The Wiener process with drift between a linear retaining and an absorbing barrier

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ABSTRACT

The Wiener process with constant drift is modified by a time-dependent retaining barrier that increases at a constant rate and by an absorbing barrier at zero. Explicit expressions in terms of series expansions are derived for the Laplace transform and the probability density function of the time of absorption.

1. INTRODUCTION

Let $X_t^0 = x + W_t + \mu t$, where $\{W_t\}$ is the standard Wiener process and μ a constant drift. Consider a retaining barrier which is at b_t at time $t \geqslant 0$, and denote by $\{X_t\}$ the modified process. Thus

$$X_t = X_t^0 - \max \{(X_s^0 - b_s)^+, 0 \leqslant s \leqslant t\}. \tag{1}$$

We shall consider the case of a linear barrier, $b_t = at + b$, where $a > 0$ and $b \geqslant x$ are constants. Let $T = \inf \{t | X_t = 0\}$ denote the time of absorption at zero,

$$L(x, b; \lambda) = E[e^{-\lambda T}], \lambda > 0, \tag{2}$$

its Laplace transform, and f(t) its probability density function. Note that in the case $\mu > 0$, T has a defective distribution.

The purpose of this note is to compute L and f. The results for the limiting case of a constant barrier (a=0) are well known, see Example 5.6., page 233, in [1]. Its discrete analogue has been treated by Weesakul [5].

In an actuarial context [2,3], X_t can be interpreted as the surplus of an insurance company at time t, T as the time when "ruin" occurs, and b_t defines the payment of "dividends". If $\mu > 0$, $b_t = at + b$ with $a > 0$ is the most simple dividend barrier that does not lead to automatic ruin of the company.

2. THE LAPLACE TRANSFORM OF THE TIME OF ABSORPTION

The function $L(x, b; \lambda)$; $0 \leqslant x \leqslant b < \infty$, can be characterized as the unique bounded solution of the partial differential equation

$$\frac{1}{2} \frac{\partial^2 L}{\partial x^2} + \mu \frac{\partial L}{\partial x} + a \frac{\partial L}{\partial b} - \lambda L = 0 \tag{3}$$

that satisfies the boundary conditions

$$\left. \frac{\partial L}{\partial x} \right|_{x=b} = 0 \tag{4}$$

and

$$L(0, b; \lambda) = 1. \tag{5}$$

In the following we shall construct this solution explicitly.

We shall expand L as a series whose first term is $L(x, \infty; \lambda) = \lim_{b \to \infty} L(x, b; \lambda)$.

$L(x, \infty; \lambda)$ is the bounded solution of the differential equation $\frac{1}{2} L'' + \mu L' - \lambda L = 0$ subject to the boundary condition $L(0, \infty; \lambda) = 1$. Thus

$$L(x, \infty; \lambda) = e^{-(\mu + u)x}, \tag{6}$$

where

$$u = (\mu^2 + 2\lambda)^{1/2}. \tag{7}$$

The subsequent terms will satisfy the equation

$$\frac{1}{2} \frac{\partial^2 k}{\partial x^2} + \mu \frac{\partial k}{\partial x} + a \frac{\partial k}{\partial b} - \lambda k = 0 \tag{8}$$

with the boundary condition $k(0,b) = 0$. The special solutions of the boundary value problem are of the form

$$k(x,b) = e^{sb} (e^{r_1 x} - e^{r_2 x}), \tag{9}$$

where, for given s, r_1 and r_2 are solutions of the equation

$$\frac{1}{2} r^2 + \mu r + as - \lambda = 0. \tag{10}$$

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Thus, we are looking for a solution of the form

$$L(x,b;\lambda)=e^{-(\mu+u)x}+\sum_{k=1}^{\infty} C_k\, e^{s^{(k)}b}[e^{r_1^{(k)}x}-e^{r_2^{(k)}x}], \tag{11}$$

with $s^{(k)} < 0$ for all k. We shall determine $s^{(k)}$, $r_1^{(k)}$ and $r_2^{(k)}$ recursively such that

$$s^{(k)}+r_2^{(k)}=s^{(k+1)}+r_1^{(k+1)},\ k \geqslant 1, \tag{12}$$

where $s^{(1)}+r_1^{(1)}=-\mu-u$. Then (4) is satisfied if we start with $C_1=\frac{\mu+u}{r_1^{(1)}}$ and set

