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TOPOLOGICAL, AFFINE AND ISOMETRIC ACTIONS ON FLAT RIEMANNIAN MANIFOLDS II

Kyung B. LEE* and Frank RAYMOND**

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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Explicit examples of finite subgroups of the group of homotopy classes of self-homotopy equivalences of some flat Riemannian manifolds which cannot be lifted to effective actions are given. It is also shown that no finite subgroups of the kernel of $\pi_0(\text{Homeo}(M)) \to \text{Out}\ \pi_1(M)$ can be lifted back to Homeo(M), for a large class of flat manifolds M. Some results of an earlier paper by the authors are refined and related to recent work of R. Schoen and S.T. Yau.

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outer automorphism group	homotopy classes of self-homotopy equivalences
crystallographic groups	obstruction to a group extension
abstract kernel	geometric extension

In this note we make certain observations and give some examples which extend and complement the results of our earlier paper [5] on actions of finite groups on flat Riemannian manifolds. We assume familiarity with [5] and retain the notations and definitions given there.

Section 1 gives some explicit examples of finite subgroups of the group of homotopy classes of self homotopy equivalences of a flat manifold which cannot be lifted to effective actions. Section 2 examines a related paper of R. Schoen and S.T. Yau [8]. In Section 3, we prove that Theorem 3 of [5] solves also the lifting problem of an abstract kernel to the group of isometries. In Section 4 we partially prove that no finite subgroup of kernel of $\pi_0\mathcal{H}(M) \to \text{Out}\ \pi_1 M$ can be lifted back to $\mathcal{H}(M)$. Section 5 is a refinement of Corollary 6 of [5] where it was shown that two affine actions are affinely equivalent if and only if the corresponding lifting sequences are isomorphic. We also show that there is a one-to-one correspondence between the strong conjugacy classes of all affine realizations of an abstract kernel $\phi : G \to \text{Out}\ \pi$ and $H^2(G; \mathfrak{z}(\pi))$.

* Present address: Dept. of Mathematics, Purdue University, W. Lafayette, IN 47907, USA.
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1. Examples

In the first part of this section we give two examples of finite subgroups of the group of homotopy classes of self homotopy equivalence of a flat manifold which do not lift to effective topological actions. In both cases, the underlying manifold M will be the flat manifold of dimension 3 with holonomy group $\Phi \cong \mathbb{Z}_2$. We then describe, for each integer $n \geq 2$, a closed flat Riemannian manifold M_n of dimension n^2+n-2 with holonomy cyclic of order n. For each of these manifolds a cyclic subgroup of order n in Out $\pi_1(M)$ is found which cannot be lifted to a topological action on M.

1.1. Example. There exists an isometry D such that D^2 is isotopic (through isometries) to the identity, but D is not homotopic to any homeomorphism H with H^2 = identity.

This follows the method of [7]. Here is a description of M. Take

$$\omega : T^2 \to T^2 \quad \text{given by} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and form M, the mapping torus of ω ($= S^1 \times_{(\omega)} T^2$). Denote points of M by $\langle r, z_1, z_2\rangle$, etc. Then $\langle r, z_1, z_2\rangle = \langle r-1, z_1^{-1}, z_2^{-1}\rangle$ for $r \in \mathbb{R}$, $(z_1, z_2) \in T^2$. Define D using the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

That is, $D\langle r, z_1, z_2\rangle = \langle -r, z_2^{-1}, z_1\rangle$. Note it is well defined and $D\langle 0, 1, 1\rangle = \langle 0, 1, 1\rangle$. Now $D^2\langle r, z_1, z_2\rangle = \langle r, z_1^{-1}, z_2^{-1}\rangle$. If we define $D_\vartheta^2\langle r, z_1, z_2\rangle = \langle r-\vartheta, z_1^{-1}, z_2^{-1}\rangle$, then $D_0^2 = D^2$ and $D_1^2 = \mathrm{id}$, which shows D^2 is isotopic to the identity. Clearly D^2 restricted to $T^2 = \{\langle 0, z_1, z_2\rangle \in M\}$ is ω. Note that $\pi_1(T^2, \langle 0, 1, 1\rangle)$ is a characteristic subgroup of $\pi = \pi_1(M)$ since it is the kernel of $(\pi \to H_1(M) \otimes \mathbb{Q})$.

If there exists H homotopic to D so that $H^2 = \mathrm{id}$, then there exists an extension diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & (H) \cong \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \mu & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi) & \longrightarrow & \mathrm{Aut}(\pi) & \longrightarrow & \mathrm{Out}(\pi) & \longrightarrow & 1
\end{array}$$

Choose $e \in E$ such that conjugation by e, μ_e, is precisely $D_* \in \mathrm{Aut}(\pi)$. Let $\eta = \{\langle s, 1, 1\rangle \mid 0 \leq s \leq 1\}$ be the element of π representing the generator of the section of the fibering $M \to S^1$, $\langle r, z_1, z_2\rangle \mapsto r$. Then $\mu_\eta = \mu_{e^2}$ so that $e^2 = c \cdot \eta$ for some c

in the center of π, $\mathfrak{z}(\pi)$. Note that $\mathfrak{z}(\pi)$ is generated by η^2. As $D_*(\eta) = \eta^{-1}$, $D_*(e^2) = e^{-2}$. On the other hand, $D_*(e^2) = \mu_e(e^2) = e^2$. Therefore we see that $e^4 = 1$. This is a contradiction because $e^2 \in \pi$, which is torsion-free.

