

Powers of Ideals Generated by Weak d -Sequences

CRAIG HUNEKE*

*Department of Mathematics, University of Michigan,
Ann Arbor, Michigan 48109*

Communicated by D. Buchsbaum

Received September 11, 1979

INTRODUCTION

This paper is concerned with establishing a general method to answer questions concerning the analytic behavior of an ideal I in a commutative Noetherian ring R . By "analytic" we mean the analysis of R/I^n for $n \geq 1$. Two specific questions are the most important: If $I = P$ is prime, when is $P^{(n)} = P^n$? (Here $P^{(n)}$ is the n th symbolic power of P .) Secondly, if R is local, what are $\text{depth } R/I^n$ for arbitrary n ? If I^n has finite projective dimension, then of course this is the same question as that of computing the projective dimension of R/I^n .

Even in specific cases which have been extensively studied these questions are extremely hard to answer. The literature concerning symbolic powers is fairly extensive; see for example, [1], [8], [14], [17], or [23]. If K is a field and $X = (x_{ij})$ is a generic $r \times s$ matrix ($r \leq s$) over K , then if we let \mathcal{A} = ideal generated by the maximal minors of X , then Hochster (unpublished) showed in characteristic 0 that $\mathcal{A}^{(n)} = \mathcal{A}^n$ for all n . Recently DeConcini *et al.* [8, 9] removed the assumptions on characteristic. We recover this result. Indeed, we find considerably more. Buchsbaum has conjectured that $\text{depth } R/\mathcal{A}^n$ ($R = K[x_{ij}]_{(x_{ij})}$) should be independent on s if $n \geq 2$ and Robbiano [21] has conjectured that the depths are independent on n if $n \geq 2$. We are able to show

COROLLARY 3.1. *For all $n \gg 0$, $\text{depth } R/\mathcal{A}^n = r^2 - 1$ and is hence independent on s and n for n large.*

In the $2 \times n$ case we recover a result of Robbiano [21].

* Partially supported by NSF.

THEOREM 4.4. $\text{depth } R/\mathcal{A}^2 = \text{depth } R/\mathcal{A}^3 = \dots = \text{depth } R/\mathcal{A}^n = \dots = 3.$

In general we show if $\text{depth } R/\mathcal{A}^2 = r^2 - 1$ in the $r \times s$ case, then all the depths are equal to $r^2 - 1$.

The maximal minors of a generic $r \times s$ matrix are an example of a weak d -sequence. d -Sequences have been studied in [15–17] to answer the same type of questions, and also to study symmetric algebras. We recall the definition:

If R is a commutative ring, then a system of elements x_1, \dots, x_n in R is said to be a d -sequence if

- (1) $(x_1, \dots, x_n) \neq R$ and x_i is not in the ideal generated by the rest of the x_j ,
- (2) $((x_1, \dots, x_i): x_{i+1}x_k) = ((x_1, \dots, x_i): x_k)$ for $k \geq i + 1$ and $i \geq 0$.

d -Sequences are a type of weak R -sequence and in fact are examples of “weak” R -sequences as in [25] and relative regular sequences as in [12]. The main fact concerning them was the ability to reduce any homogeneous relation on them to a linear relation. Specifically the following was shown in [16].

THEOREM [16]. *Let R be a commutative ring and x_1, \dots, x_n a d -sequence. Let q be the ideal in $R[T_1, \dots, T_n]$ generated by all forms $F(T_1, \dots, T_n)$ such that $F(x_1, \dots, x_n) = 0$. Let q_1 be the ideal in $R[T_1, \dots, T_n]$ generated by all linear forms $L(T_1, \dots, T_n)$ such that $L(x_1, \dots, x_n) = 0$. Then $q = q_1$.*

