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THE SMALLEST 2-CONNECTED CUBIC BIPARTITE PLANAR NONHAMILTONIAN GRAPH

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Tutte produced the first example of a 3-connected cubic planar nonhamiltonian graph. On adding the condition that the graph must be bipartite and admitting 2-connected graphs. We prove that the smallest possible such graph has 26 points and is unique.

One of the many problems which arose out of attempts to prove the Four Color Theorem is the determination of conditions sufficient to ensure that a planar graph, especially a cubic planar graph, is hamiltonian. A result of this type was given by Tutte [5] who proved that every 4-connected planar graph is hamiltonian. Tutte had previously shown that this condition could not be weakened to include all 3-connected graphs by constructing his now famous 3-connected cubic planar nonhamiltonian graph [4]. The smallest such graph found to date is due to Joshua Lederberg; [2, p. 168]. Tutte then conjectured that all bipartite 3-connected graphs are hamiltonian. As reported in [2, p. 170], Horton produced a nonplanar counterexample. Still open is the conjecture of Barnette [1] that all bipartite 3-connected cubic graphs are hamiltonian. All these graphs exclude multiple edges, as do those we now study.

In this paper we shall be dealing with 2-connected graphs and we will show that the graph A in Fig. 1, which has order 26, is the smallest nonhamiltonian

A:

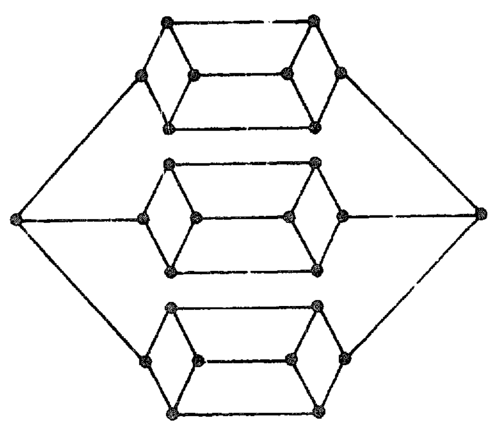

Fig. 1.

2-connected cubic bipartite planar graph. Our notation and terminology will follow [3]. In particular p and q will denote the number of points and lines in a graph; and in a plane graph r is the number of faces and r_n the number of faces bounded by n-cycles. Rather than repeatedly refer to 2-connected cubic bipartite planar graphs we shall say that such a graph is *topical*. Note that the '2-' is superfluous since no connected cubic bipartite graph has a cutpoint.

If G is topical, then we can use Euler's formula to derive three useful equations:

$$\sum_{n\geq 2} 2mr_{2m} = 2q = 3p, \tag{1}$$

$$r = \sum_{m\geq 2} r_{2m} = \tfrac{1}{2}(p+4) \tag{2}$$

$$r_4 = 6 + \sum_{m\geq 1} mr_{2m+6}. \tag{3}$$

The first equation is derived by counting the incident point-face pairs in two ways. The second follows directly from Euler's formula. The third is obtained by multiplying (2) by six, subtracting the result from (1), and dividing by two.

Before beginning the proof of our theorem we describe three transformations which are applied to certain topical graphs. These transformations, f_1, f_2 and f_3, act on such a graph G by removing a certain subgraph and replacing it with a smaller subgraph. In Fig. 2(a), we show three graphs H_1, H_2 and H_3 which can occur as subgraphs of topical graphs. The endpoints of H_i are called its *points of attachment*. In Fig. 2(b), three smaller graphs H_1^-, H_2^- and H_3^- are shown. Now suppose that G contains H_i as a subgraph and if $i = 3$ suppose also that its points