Consequently, no such H can exist, since no such E can exist. Compare these examples with Example 2.8 of Heil–Tollefson, Deforming Homotopy Involutions, Topology 17 (1978) 349–365. Obviously, this example is just a prototype of many such examples on flat manifolds all working the same way: One needs only to choose a flat manifold T with a period 4 automorphism γ with non-empty fixed points and whose powers $\not\equiv 0$ (4) are not homotopic to the identity. Use $\omega = \gamma^2$ to make the mapping torus, M. On this new flat manifold one can define the analogue of D. Note, incidentally, that D^4 is the identity [5, Theorem 5]. This would *correspond to enlarging the abstract kernel* $(\mathbb{Z}_2, \pi, \phi)$, where D_* generates $\mathbb{Z}_2$, to $(\mathbb{Z}_4, \pi, \bar{\phi})$, where $\bar{\phi}: \mathbb{Z}_4 \to \mathbb{Z}_2 \xrightarrow{\phi} \operatorname{Out} \pi$. To obtain the conclusion of the previous argument, one needs to know how D_* acts on $\mathfrak{z}(\pi)$. The simplest way to guarantee the non-existence of an extension is to have $\mathfrak{z}(\pi_1(R))^{\omega} = 1$.

1.2. Example. The following is another example on M. We will show that a certain abstract kernel fails to admit an extension by showing directly that its obstruction class does not vanish. Recall

$$\pi = \langle \alpha, t_1, t_2, t_3 \mid \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}, \alpha t_3 \alpha^{-1} = t_3^{-1}, [t_i, t_j] = 1 \rangle.$$

Consider the following automorphisms of π

$$a: \quad \alpha \mapsto \alpha, \quad t_2 \mapsto t_3, \quad t_3 \mapsto t_2^{-1},$$

$$b: \quad \alpha \mapsto \alpha, \quad t_2 \mapsto t_3, \quad t_3 \mapsto t_2,$$

$$c: \quad \alpha \mapsto \alpha, \quad t_2 \mapsto t_2, \quad t_3 \mapsto t_3^{-1}.$$

The following table shows these automorphisms together with the identity map 1 make a subgroup G of $\operatorname{Out}(\pi)$, which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (e.g. $ab = \mu c$).

	1	a	b	c
1	1	a	b	c
a	a	μ	μc	b
b	b	c	1	a
c	c	μb	μa	1

Here μ denotes the automorphism

$$\mu: \quad \alpha \mapsto \alpha, \quad t_2 \mapsto t_2^{-1}, \quad t_3 \mapsto t_3^{-1}$$

which is conjugation by any odd power of α. In fact, these are the *only* possible choices to denote μ by inner automorphisms since α^2 generates the infinite cyclic center. This fact will be useful later.

Let us denote the automorphism classes containing 1, a, b, c by $\bar{1}$, $\bar{a}$, $\bar{b}$, $\bar{c}$ respectively, so

$$G=\{\bar{1}, \bar{a}, \bar{b}, \bar{c}\} \overset{\phi}{\hookrightarrow} \operatorname{Out} \pi$$

is our abstract kernel.

At this point we recall how we get the obstruction class in $H^3(G, \mathfrak{z}(\pi))$ for an abstract kernel $\phi: G \to \operatorname{Out} \pi$ to have an extension. In each automorphism class $\phi(x)$ choose an automorphism $u(x)$, taking care that $u(1)=1$, once and for all. Since ϕ is a homomorphism into Out π, $u(x) \cdot u(y) \cdot u(xy)^{-1}$ is an inner automorphism. For each $x, y \in G$ choose an element $f(x, y)$ in π yielding this inner automorphism, in particular $f(x, 1)=0=f(1, y)$. (This choice of f is not unique, of course). For all $x, y, z \in G$,

$$k(x, y, z)=u(x)(f(y, z)) \cdot f(x, yz) \cdot f(xy, z)^{-1} \cdot f(x, y)^{-1}$$

defines a normalized 3-cochain of G with coefficients in $\mathfrak{z}(\pi)$, which yields the obstruction class $\operatorname{obs}(G, \pi, \phi)=[k] \in H^3(G, \mathfrak{z}(\pi))$. It is known that $\operatorname{obs}(G, \pi, \phi)=0$ if and only if k is *identically* 0 for a suitable choice of f.

Now we go back to our example. For $\bar{1}, \bar{a}, \bar{b}, \bar{c} \in G$, we choose $u(\bar{x})=x$. That is $u(\bar{a})=a$, etc. Since $u(\bar{c}) \cdot u(\bar{b}) \cdot u(\overline{cb})^{-1}=c \cdot b \cdot a^{-1}=\mu$, conjugation by $f(\bar{c}, \bar{b})$ should be μ. This means all the possible choices for $f(\bar{c}, \bar{b})$ are odd powers of α as remarked earlier. Also $u(\bar{b}) \cdot u(\bar{c}) \cdot u(\overline{bc})^{-1}=b \cdot c \cdot a^{-1}=1$ shows that conjugation by $f(\bar{b}, \bar{c})$ is the identity. So $f(\bar{b}, \bar{c})$ is a central element, and hence should be an even power of α.

Assume now that $G=\mathbb{Z}_2 \oplus \mathbb{Z}_2 \overset{\phi}{\hookrightarrow} \operatorname{Out} \pi$ admits an extension. Then for some choice of f, k is identically 0. An easy computation shows

$$1=k(\bar{a}, \bar{b}, \bar{a}) \cdot k(\bar{b}, \bar{c}, \bar{b})^{-1} \cdot k(\bar{c}, \bar{a}, \bar{c})^{-1}=f(\bar{b}, \bar{c})^2 \cdot f(\bar{c}, \bar{b})^{-2}.$$

[For any $\bar{x}$, $\bar{y} \in G$, conjugation by $f(\bar{x}, \bar{y})$ is either 1 or μ so that $f(\bar{x}, \bar{y})$ is some power of α. This implies all $f(\bar{x}, \bar{y})$ commute with each other.] Both $f(\bar{b}, \bar{c})$ and $f(\bar{c}, \bar{b})$ are powers of α. The identity above implies $f(\bar{b}, \bar{c})=f(\bar{c}, \bar{b})$. This is a contradiction because $f(\bar{b}, \bar{c})$ is an even power of α, while $f(\bar{c}, \bar{b})$ is an odd power and α is, of course, of infinite order.