A weak d -sequence (see Definition 1.1) is basically a sequence of elements which behave well in terms of their linear and quadratic relations. Weak d -sequences $\{x_\alpha\}$ are indexed by a partially ordered set H ; we define an H -ideal I to be an ideal generated by $\{x_\lambda \mid \lambda \in A \subset H\}$ such that if $\beta \in A$ and $\alpha \leq \beta$, then $\alpha \in A$. An ideal J is said to be *related* to the weak d -sequence $\{x_\alpha\}$ if J is of the form $((I : x_\alpha), \mathcal{A})$, where I is an H -ideal containing all x_β for $\beta < \alpha$ but not x_α , and \mathcal{A} is the ideal generated by all the x_α . Clearly the related ideals depend only on the linear syzygies of the weak d -sequence. Our main result is:

THEOREM 2.2. *Let R be a commutative ring and $\{x_\beta\}$ a weak d -sequence in R on the partially ordered set H . Let \mathcal{A} be the ideal generated by all the $\{x_\beta\}$. Then for all $n \geq 1$, R/\mathcal{A}^n has a filtration, $R/\mathcal{A}^n = M_0 \supset \dots \supset M_d = (0)$, of R -modules such that M_k/M_{k+1} is isomorphic to R/J , where J is a related ideal of $\{x_\beta\}$.*

It is clear from this that information concerning the powers of \mathcal{A} can be deduced solely from the linear relations of the weak d -sequence.

Of course, studying weak d -sequences would not be useful unless there were many and varied examples of them. We show there are indeed many

examples, including the maximal minors of a generic $r \times s$ matrix, the highest-order Pfaffians of a generic alternating odd-order matrix, various Veronesean ideals, almost complete intersections, and many particular examples such as the example $k[u^2, u^3, uv, v]$ considered in [14]. The number of examples is too great to be listed here, but we note one result, some of which was done independently in [1].

THEOREM 3.2. *Let k be a field, X a generic $(2n + 1) \times (2n + 1)$ alternating matrix and \mathcal{A} the ideal generated by the Pfaffians of rank $2n$. Then \mathcal{A} is prime, $\mathcal{A}^{(n)} = \mathcal{A}^n$ for all n , and if $A = k[x_{ij}]_{(x_{ij})}$, then $\inf_m \text{depth}(A/\mathcal{A}^m) = (2n + 1)(n - 1)$. Further, once $\text{depth} A/\mathcal{A}^m = (2n + 1)(n - 1)$, $\text{depth} A/\mathcal{A}^k = (2n + 1)(n - 1)$ for all $k \geq m$.*

The largest class of weak d -sequences (outside of d -sequences) come from algebras with straightening law. These have been recently defined and studied by DeConcini *et al.* [8, 9] and we rely heavily upon their work. We recall their definition:

Let H be a finite partially ordered set (poset) and let A be a ring with $H \subset A$. We say a monomial $m = \alpha_1 \cdots \alpha_k$ of elements of H is *standard* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. We will write $m \leq n = \beta_1 \cdots \beta_e$ if either $\alpha_1 \cdots \alpha_k$ is an initial subsequence of $\beta_1 \cdots \beta_e$ or if, for the first i with $\alpha_i \neq \beta_i$, we have $\alpha_i < \beta_i$.

DEFINITION [6]. Let R be a ring, A a commutative R -algebra and H a finite poset. A straightening law on H for A is a set of distinct algebra generators $\{\bar{\alpha} \mid \alpha \in H\}$ for A over R , indexed by H such that any monomial $n = \bar{\alpha}_1 \cdots \bar{\alpha}_k$ in A can be written *uniquely* as an R -linear combination of standard monomials m_i with $m_i \leq n$.

We show,

PROPOSITION 1.3. *Let A be an algebra with straightening law on H , and $I \subset H$ a subset such that if $\beta \in I$ and $\alpha \leq \beta$ then $\alpha \in I$. Let \bar{I} be the ideal generated by all $\bar{\alpha}$, $\alpha \in I$. Suppose if $\alpha, \beta \in I$ are noncomparable and $\bar{\alpha}\bar{\beta} = \sum r_i \bar{\gamma}_i \bar{m}_i$ is the straightening of $\bar{\alpha}\bar{\beta}$ with $\gamma_i < \alpha$, $\gamma_i < \beta$, then $\bar{m}_i \in I$. Then $\{\bar{\alpha} \mid \alpha \in I\}$ form a weak d -sequence.*

We discuss several of the many examples of weak d -sequences this gives in Section 1 and throughout this paper.

