle^2 (1 - \lambda_i) + \|h_0\|^2, \tag{2.5}$$

where $h_0 = h - \sum_i \langle he_i \rangle e_i$ (cf. [7, p. 392]). Now since L is completely continuous, the λ_i converge to θ if there are infinitely many. Thus in any case there are at most a finite number N of λ_i satisfying

$$(1 - \lambda_i) \leqslant 0. \tag{2.6}$$

If there are such λ_i we may assume that they are $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_N$. But since for $x = \theta$ the left member of (2.3) is a nondegenerate bilinear form, it follows from (2.3) and (2.4) that the L–S operator l is nonsingular, i.e., $\lambda = 1$ is not an eigenvalue of L. Thus the equality cannot hold in (2.6). This shows that N is the index of quadratic form (2.5).

Now if $\mu_1 > \mu_2 > \cdots > \mu_r$ are distinct among $\lambda_1 \lambda_2, ..., \lambda_N$ and if $m_\rho = m(\mu_\rho)$ is the multiplicity of the eigenvalue μ_ρ, then

$$\text{index of } (2.5) = N = \sum_{\rho=1}^{r} m_\rho \tag{2.7}$$

since, by definition, $m(\mu_\rho)$ is the dimension of the eigenspace belonging to μ_ρ.

If there are no eigenvalues λ_i satisfying (2.6), then it is clear from (2.5) that the left part of (2.7) holds with $N = 0$.

3. Background Material from the Intrinsic Leray–Schauder Theory

Let

$$l(h) = h - L(h) \tag{3.1}$$

be an arbitrary linear nonsingular (not necessarily symmetric) L–S map $E \to E$. We first recall the definition of the index $j(l)$ of l. As in Section 2 we see from the nonsingularity of l that 1 is not an eigenvalue of L and from the complete continuity of the operator L that at most a finite number of its eigenvalues λ_i satisfy (2.6). (Since E is a real Hilbert space, by eigenvalue we always mean "real eigenvalue.")

If L has eigenvalues greater than 1 we denote them by $\mu_1 > \mu_2 > \cdots > \mu_r$ and by $v_\rho = v(\mu_\rho)$ the generalized multiplicity of μ_ρ, i.e., the dimension of

$$E_\rho = \{x \in E \mid (\mu_\rho I - L)^n x = \theta\}$$

for some $n = 1, 2, \ldots,$ where I denotes the identiy map on E. It is well known that v_ρ is finite. We define

$$j(l) = (-1)^{\sum_{\rho=1}^{r} v_\rho}. \tag{3.2}$$

If l has no eigenvalues greater than 1, we set $j(l) = 1$. (See [6, Definition 6.2] and [2, p. 383].)

Now let Ω be a bounded open subset of E and let $\phi \in C'(\bar{\Omega})$ be an L–S map $\bar{\Omega} \to E$. Let y_0 be a point of E and suppose that $x_0 \subset \Omega$ is the only solution in $\bar{\Omega}$ of the equation $\phi(x) = y_0$. Suppose, moreover, that $D\phi(x_0; h)$ is nonsingular. Then the intrinsic definition of the Leray–Schauder degree $d(\phi, \Omega, y_0)$ is given by

$$d(\phi, \Omega, y_0) = j(D\phi(x_0; h)). \tag{3.3}$$

4. Proof of Assertion (1.4)

Let f, g, $\mu_1, \mu_2, \ldots, \mu_r$ be as in Section 2, and let the ϕ, x_i, y_0, and l of Section 3 be given by $\phi = g$, $x_0 = y_0 = \theta$, and (2.4). Then $D\phi(\theta; h) = l$ is nonsingular and (3.3) holds. Thus by (1.3), (3.3), and (3.2)

$$\lambda(\theta; g) = d(g, \Omega, \theta) = j(l) = (-1)^{\sum_{\rho=1}^{r} v_\rho}. \tag{4.1}$$

But the operator L in (2.4) is not only completely continuous but also symmetric, and it is well known that this implies that the generalized multiplicity v_ρ of μ_ρ equals the multiplicity m_ρ of that eigenvalue. Therefore by (4.1) and (2.7)

$$\lambda(\theta; g) = (-1)^{\nu}. \tag{4.2}$$

But by definition (2.2) of the Morse numbers M_i our assertion (1.4) is equivalent to relation (4.2).

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