$$C_{k+1}=\frac{r_2^{(k)}}{r_1^{(k+1)}}C_k. \tag{13}$$

We rewrite (10) in the form

$$\frac{1}{2}r^2+(\mu-a)\,r+a(s+r)-\lambda=0, \tag{14}$$

which is useful when the value of $s+r$ is given, such as in (12). Using this, we see that

$$r_1^{(1)}=-\mu+u+2a \tag{15}$$

and thus

$$s^{(1)}=-2u-2a \tag{16}$$

$$r_2^{(1)}=-\mu-u-2a. \tag{17}$$

From (10), (14) and Vieta's rule we find that

$$r_1^{(k+1)}=2\,\frac{a(s^{(k)}+r_2^{(k)})-\lambda}{r_2^{(k)}}$$

$$=2a+2\,\frac{as^{(k)}-\lambda}{r_2^{(k)}}=2a+r_1^{(k)}. \tag{18}$$

Similarly,

$$r_2^{(k+1)}=2\,\frac{a\,s^{(k+1)}-\lambda}{r_1^{(k+1)}}$$

$$=2\,\frac{a[s^{(k)}+r_2^{(k)}-r_1^{(k+1)}]-\lambda}{r_1^{(k+1)}}=-2a+r_2^{(k)} \tag{19}$$

Hence,

$$r_1^{(k)}=-\mu+u+2\,ak \tag{20}$$

$$r_2^{(k)}=-\mu-u-2\,ak \tag{21}$$

From this and (12) we get

$$s^{(k+1)}=s^{(k)}+r_2^{(k)}-r_1^{(k+1)}=s^{(k)}-2u-2a(2k+1), \tag{22}$$

and thus

$$s^{(k)}=-2\,uk-2\,ak^2 \tag{23}$$

From (13) we obtain

$$C_k=\frac{\mu+u}{r_1^{(1)}}\cdot\frac{r_2^{(1)}\ldots r_2^{(k-1)}}{r_1^{(2)}\ldots r_1^{(k)}}=(-1)^{k-1}\frac{(\frac{\mu+u}{2a})_k}{(\frac{-\mu+u}{2a}+1)_k} \tag{24}$$

with the notation $(r)_k=r(r+1)\ldots(r+k-1)$. Finally, (20), (21), (23) and (24) are substitued in (11) to obtain $L(x,b;\lambda)$.

Remarks

(1) Setting $\lambda=0$ in formula (11) we obtain the probability for absorption at zero (the probability of "ruin" in actuarial terminology). The resulting series in the nontrivial case $\mu>0$ has been obtained in [3].

(2) In the limiting case $a=0$, the series in (11) is a geometric series that can be simplified. Of course, the resulting expression for L can be obtained directly from (3), (4), (5) as it is done in [1].

3. THE P.D.F. OF THE TIME OF ABSORPTION

Let $\mathcal{L}$ denote the Laplace transform, and $\mathcal{L}^{-1}$ its in-inverse. We want to find $f(t)=\mathcal{L}^{-1}\,L(x,b;\lambda)$. For this purpose we write the series that was obtained for L in the following form :

$$L(x,b;\lambda)=e^{-\mu x}\{e^{-ux}+\sum_{k=1}^{\infty}[\phi_k(x,b,u)-\phi_k(-x,b,u)]\}, \tag{25}$$

where $u=u(\lambda)$ is defined in formula (7) and

$$\phi_k(x,b,u)=(-1)^{k-1}\,e^{-2\,ak(kb-x)}\,e^{-(2\,kb-x)u}\cdot\frac{(\frac{\mu+u}{2a})_k}{(\frac{-\mu+u}{2a}+1)_k} \tag{26}$$

The last term can be expanded by partial fractions as follows :

$$\frac{(\frac{\mu+u}{2a})_k}{(\frac{-\mu+u}{2a}+1)_k}=1+\sum_{j=0}^{k-1}(-1)^j\frac{(\frac{\mu}{a}-1-j)_k}{j!\,(k-j-1)!}\,\frac{1}{\frac{-\mu+u}{2a}+1+j} \tag{27}$$