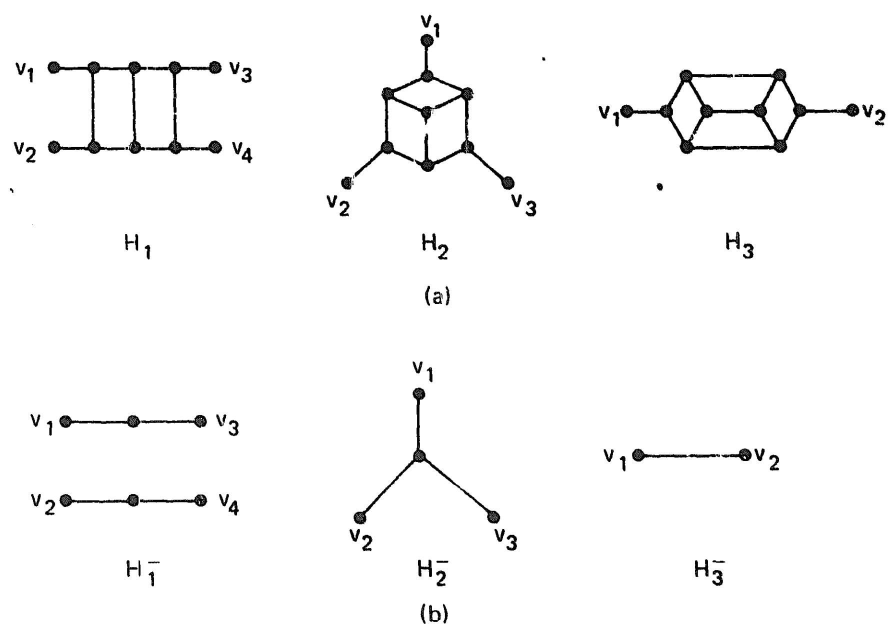

Fig. 2.

of attachment are not adjacent. Then $f_i(G)$ is the graph obtained by removing H_i and replacing it with H_i^- in such a way that the points of attachment correspond as in Fig. 2. Of course $f_i(G)$ may not be unique, as G might contain several copies of H_i. However, this will cause no problem as any one choice of H_i in G will do. Thus we feel free to abuse notation and presume that $f_i(G)$ is well defined.

It is nearly obvious that if G is topical and admits the transformation f_i, then $f_i(G)$ is also topical. The only fact that needs to be checked is that $f_i(G)$ does not have a bridge. But, as noted above a connected cubic bipartite planar graph is necessarily 2-connected. Next we show that f_1 and f_2 preserve nonhamiltonicity.

Lemma. *If G is a nonhamiltonian connected cubic bipartite planar graph admitting the transformation f_1 or f_2, then $f_i(G)$ is nonhamiltonian.*

Proof. The proof is indicated by Fig. 3 in which we show how to use a hamiltonian cycle of $f_i(G)$ to construct one in G. The heavy lines in this figure are on the hamiltonian cycles.

Theorem. *The graph A of Fig.* 1 *is the smallest* 2*-connected cubic bipartite planar nonhamiltonian graph.*

Proof. Let G be a smallest nonhamiltonian topical graph. Since A has order 26, G has order $p \leq 26$. By the minimality of G and Lemma, we know that G does not admit either of the transformations f_1 or f_2.

Consider G as having a fixed plane drawing. If G has two intersecting 4-faces F_1 and F_2, then we will now show that F_1 and F_2 share one line (and two points). To prove this, observe that F_1 and F_2 cannot share just one point since G is cubic. If they share two adjacent lines, then G is not 2-connected. Similarly, these two faces cannot share two nonadjacent points or lines. As these are the only other possibilities, the assertion is proved.

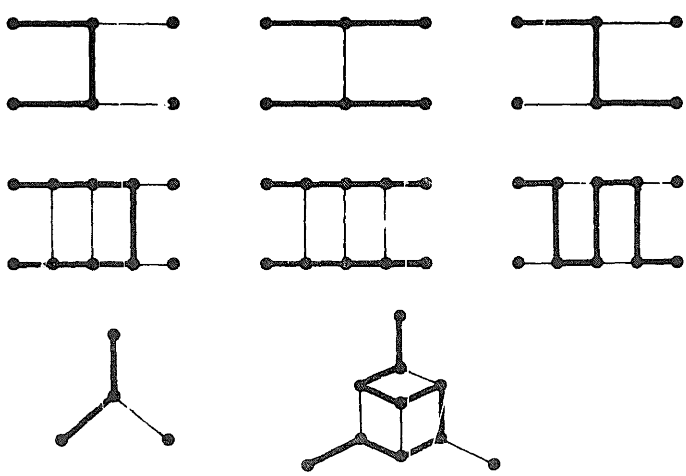

Fig. 3.