Note that we can enlarge this abstract kernel to $\phi_H: \mathbb{Z}_4 \oplus \mathbb{Z}_2 \to \operatorname{Out} \pi$ so that this new abstract kernel has an admissible extension as in [5, Theorem 5]. Here $\mathbb{Z}_4$ is generated by a (with $a^4=\mu^2=1$) and $\mathbb{Z}_2$ by b. The kernel of $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is $\mathbb{Z}_2=\{1, \mu\}$. The group of affine diffeomorphisms realizing ϕ_H effectively is generated by

$$a(x)=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} x \quad \text{and} \quad \mu(x)=x+\begin{bmatrix}\frac{1}{2}\\ 0\\ 0\end{bmatrix}.$$

1.3. Examples. For each $n \geq 2$, there exists a closed orientable flat Riemannian manifold M_n such that a subgroup of $\mathrm{Out}(\pi_1 M_n)$, isomorphic to $\mathbb{Z}_n$, cannot be realized as a group of homeomorphisms of M_n.

The proof is by direct construction. First we construct a torsion free, discrete uniform subgroup Γ_n of $E(n^2+n-2)$, i.e., an (n^2+n-2)-dimensional Bieberbach group. This will define the desired manifold $M_n = \mathbb{R}^{n^2+n-2}/\Gamma_n$. Next, we find a certain element of $E(n^2+n-2)$ which normalizes Γ_n, and show that the outer automorphism induced by that element generates a subgroup of order n which has no geometric realization.

Let $n \geq 3$. B_n will denote the companion matrix of the polynomial $1+x+x^2+\cdots+x^{n-1}$ so that $B_n \in \mathrm{GL}(n-1, \mathbb{Z})$. For example,

$$B_3 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad \text{etc.}$$

This matrix has the following properties:

(i) $\det(B_n) = (-1)^{n-1}$,

(ii) $I + B_n + B_n^2 + \cdots + B_n^{n-1} = 0$,

and hence, in particular, $B_n^n = I$.

(iii) $(I - B_n) b_n = e_2$,

where

$$b_n = \frac{1}{n}\,{}^t[-1, n-2, \ldots, 3, 2, 1],$$

$$e_2 = {}^t[0, 1, 0, \ldots, 0]$$

are elements of $\mathbb{R}^{n-1}$.

Let Γ_n be the subgroup of $E(n-1) \times E(n^2-1) \subset E(n^2+n-2)$ generated by

$$t_1, t_2, \ldots, t_{n-1}; \qquad s_1, s_2, \ldots, s_{n^2-1}; \qquad \alpha$$

where $\{t_i \mid 1 \leq i \leq n-1\}$ is the standard unit translations of $\mathbb{R}^{n-1}$, $\{s_j \mid 1 \leq j \leq n^2-1\}$ the standard unit translations of $\mathbb{R}^{n^2-1}$ and, $\alpha = (A, a) \in E(n^2+n-2)$ with

$$A = \begin{bmatrix} I & 0 \\ 0 & (B_{n^2})^2 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} b_n \\ 0 \end{bmatrix} \in \mathbb{R}^{n-1} \times \mathbb{R}^{n^2-1}.$$

Let Λ be the subgroup of Γ_n generated by $\{t_1, \ldots, t_{n-1}, s_1, \ldots, s_{n^2-1}\}$. Clearly Λ is a free abelian group of rank n^2+n-2. Since $A \in \mathrm{GL}(n^2+n-2, \mathbb{Z})$, Λ is normal in Γ_n. Now (order of A) $= n$ implies that $\alpha^k \notin \Lambda$ for $0 < k < n$. We want to show $\alpha^n \in \Lambda$. Since

$$\alpha^n = (A^n, (I + A + \cdots + A^{n-1})a)$$

and

$$I + A + \cdots + A^{n-1} = \begin{bmatrix} nI & 0 \\ 0 & * \end{bmatrix}$$

we have

$$\alpha^n = (I, nb_n) = t_1^{-1} t_2^{n-2} \cdots t_{n-2}^2 t_{n-1}$$

which is in A, as we expected.

It is clear that A is maximal abelian in Γ_n and $[A : \Gamma_n] = n$. Now we claim that Γ_n is torsion free. Suppose not, so that

$$(t \cdot s \cdot \alpha^k)^r = 1$$

for some $0 < k < n$, $r > 0$, $t \in \langle t_1, \ldots, t_{n-1} \rangle$, $s \in \langle s_1, \ldots, s_{n^2-1} \rangle$. Then

$$1 = (t \cdot s \cdot \alpha^k)^r = t^r \cdot (s \cdot {}^{\alpha^k}s \cdots {}^{\alpha^{k(r-1)}}s) \cdot \alpha^{kr}$$

since t is central in Γ_n. Here we are using the notation ${}^x y = xyx^{-1}$. Since A is normal in Γ_n, the first two factors are in A, and hence $\alpha^{kr} \in A$. However, we have seen that n is the least positive integer for which $\alpha^n \in A$. This implies $kr = np$ for some integer p. We compare the number of t_1-factors of both sides in the last equation. If t has d t_1-factors, then

$$0 = rd + 0 + (-1)p,$$

since the middle term has no t_1 factor. Therefore r divides p, say, $p = qr$. Then $kr = qnr$ so that n divides k, which is impossible.