Hence, by the linearity of $\mathcal{L}^{-1}$,

$$f(t)=e^{-\mu x}\{g(x,1,0;t)+\sum_{k=1}^{\infty}[\Phi_k(x,b;t)-\Phi_k(-x,b;t)]\}, \tag{28}$$

where

$$g(r, p, q; t) = \mathcal{L}^{-1}\left[\frac{e^{-ru}}{p+qu}\right] \quad r > 0, p > 0, q \geqslant 0, \qquad (29)$$

and $\Phi_k = \mathcal{L}^{-1}\,\phi_k$ is

$$\Phi_k(x, b; t) = (-1)^{k-1}\, e^{-2\,ak(kb-x)}\{g(2\,kb - x, 1, 0; t)$$
$$+ \sum_{j=0}^{k-1} (-1)^j \frac{(\frac{\mu}{a} - 1 - j)_k}{j!\,(k-1-j)!}\, g(2\,kb - x, -\frac{\mu}{2a} + 1 + j, \frac{1}{2a}; t)\} \qquad (30)$$

The remaining problem is to compute $g(r, p, q; t)$. First it is easily verified that

$$\mathcal{L}^{-1}\,[e^{-r\lambda}] = \delta(t - r) \qquad (31)$$

and, for $q > 0$, that

$$\mathcal{L}^{-1}\left[\frac{e^{-r\lambda}}{p+q\lambda}\right] = \begin{cases} 0. & \text{for } t < r \\ \frac{1}{q}\, e^{-p(t-r)/q} & \text{for } t > r \end{cases} \qquad (32)$$

How do these relationships have to be modified if we replace λ by $u = u(\lambda)$ on the left ? The answer is given by the following lemma.

Lemma

If $g(t) = \mathcal{L}^{-1}\,[h\,(\lambda)]$, it follows that

$$\mathcal{L}^{-1}\,[h\,(u)] = \frac{t^{-3/2}}{\sqrt{2\pi}}\, e^{-\mu^2 t/2} \int_0^\infty s\, e^{-s^2/2t}\, g(s)\, ds, \qquad (33)$$

where $u = u(\lambda)$ is defined in formula (7).

Proof

The proof follows from the relation

$$\mathcal{L}^{-1}\,[h(\sqrt{\lambda})] = \frac{t^{-3/2}}{2\sqrt{\pi}} \int_0^\infty s\, e^{-s^2/4t}\, g(s)\, ds,$$

– see for example formula (1.27.), page 210, in [4] – and the more trivial relation

$$\mathcal{L}^{-1}\,[h(\alpha + \beta\lambda)] = \frac{1}{\beta}\, e^{-\alpha t/\beta}\, g(\frac{t}{\beta}) \quad \text{for } \beta > 0.$$

From formula (31) and the lemma we obtain

$$g(r, 1, 0; t) = \frac{t^{-3/2}}{\sqrt{2\pi}}\, e^{-\mu^2 t/2}\, r\, e^{-r^2/2t}. \qquad (34)$$

From (32) and (33) we get

$$g(r, p, q; t) = \frac{1}{q}\,\frac{t^{-3/2}}{\sqrt{2\pi}}\, e^{-\mu^2 t/2} \int_r^\infty s\, e^{-s^2/2t - p/q\,(s-r)} ds. \qquad (35)$$

The integral can be simplified to

$$t\, e^{-r^2/2t} - \frac{p}{q}\sqrt{2\pi}\, t^{3/2}\, e^{r\,p/q + (p/q)^2\, t/2}\, \bar{\Phi}\left(\frac{r + \frac{p}{q}t}{\sqrt{t}}\right), \qquad (36)$$

where $\bar{\Phi}(\cdot) = 1 - \Phi(\cdot)$, and Φ is the standard normal distribution. Thus

$$g(r, p, q, t) = \frac{1}{q}\,\frac{1}{\sqrt{2\pi t}}\, e^{-r^2/2t - \mu^2 t/2}$$
$$- \frac{p}{q^2}\, e^{r\,p/q + [(p/q)^2 - \mu^2]\, t/2}\, \bar{\Phi}\left(\frac{r + \frac{p}{q}t}{\sqrt{t}}\right) \qquad (37)$$

In summary, formula (28), together with formulas (30), (34) and (37), gives the p.d.f. of the time of absorption at zero.

4. REFERENCES

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