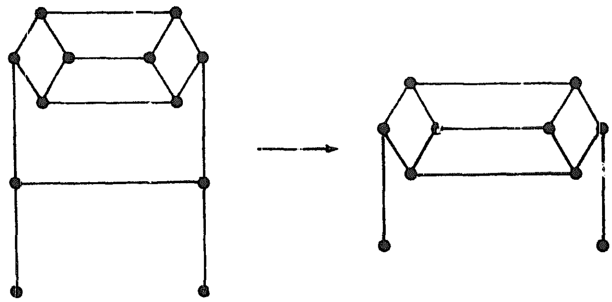

Fig. 4.

Next we claim that if two 4-faces F_1 and F_2 intersect, then the graph H_3 of Fig. 2(a) occurs as a subgraph of G and G admits f_3. As F_1 and F_2 intersect, they share exactly one line and two points as just shown. By considering all possibilities for the remaining lines incident with the points of F_1 and F_2, it is clear that G contains H_1, H_2 or H_3 as a subgraph and so must contain H_3. If the points of attachment of H_3 were adjacent in G, then G could be transformed as in Fig. 4 to a smaller nonhamiltonian topical graph. Thus they are not adjacent and G admits F_3.

We shall now determine the possible values for $r_4^{(G)}$. From Eq. (3) it follows that $r_4 \geq 6$ for any topical graph. It is clear that the transformation f_3 reduces the value of r_4 by at least two. Therefore when G admits f_3, $r_4(f_3(G)) \geq 6$, which implies $r_4(G) \geq 8$.

The proof is completed by considering three cases: $r_4 = 6$, $7 \leq r_4 \leq 8$, and $r_4 \geq 9$. In both of the first two cases we shall show $p > 26$, and in the third case we show that if $p \leq 26$, then $G = A$.

Case 1. $r_4 = 6$.

Our earlier observations indicated that in this case G does not admit f_3 and therefore the 4-faces of G do not intersect. Since $r_4 = 6$, we immediately obtain $p \geq 24$. By (3) it follows that $r_{2m} = 0$ for $m \geq 4$ and therefore $r_6 = \frac{1}{2}(p-8)$.

If $p = 24$, then every point is on exactly one 4-face. In this case G must be the graph in Fig. 5 which is hamiltonian. Showing that G must be this graph is simply a matter of exhausting all possibilities in a perfectly straightforward manner.

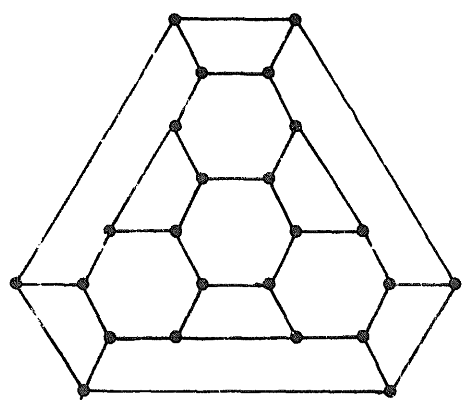

Fig. 5.

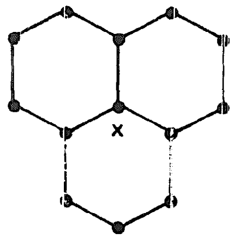

Fig. 6.

Next suppose that G has order 26. Then G has exactly two points, x and y, not on 4-faces. Each of these points is on three 6-faces. Let F_1, F_2 and F_3 be the three 6-faces having x on their boundary, as in Fig. 6. Since every point of G other than x and y is on exactly one 4-face, four of the points on each of the faces F_1, F_2 and F_3 are on 4-faces. Therefore y is also on each of these 6-faces, which is easily shown to be impossible. This dispenses with the first case, $r_4 = 6$.

Case 2. $7 \leq r_4 \leq 8$.