We have just proved that Γ_n is an (n^2+n-2)-dimensional Bieberbach group with holonomy group isomorphic to $\mathbb{Z}_n$ so that $1 \to A \to \Gamma_n \to \mathbb{Z}_n \to 1$ is exact. It is well known that the Γ_n action on $\mathbb{R}^{n^2+n-2}$ is free and properly distoncinuous, yielding a closed flat Riemannian manifold, $M_n = \mathbb{R}^{n^2+n-2}/\Gamma_n$. It is also easy to see that M_n is a nil-manifold. In fact, it is a T^{n^2-1} bundle over T^{n-1} with structure group $\mathbb{Z}_n$. At this point, the picture of the manifold M_n as well as the group Γ_n is clear. But let us write down a presentation of Γ_n:

$$[t_i, t_j] = [s_i, s_j] = [t_i, s_j] = 1, \qquad \alpha t_i \alpha^{-1} = t_i.$$

$\alpha s_j \alpha^{-1}$ = polynomial in s_i's with coefficients from the jth column of the matrix $B_{n^2}^n$

$$\alpha^n = t_1^{-1} t_2^{n-2} \cdots t_{n-2}^2 t_{n-1}.$$

In order to find a subgroup G of Out Γ_n we look at the element $\beta = (B, 0)$ of $\mathfrak{A} = A(n^2+n-2)$, the (n^2+n-2)-dimensional affine group, where

$$B = \begin{bmatrix} B_n & 0 \\ 0 & B_{n^2} \end{bmatrix}$$

we claim that $\beta \in N_{\mathfrak{A}}(\Gamma_n)$, the normalizer of Γ_n in $\mathfrak{A}$. It will be enough to show $\beta\alpha\beta^{-1} \in \Gamma_n$. Since $B^n = A$, $BAB^{-1} = A$. By (iii),

$$Ba = \begin{bmatrix} B_n b_n \\ 0 \end{bmatrix} = \begin{bmatrix} b_n - e_2 \\ 0 \end{bmatrix} = a - e_2$$

Therefore,

$$\begin{aligned} \beta\alpha\beta^{-1} &= (B, 0)(A, a)(B, 0)^{-1} \\ &= (BAB^{-1}, Ba) = (A, a - e_2) = t_2^{-1}\alpha \end{aligned}$$

so that $\beta \in N_{\mathfrak{A}}(\Gamma_n)$.

Look at the commutative diagram of exact rows and columns.

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathfrak{z}(\Gamma_n) & \longrightarrow & C_{\mathfrak{A}}(\Gamma_n) & \longrightarrow & A_0(M) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Gamma_n & \longrightarrow & N_{\mathfrak{A}}(\Gamma_n) & \xrightarrow{\nu} & A(M) & \longrightarrow & 1 \\ & & \downarrow & & \mu\downarrow & \searrow^{\mu'} & \downarrow\Psi & & \\ 1 & \longrightarrow & \operatorname{Inn}\Gamma_n & \longrightarrow & \operatorname{Aut}\Gamma_n & \longrightarrow & \operatorname{Out}\Gamma_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

we claim that $\mu'(\beta^n) = \mu(\alpha)$. For,

$$\alpha \cdot \beta^{-n} = (A, a)(B, 0)^{-n} = (I, a)$$

so that $a \in \mathbb{R}^{n-1} \subset C_{\mathfrak{A}}(\Gamma_n)$. Note that $C_{\mathfrak{A}}(\Gamma_n)$ is generally bigger than $\mathbb{R}^{n-1}$. Let $\bar{D} = \mu'(\beta)$. The above argument shows that the subgroup G of $\operatorname{Out}\Gamma_n$ generated by $\bar{D}$ has order n.

We prove now that $G \hookrightarrow \operatorname{Out}\Gamma_n$ does not have any affine (and hence, topological (see [5, Theorem 3]) realization. Suppose there exists a subgroup G' of $A(M)$ so that $\psi|_{G'}$ is an isomorphism of G' onto G. Form the lifting sequence of G', $1 \to \Gamma_n \to E \xrightarrow{\nu} G' \to 1$. This is just a restriction of the middle row in the diagram so that $\Gamma \subset N_{\mathfrak{A}}(\Gamma_n)$. Pick $\gamma \in E$ so that $\Psi(\nu(\gamma)) = \bar{D}$. Since $\mu'(\beta) = \bar{D} = \mu'(\gamma)$ and $\ker(\mu')$ is generated by Γ_n and $C_{\mathfrak{A}}(\Gamma_n)$, $\gamma = \sigma c \beta$ for some $\sigma \in \Gamma_n$ and $c \in C_{\mathfrak{A}}(\Gamma_n)$. For $\Psi : G' \to G$ to be an isomorphism, we should have $\gamma^n \in \Gamma_n$ or $(\sigma c\beta)^n \in \Gamma_n$. But this is equivalent to:

(*) $\qquad (c\beta)^n \in \Gamma_n \quad$ for some $c \in C_{\mathfrak{A}}(\Gamma_n)$.

We will prove (*) is impossible. Let $c=(I, c)$ with $c \in \mathbb{R}^{n^2+n-2}$ (this is an abuse of notation). We need a little computation

$$(c\beta)^n = ((I, c)(B, 0))^n = (B^n, (I+B+\cdots+B^{n-1})c).$$

By the definition of B, we have $B^n = A$. Also,

$$(I+B+\cdots+B^{n-1})c = \begin{bmatrix} I+B_n+\cdots+B_n^{n-1} & 0 \\ 0 & * \end{bmatrix} c$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} c \qquad \text{by (ii)}$$

$$= \begin{bmatrix} 0 \\ *' \end{bmatrix} \overset{let}{=} b \in \mathbb{R}^{n^2+n-2}.$$

Therefore, we have $(c\beta)^n = (A, b)$. Since

$$(c\beta)^n \alpha^{-1} = (A, b)(A, a)^{-1} = (I, b-a) \notin \Gamma_n,$$

$(c\beta)^n$ cannot be an element of Γ_n, showing (*) is not possible. This completes the proof for $n \geq 3$.

