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Variational Principles for Eigenvalues of Compact Nonselfadjoint Operators*

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Variational principles for eigenvalues of compact nonselfadjoint operators are given.

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Let T be a linear compact operator on a Hilbert space H, λ_j be its eigenvalues, $|\lambda_1| \geqslant |\lambda_2| \geqslant ,...,$ r_j be the moduli of the real parts of the eigenvalues ordered so that $r_1 \geqslant r_2 \cdots$. Note that r_j is not necessarily equal to $|\mathrm{Re}\,\lambda_j|$. Let L_j be the eigensubspace of T corresponding to λ_j, M_j be the eigensubspace of T corresponding to r_j, $\tilde{L}_j = \sum_{k=1}^{j} \dotplus L_k$, $\tilde{M}_j = \sum_{k=1}^{j} \dotplus M_k$. Let t_j be the moduli of the imaginary parts of the eigenvalues, $t_1 \geqslant t_2 \geqslant ,...,$ $\tilde{N}_j = \sum_{k=1}^{j} \dotplus N_k$, N_j be the eigensubspace of T corresponding to t_j, $\tilde{L}_j^{\amalg} \dotplus \tilde{L}_j = H$, the sign $\dotplus$ denotes the direct sum, $\amalg$ denotes the direct complement in H.

THEOREM 1. *Under the above assumptions the following formulas hold*:

$$|\lambda_j| = \max_{x \in \tilde{L}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |(Tx, y)|, \tag{1}$$

$$r_j = \max_{x \in \tilde{M}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Re}(Tx, y)|, \tag{2}$$

$$t_j = \max_{x \in \tilde{N}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Im}(Tx, y)|. \tag{2'}$$

Proof of Theorem 1. The proof of (2′) is similar to the proof of (2). So let us prove (1) and (2). First let us prove formulas (1) and (2) for $j = 1$, i.e.,

$$|\lambda_1| = \max_{x} \min_{\substack{y \\ (x,y)=1}} |(Tx, y)|, \tag{3}$$

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$$r_1 = \max_x \min_{\substack{y \\ (x,y)=1}} |\mathrm{Re}(Tx, y)|. \tag{4}$$

For a fixed x it is possible to write $Tx = \lambda x + z$, where $z \in x^\perp$, $x^\perp$ is the subspace of all vectors orthogonal to x and λ is a number. Thus $(Tx, y) = \lambda + (z, y)$. Let us represent y in the form $y = x|x|^{-2} + u$, $u \in x^\perp$. Here the condition $(x, y) = 1$ was taken into account. Finally one gets $(Tx, y) = \lambda + (z, u)$. From here is follows that

$$\min_{\substack{y \\ (x,y)=1}} |(Tx, y)| = \min_u |\lambda + (z, u)| = |\lambda| \quad \text{if} \quad z = 0$$
$$= 0 \quad \text{if} \quad z \neq 0, \tag{5}$$

$$\max_x \min_{\substack{y \\ (x,y)=1}} |(Tx, y)| = |\lambda_1|, \tag{6}$$

$$\min_{\substack{y \\ (x,y)=1}} |\mathrm{Re}(Tx, y)| = \min_u |\{\mathrm{Re}\,\lambda + \mathrm{Re}(z, u)\}|$$
$$= |\mathrm{Re}\,\lambda| \quad \text{if} \quad z = 0$$
$$= 0 \quad \text{if} \quad z \neq 0. \tag{7}$$

From (7) formula (4) follows. Suppose that formulas (1), (2) are proved for $j \leqslant n$. Then we can follow the same line of reasoning and take into account that all the eigenvalues of T in the subspace $\tilde{L}_n^{\perp\!\!\perp}$ have moduli $\leqslant |\lambda_{n+1}|$ and in the subspace $\tilde{M}_n^{\perp\!\!\perp}$, the operator T has $\max |\mathrm{Re}\lambda_j| = r_{n+1}$. For example,

$$\max_{x \in \tilde{L}_{j-1}^{\perp\!\!\perp}} \min_{\substack{y \in H \\ (x,y)=1}} |(Tx, y)| = \max_{\substack{x \in \tilde{L}_{j-1}^{\perp\!\!\perp} \\ Tx = \lambda x}} |\lambda| = |\lambda_j|.$$

Remark 1. There is a one-to-one correspondence between $\{M_j\}$ and $\{L_j\}$. Namely, take $M_i = L_{j(i)}$, where $j(i)$ is so chosen that the eigenvalue $\lambda_{j(i)}$ has $|\mathrm{Re}\,\lambda_{j(i)}| = r_i$.

Remark 2. If $\|T_n - T\| \to 0$ and T_n are compact, then $\lambda_j(T_n) \to \lambda_j(T)$, $\forall j$. Thus $|\lambda_j(T_n)| \to |\lambda_j(T)|$, $\mathrm{Re}\,\lambda_j(T_n) \to \mathrm{Re}\,\lambda_j(T)$. This fact permits an approximate calculation of the spectrum of T using in (1), (2) the operator T_n instead of T. One can take, e.g., n-dimensional operator T_n, i.e., dim range $T_n = n$.

Remark 3. Principles similar to (1), (2) were announced by Popov [1] for the case $\mathrm{Re}\,\lambda_j > 0$, but instead of $\tilde{M}_{n-1}^{\perp\!\!\perp}$ in (2) in [1] appears $\tilde{L}_{n-1}^{\perp\!\!\perp}$ which seems to be erroneous. The line of arguments in Popov [1] is quite different from the ones given above. In [1] L_j denotes the root space corresponding to λ_j and minimization over y in (1) is taken over a different set.

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The same arguments lead to variational principles for the spectrum of an unbounded linear operator with a discrete spectrum. Let operator A be a closed linear densely define operator on H, $\lambda_j = \lambda_j(A)$ its eigenvalues, $\sigma(A) = \{\lambda_j\}$, $\sigma(A)$ denotes the spectrum of A. Each λ_j is an isolated eigenvalue of finite algebraic multiplicity. The eigenvalues are ordered so that $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$. Let $r_1 \leqslant r_2 \leqslant \cdots$ be the moduli of real parts of the eigenvalues of A. Again we emphasize the fact that r_j is not necessarily equal to $|\mathrm{Re}\,\lambda_j|$, but it is possible to establish a one-to-one correspondence between $\{\lambda_j\}$ and $\{r_j\}$ by setting $|\mathrm{Re}\,\lambda_{j(i)}| = r_i$, as above. The variational principles read

$$|\lambda_j| = \min_{x \in \tilde{L}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |(Tx, y)|, \tag{8}$$

$$r_j = \min_{x \in \tilde{M}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Re}(Tx, y)|. \tag{9}$$

Here, as in Section 1, $\tilde{L}_j = \sum_{k=1}^{j} \dotplus L_k$ and K_k is the eigensubspace corresponding to λ_k. A similar meaning is ascribed to M_j. The proof of principles (8), (9) is similar to that of (1), (2).

Remark 4. For the moduli of the imaginary parts of the eigenvalues $t_1 \leqslant t_2 \leqslant \cdots$, the following formula holds:

$$t_j = \min_{x \in \tilde{N}_{j-1}^{\amalg}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Im}(Tx, y)|. \tag{9'}$$

and can be proved similarly.

REFERENCE

1. P. POPOV, Variational principles for spectrum of nonselfadjoint operators, *Soviet Math. Dokl.* **208** (1973), 290–292.