In this case, if $p \leq 26$, then G has two intersecting 4-faces, and hence it contains H_3 as a subgraph and admits f_3. Therefore $r_4(G) = 8$ since we must have $r_4(f_3(G)) = 6$ by (3). It is evident that applying f_3 to G eliminates four 4-faces and thus creates two new ones. Let F_1 and F_2 be the two new 4-faces in $f_3(G)$ and let F_3–F_6 be the remaining four.

If any two of the faces F_3–F_6 intersect, then $f_3(G)$ contains a copy of H_3. If $f_3(G)$ also admits f_3, then we have $r_4(G) \geq 10$, arguing as above. So the points of attachment of H_3 in $f_3(G)$ are adjacent. But in order for this to be the case, the two copies of H_3 in G must have the same points of attachment and clearly this implies that $r_4(G) > 8$ if $p \leq 26$.

Therefore F_3–F_6 are disjoint in $f_3(G)$. This means that $f_3(G)$ has at least 16 points and that G has at least 24. If $p = 24$, then each point of $f_3(G)$ lies on the boundary of exactly one of the faces F_3–F_6. Therefore G has the structure indicated in Fig. 7. But now G cannot be completed in such a way as to satisfy all the necessary conditions.

If $p = 26$ and if u and v are the points of attachment, then there are three possibilities according to whether both, one or none of the points of attachment of

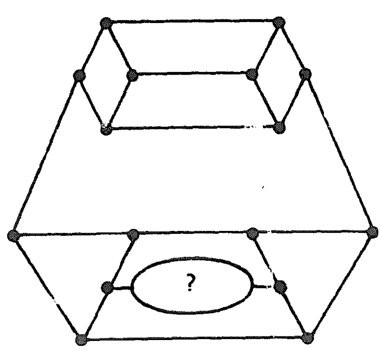

Fig. 7.

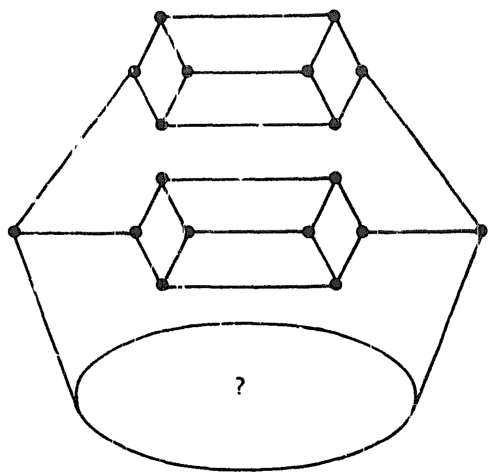

Fig. 8.

H_3 are in F_3–F_6. In each case a contradiction is obtained in a manner similar to the case $p = 24$.

Case 3. $r_4 \geq 9$.

In this case G has at least two induced subgraphs $H_{3,1}$ and $H_{3,2}$ isomorphic to H_3. Let u_i and v_i be the points of attachment of $H_{3,i}$.

If $\{u_1, v_1\} \neq \{u_2, v_2\}$, then $f_3(f_3(G))$ is a topical graph and so must have at least 8 points, implying $p \geq 24$. If $p = 24$, then $f_3(f_3(G))$ is the cube, Q_3. But Q_3 has the property that every pair of its lines lie on a hamiltonian cycle. Thus G is hamiltonian by Lemma, and so $p \geq 26$. If $p = 26$, then $f_3(f_3(G))$ is a topical graph of order 10. However no such graph exists, implying that $\{u_1, v_1\} = \{u_2, v_2\}$. So we can draw G as in Fig. 8, from which it is easy to see that G is the graph A of Fig. 1.

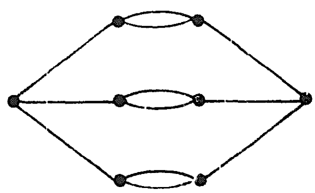

Fig. 9

We are grateful to the referee for pointing out that if multiple edges are permitted, then the well-known example of Fig. 9 gives the smallest nonhamiltonian planar bipartite cubic multigraph.

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