For $n=2$, we take $B_2 = [-1]$ and $b_2 = \frac{1}{2}$. Then all of the proof goes through just as above. The only modification needed is $(I - B_2)b_2 = e_1$ since there is only one basis element. Thus Γ_2 is generated by t_1, s_1, s_2, s_3 and $\alpha = (A, a)$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \qquad a = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The subgroup $G \cong \mathbb{Z}_2$ of Out Γ_2 is generated by $\bar{D} = \mu'(\beta)$, where $\beta = (B, 0)$ and

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

For the orientation argument, note that, by (i), $\det A = \det B = 1$, for any $n \geq 2$. This completes the proof.

Remark. By a slight modification of Example 1.3, we can produce many others. As an example, *we claim that*:

For each dimension $n \geq 3$, there exists a closed n-manifold N_n, of non-positive sectional curvature, on which Nielsen's theorem fails to hold. That is, there exists a cyclic subgroup of Out$(\pi_1 N_n)$, of order $n-1$, which cannot be realized as a group of homeomorphisms.

Proof. Let S be a surface of genus $(n-1)^2$ with an isometry δ of order $(n-1)^2$. We define an action of $\mathbb{Z}_{n-1} = \langle\alpha\rangle$ on $T^{n-2}\times S$ by $\alpha(x,s) = (x + b_{n-1}, \delta^{n-1}(s))$ This is free since it acts freely on T^{n-2} already. Let $N_n = T^{n-2}\times S/\langle\alpha\rangle$.

Let $\bar{\beta}$ be the diffeomorphism on $T^{n-2}\times S$ defined by $\bar{\beta}(x,s) = (B_{n-1}x, \delta(s))$. Passing to the quotient, $\bar{\beta}$ defines a diffeomorphism β on N_n; of order $(n-1)^2$. In turn, β defines a subgroup G of $\mathrm{Out}(\pi_1 N_n)$, isomorphic to $\mathbb{Z}_{n-1}$. We claim that $G \hookrightarrow \mathrm{Out}(\pi_1 N_n)$ has no realization.

2. Remarks on a paper of R. Schoen and S.T. Yau

In a recent paper of R. Schoen and S.T. Yau, [8], one finds some results that are related to those of [5]. However we caution the reader that their Corollary 6, Theorems 11, 12 and 13 need an additional hypothesis – the G-action on M leaves the kernel of $f_*: \pi_1(M) \to \pi_1(N)$ invariant – in order to be correct. Of course this condition, in general, will not hold and must be assumed. (Also, their Theorem 10 is incorrect since one cannot, in general, lift homomorphisms of G into $\mathrm{Out}\,\pi$ back up to the affine diffeomorphisms of $M(\pi)$. Theorem 3 of [5] is an independent and different treatment of the same problem.)

By combining our techniques with their theorems one may obtain extensions of some of their theorems. Here is an illustration of a generalization of their Theorem 13 (after the hypothesis on the fundamental group has been strengthened according to our suggestion).

2.1. Theorem. *Let M be a compact manifold with a finite group G acting effectively and differentiably. Suppose there is a degree 1 map f from M to a closed flat Riemannian manifold N such that K = kernel of f_* is a characteristic subgroup of $\pi_1 M$. Then G has a faithful representation in $A(N)$.*

Proof. There exists a natural diagram of extensions:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & \xrightarrow{\cong} & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \pi_1(N) & \longrightarrow & E/K & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Therefore by our result (Corollary to Theorem 4), there exists an affine geometric realization of the abstract kernel arising from the extension of $\pi_1(N)$ by G. However, if we now apply their Theorem 8, we see that the constructed action of G will act effectively since the degree of the map is 1. Consequently, under the strengthened hypothesis to Theorem 13 to make the conclusion correct (the invariant kernel condition) one needs only assume that N is a compact flat manifold. The condition on the center of $\pi_1(N)$ or the first Betti number of M is not necessary. Of course, one cannot assert a conclusion about the faithful representation into the outer automorphism group of N in case $b_1(M) \neq 0$ and $b_1(N) \neq 0$.

At the end of Theorems 4, 5 and 6 of [5] we pointed out that affine actions on M must qualitatively describe the possible topological actions on M. Nevertheless, as an example (end of Theorem 6 in [5]) showed, even smooth actions, in the presence of non-trivial isotropy subgroups, can differ from affine actions in some essentials. Still, Schoen and Yau's results, coupled with 2.1 above, compares, up to homotopy, each smooth action with an affine action. Specifically:

Given a homotopy equivalence f between a closed manifold M and a closed flat manifold N and a smooth effective action of a finite G on M, there exists an affine G-action on N and a smooth f' homotopic to f so that $f'(gx) = g \cdot f'(x)$.

3. Realization by isometries

For a Riemannian manifold there are many lifting problems to explore. Let us consider the inclusions and quotient groups

$$\begin{array}{ccccccccccc}
\mathscr{I}(M) & \subset & A(M) & \subset & \mathrm{Diffeo}(M) & \subset & \mathscr{H}(M) & \subset & \mathscr{E}(M) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & \\
& & & & & & & & & & \mathrm{Out}\,\pi_1 M \\
& & & & & & & & & \nearrow & \\
\pi_0\mathscr{I}(M) & \longrightarrow & \pi_0 A(M) & \longrightarrow & \pi_0\,\mathrm{Diffeo}(M) & \longrightarrow & \pi_0\mathscr{H}(M) & \longrightarrow & \pi_0\mathscr{E}(M) & &
\end{array}$$

where $\mathscr{I}(M)$ denotes the group of isometries of M, $A(M)$ the group of affine diffeomorphisms of M. One might try to start with, say a finite group at any stage and attempt to lift upwards and/or backwards. So far we have concentrated on lifting Out $\pi_1(M)$ to $A(M)$ for flat Riemannian manifolds. In this section we will treat the problem of lifting back to $\mathscr{I}(M)$. Note that for closed Riemannian manifolds $\pi_0\mathscr{I}(M) \to \pi_0 A(M)$ is always injective but $\pi_0\,\mathrm{Diffeo}(M) \to \pi_0\mathscr{H}(M) \to \pi_0\mathscr{E}(M) \to \mathrm{Out}\,\pi_1(M)$ may neither be injective nor surjective.

Let M be a closed flat Riemannian manifold. Let $\mathrm{out}(\pi_1 M)$ denote the image of the injection $\pi_0(\mathscr{I}(M)) \to \mathrm{Out}(\pi_1 M)$. Using the surjection $\mu : N_{A(n)}(\pi) \to \mathrm{Aut}(\pi_1 M)$, the conjugation map, we define $\mathrm{aut}(\pi_1 M)$ by $\mathrm{aut}(\pi_1 M) = \mu(N_{E(n)}(\pi))$. Notice that $\mathrm{out}(\pi_1 M) \cong \mathrm{aut}(\pi_1 M)/\mathrm{Inn}(\pi_1 M)$ and $\mathrm{aut}(\pi_1 M)$ depend on the Riemannian metric on M, while $\mathrm{Out}(\pi_1 M)$ and $\mathrm{Aut}(\pi_1 M)$ do not. The latter are purely algebraic objects depending only on the group $\pi_1 M$.

Out π may not be a finite group even though π has trivial center [2, Example 2]. Notice that out π is always finite.

Even though out$(\pi_1 M)$ varies according to the metric of M, there is a "common" subgroup which is very convenient for our lifting problems.

3.1. Theorem [1]. *Let M be a closed flat Riemannian manifold. Let Z be the maximal abelian subgroup of $\pi_1 M$ with quotient Φ. Then $H^1(\Phi, Z)$ is naturally embedded in* out$(\pi_1 M)$ *and can be realized as a group of isometries of M.*

The realization problem of an abstract kernel by isometries is completely settled by applying [5, Theorem 3].

3.2. Theorem. *Let M be a closed flat Riemannian manifold. Then an abstract kernel (G, π, ϕ) with G finite can be realized as a group of isometries if and only if it admits an admissible extension and $\phi(G) \subset$ out$(\pi_1 M)$.*

Proof. Suppose (G, π, ϕ) admits an admissible extension. By [5, Theorem 3], there is a realization of this abstract kernel, say, $\theta : G \to A(M)$, so that $\Psi \circ \theta$ is the identity on G, where Ψ is the natural homomorphism $A(M) \hookrightarrow \mathscr{E}(M) \to \text{Out}\ \pi$. Since $\Psi^{-1}(\text{out}(\pi_1 M)) \subset \mathscr{I}(M)$, $\theta(G)$ should be inside $\mathscr{I}(M)$ if $G \subset \text{out}(\pi_1 M)$. Another implication is easy.

3.3. Proposition. *Let M be a closed flat Riemannian manifold. For any finite subgroup G of* Out$(\pi_1 M)$, *there exists a flat Riemannian manifold M' which is affinely diffeomorphic to M so that $G \subset$* out$(\pi_1 M')$.

Proof. This is non-trivial, because G cannot be realized in general. In [4], it is proved that any such G has an inflation $G^* \to \text{Out}\ \pi$, with a finite abelian kernel $H^1(M, \mathbb{Z})/\text{Center}\,(\pi_1 M)$, which can be realized as a group of affine diffeomorphisms of M. So, we may assume that $G^* \subset A(M)$. Now, by [5, Theorem 6], there exists a flat Riemannian manifold M' with an action of a group of isometries $(M', G^{*\prime})$ which is affinely equivalent to the original (M, G^*). Therefore, $G^{*\prime}$ maps to G under $A(M') \to \text{Out}(\pi_1 M') \xrightarrow{\cong} \text{Out}(\pi_1 M)$ and certainly $G \hookrightarrow \text{out}(\pi_1 M')$.

4. Can the kernel K of $\pi_0 \mathscr{H}(M) \to$ Out π be realized?

It is known for certain aspherical manifolds that K is very large (all 2-torsions). The explicit description depends upon A. Hatcher's theory, where certain aspects are still in doubt, but enough is verified so no doubt exists for tori of dimension greater than 4, and most asphericals of sufficiently large dimensions.

We *conjecture* that for any closed aspherical manifold no non-trivial finite subgroup of K can be lifted back to $\mathscr{H}(M)$.

For M with $\mathfrak{z}(\pi_1(M)) = 1$, this is true since any finite group G acting effectively on M necessarily injects into Out $\pi_1(M)$.

We shall show that our conjecture holds at least for certain kinds of flat manifolds.

4.1. Theorem. *Let M^n be a closed flat Riemannian manifold, and G be any finite subgroup of $\pi_0(\mathscr{H}(M))$. Suppose either:*

(i) *$\mathfrak{z}(\pi_1(M))$ is a "summand" of $\pi_1(M)$, or*

(ii) *$n > 4$, and the holonomy of M is of odd order [this case depends upon the kernel of $\pi_0\mathscr{H}(M) \to \mathrm{Out}\ \pi_1(M)$ having only 2-torsion]. Then G lifts to $\mathscr{H}(M)$ if and only if $G \to \mathrm{Out}\ \pi_1(M)$ is injective and admits an extension (and hence admissible).*

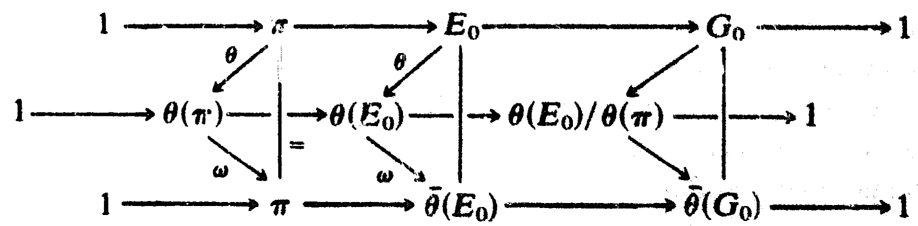
Proof. We have already shown the if part. So let us assume that G acts effectively on M. We get the admissible extension:

$$1 \to \pi \to E \to G \to 1,$$

and by restricting to $G_0 =$ kernel of $(G \to \mathrm{Out}\ \pi)$, get

$$1 \to \pi \to E_0 \to G_0 \to 1.$$

We claim E_0 is torsion free. If this is the case, then E_0 is a torsion free extension of a Bieberbach group and hence, by Proposition 2 of [5], is an abstract Bieberbach group. Note that $E_0 \subset \mathscr{H}(\mathbb{R}^n)$ by construction and so it acts freely and properly discontinuously on $\mathbb{R}^n$. By Farrell–Hsiang, [3], there exists $h \in \mathscr{H}(\mathbb{R}^n)$ such that $hE_0h^{-1} < E(n)$. Denote $E_0 \xrightarrow{\theta} \theta(E_0)$ ($\theta =$ conjugation by h). Since $\theta|\pi : \pi \to \theta(\pi)$ is an isomorphism between Bieberbach groups, there is an affine map $f \in A(n)$ so that $h \circ \sigma \circ h^{-1} = f \circ \sigma \circ f^{-1}$ for all $\sigma \in \pi$. Let ω be the conjugation by f^{-1}. Then



Note that $\theta(E_0) \subset E(n)$, and hence $\bar{\theta}(E_0) \subset A(n)$. This implies that $\bar{\theta}(G) = \bar{\theta}(E_0)/\pi$ is in $A(M)$. Originally, G_0 was mapped to $\mathrm{id} \in \mathrm{Out}\ \pi$, so the extension $1 \to \pi \to \bar{\theta}(E_0) \to \bar{\theta}(G_0) \to 1$ is an extension realizing the abstract kernel $\bar{\theta}(G_0) \to 1 \in \mathrm{Out}\ \pi$. That is, in the induced diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi & \longrightarrow & \bar{\theta}(E_0) & \longrightarrow & \bar{\theta}(G_0) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \\
1 & \longrightarrow & \mathrm{Inn}\ \pi & \longrightarrow & \mathrm{Aut}\ \pi & \longrightarrow & \mathrm{Out}\ \pi & \longrightarrow & 1
\end{array}$$

$\theta(G_0) = \ker \bar{\phi} \subset T^k$, the connected component of $\mathcal{I}(M)$. This means every element of $\bar{\theta}(G_0)$ is isotopic to the identity, which in turn implies that every element of $G_0 = h^{-1}f\bar{\theta}(G_0)f^{-1}h$ is isotopic to the identity. Thus, G_0 must be trivial.

It remains to check that E_0 is necessarily torsion free. We have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathfrak{z}(\pi) & \longrightarrow & C_{E_0}(\pi) & \longrightarrow & G_0 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \pi & \longrightarrow & E_0 & \longrightarrow & G_0 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn}\pi & \xrightarrow{=} & \operatorname{Inn}\pi & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

with exact rows and columns. Note that in case (i), $\operatorname{Inn}\pi =$ the complementary summand. (In fact, the splitting of the center implies that $M = T^k \times N$ and $\operatorname{Inn}\pi = \pi_1(N)$). If E_0 has a torsion element e_0, then it injects into $\operatorname{Inn}\pi$ since $C_{E_0}(\pi)$ is torsion free (or trivial). Now in case (i), $\operatorname{Inn}\pi$ is torsion free. In case (ii), the order of e_0 in $\operatorname{Inn}\pi$ is 2 since it injects into G_0 because π is torsion free, and also injects into $\operatorname{Inn}\pi$ because $C_{E_0}(\pi)$ is torsion free. Suppose $\mu(\sigma) = \text{image } e_0 \subseteq \operatorname{Inn}\pi$, $\sigma \in \pi$. Then $\mu(\sigma^2) = (\mu(\sigma))^2 = 1$ implies $\sigma^2 \in \mathfrak{z}(\pi)$. This implies that M has even order holomony. For, if we denote $\sigma = (A, a) \in A(n)$, then $(A^2, (I + A)a) = \sigma^2 \in \mathfrak{z}(\pi)$ implies $A^2 = I$. This contradiction shows E_0 is torsion free in case (ii) and completes the argument.

Corollary 1. *Let T^n be a flat torus. $K =$ the kernel of $\pi_0\mathcal{H}(T^n) \to \mathrm{GL}(n, \mathbb{Z}) = \mathrm{Aut}\,\mathbb{Z}^n$. Then no non-trivial finite subgroup of K can be geometrically realized as an effective group of homeomorphisms of T^n, $n \neq 4$.*

Corollary 2. *Let M^n be flat and G a finite subgroup of K. Then no nontrival element of G can be lifted to act freely on M if $n \neq 4$.*

Proof. Examining the argument used in the proof of Theorem 4.1, one sees that what is needed to carry out the argument is that the lifted action of E on $M(\pi)$ is free (equivalently, E is torsion free). This is guaranteed if G acts freely.

5. Classifications of finite affine actions

An affine action (G_1, M) is said to be *strongly conjugate* to (G_2, M) if there is h in $A_0(M)$, the connected component of identity of $A(M)$, such that $\mu(h)$ maps G_1

onto G_2 isomorphically. We shall now classify all affine realizations of a given abstract kernel ϕ up to strong conjugacy.

5.1. Theorem. *Suppose (G, π, ϕ) is an injective finite abstract kernel with an affine realization. Then there is an isomorphism of the strong conjugacy classes of all affine realizations of (G, π, ϕ) onto $H^2(G, \mathfrak{z}(\pi))$.*

Proof. We pick a map $w: G \to \mathrm{Aut}\, \pi$ so that w composed with the natural homomorphism $\mathrm{Aut}\, \pi \to \mathrm{Out}\, \pi$ is the same as ϕ, once and for all. Let θ_0 be the realization given by the hypothesis, and $1 \to \pi \to E_0 \to \theta_0(G) \to 1$ the lifting sequence of $\theta_0(G)$. Again we choose a map $f_0: G \to E_0$ so that $\mu \circ f_0 = w$.

For any other affine realization θ with lifting sequence, $1 \to \pi \to E \to \theta(G) \to 1$, and any map $f: G \to E$ with $\mu \circ f = w$, it is readily verified that $f - f_0: G \to \mathbb{R}^k = C_{A(n)}(\pi)$ yields a 1-cocycle $g: G \to T^k = \mathcal{I}_0(M)$. Now it is not hard to see that θ is strongly conjugate to θ' if and only if $g - g'$ is principal. Thus we have shown that there is an injective homomorphism of all strong conjucacy classes of affine realizations into $H^1(G, T^k)$. Conversely, given a 1-cocycle $g: G \to T^k = \mathcal{I}_0(M)$ we can lift g to $\tilde{g}: G \to \mathbb{R}^k$ and define a map $f = f_0 + \tilde{g}: G \to A(n)$. The subgroup E of $A(n)$ generated by π and $f(G)$ induces an affine realization $\theta: G \to \theta(G) = E/\pi$ so that $\Psi \circ \theta = \phi$, showing {realizations} $\to H^1(G, T^k)$ is surjective.

Since $H^i(G, \mathbb{R}^k) = 0$ for $i > 0$, we have an isomorphism $\delta: H^1(G, T^k) \to H^2(G, \mathfrak{z}(\pi))$. Therefore, by composing δ with the isomorphism above, we have the desired isomorphism of all strong conjugacy classes of affine realizations of (G, π, ϕ) onto $H^2(G, \mathfrak{z}(\pi))$. Note that θ_0 maps to 0 under this isomorphism.

5.2. Example. On M of 1.2, the automorphism of π given by

$$\alpha \to \alpha, \qquad t_2 \to t_3, \qquad t_3 \to t_2$$

defines an abstract kernel $\{1, \tau\} \cong \mathbb{Z}_2 < \mathrm{Out}\, \pi$. We have an obvious realization of this by $\beta_1 = (B, 0) \in A(3)$, where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now $H^2(\mathbb{Z}_2, \mathfrak{z}(\pi))$ is isomorphic to $\mathbb{Z}_2$, and a non-trivial cocycle is $g(\tau) = \frac{1}{2} \in T^1$ so that $g(\tau) = \frac{1}{2} \in \mathbb{R}^1$. Therefore our new realization is given by $\beta_2 = (B, {}^t[\frac{1}{2}, 0, 0])$. Of course, these two actions are not strongly conjugate to each other.

Two effective affine (respectively; topological) actions (G_1, M) and (G_2, M) are *affinely* (resp.; topologically) *equivalent* if there is $h \in A(M)$ (resp. $h \in \mathcal{H}(M)$) so that $\theta = \mu(h)$, conjugation by h, is an isomorphism of G_1 onto G_2; i.e., $\theta(g) \cdot h(x) = h(g \cdot x)$, for all $g \in G_1$ and $x \in M$. We say that two extensions E and E' of π are *isomorphic* if there is an isomorphism between them inducing an automorphism of π.

We shall restate the Corollary to Theorem 6 of [5], (which essentially summarized many of our results there), in a slightly altered form to conform with our effective requirement here:

If (G_1, M) and (G_2, M) are finite effective equivalent actions then their lifting sequences are isomorphic. Conversely, two admissable isomorphic extensions $1 \to \pi \to E \to G \to 1$ and $1 \to \pi \to E' \to G' \to 1$ yield affinely equivalent affine realizations.

Hence, affine actions are topologically equivalent if and only if they are affinely equivalent.

The group of all affine actions which are affinely equivalent to a given effective affine action $(\theta(G), M)$ can be measured. It is isomorphic to $N(\pi)/N(\pi) \cap N(E)$, where E is the lifting of $\theta(G)$ to $\tilde{M}$, $N(\pi)$ and $N(E)$ denote the normalizers of π and E in $A(n)$. Note that isomorphic extensions do not, in general, yield the same abstract kernel.

5.3. Example. Even though two affine actions realize the same abstract kernel and are affinely equivalent, they are not, in general, strongly conjugate. Consider the two actions in 5.2. Let $\theta = \mu(C, 0)$, where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then one easily checks that $\theta(t_2) = t_3^{-1}$, $\theta(t_3) = t_2$, $\theta(\alpha) = \alpha$ and $\theta(\beta_1) = \beta_2\alpha^{-1}$. Therefore, θ is an isomorphism of the lifting of the first action to that of the second one leaving π invariant, making the two actions affinely equivalent. Certainly the isomorphism $\bar{\theta}$ of G_1 and G_2 induced by θ preserves the abstract kernel. However, we have seen in 5.2 that they are not strongly conjugate.

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