The importance of $\text{depth } R/I^n$ has been apparent for some time. The depth I/I^2 plays a central role in the deformations of R/I , while the depth R/I^n for all n have a great deal to do with the analytic spread, $l(I)$, of I . The analytic spread was first studied by Northcott and Rees [20]; if R is local with maximal ideal m , the analytic spread I , denoted $l(I)$, is defined to be the degree +1 of the polynomial over the rationals which gives $\dim p^n/mp^n$ for large n . L. Burch showed:

PROPOSITION 3.0 [5]. *Let R be a Noetherian local ring, I and ideal. Then*

$$\dim R - \inf \text{depth } R/I^n \geq l(I).$$

See Brodmann [3] for an improvement of this.
Cowsik and Nori used this to show the following result.

THEOREM [6]. *If I is a self-radical ideal in a Cohen–Macaulay local ring R such that*

- (1) R_p is regular for each minimal prime p containing I , and
- (2) R/I^n is Cohen–Macaulay for every n ,

then I is a complete intersection.

Thus the behavior of $\text{depth } R/I^n$ does indeed reflect the nature of the ideal I .

We now describe the contents of this paper more precisely.

Section 1 deals with the definition of a weak d -sequence and the basic properties which such sequences have. Then we turn to the question of examples. We list these in Section 1 since we feel that the wealth of examples is the most important consideration.

Section 2 proves the main theorem listed above (Theorem 2.2) and the rest of the section deals with corollaries of this result.

Section 3 applies the results of Section 2 to many of the examples of Section 1, analyzing the depths and the symbolic powers. Besides the new results we can recover virtually all of the known examples of prime ideals whose symbolic powers are equal to their regular powers, but we only deal with the major classes.

Section 4 is concerned with the behavior of weak d -sequences under ring extension and concentrates in particular on the map $R \rightarrow R/I$, especially when I is a complete intersection. We show how some “almost” generic primes arise in such a way and we evaluate their powers. We use this to recover the result of Robbiano listed above (Theorem 4.4).

The reliance on the theory of algebras with straightening law is heavy throughout the paper; this theory basically allows one to actually evaluate the syzygies of weak d -sequences. This author extends his gratitude to David Eisenbud for allowing him to see a preliminary draft. The results on algebras with straightening laws are denoted throughout the paper by capital letters, i.e., Proposition 4A, etc.

For terminology and basic results, the reader is referred to Matsumura [19]. “Ring” will always mean commutative with identity. “Local ring” will mean a Noetherian ring with unique maximal ideal.

1. WEAK d -SEQUENCE

In this section we introduce the concept of a weak d -sequence and prove several elementary remarks concerning them. The bulk of this section is devoted to recognizing a wide class of examples as weak d -sequences to which we will apply the results of the latter sections.

Let H be a finite partially ordered set. Elements of H will always be denoted by Greek letters. Suppose $\{x_\alpha\}$ is a set of elements in a commutative ring R indexed by α in H . If I is an ideal of R , I is said to be an H -ideal if I is generated by some subset $\{x_\lambda \mid \lambda \in A\}$ of the $\{x_\alpha\}$, and A has the property that β in A and $\alpha \leq \beta$ implies α in A . For every $\alpha \in H$, I_α will henceforth denote the ideal generated by all x_β , $\beta < \alpha$. Clearly I_α is an H -ideal. If J is an ideal of R , J^* will always denote the ideal generated by all x_β , $x_\beta \in J$. Finally the ideal generated by all the x_α will always be denoted by \mathcal{A} .

DEFINITION 1.1. Let $R, H, \{x_\alpha\}$ be as above. We will say the $\{x_\alpha\}$ form a *weak d -sequence* if the following hold. Suppose I is an H -ideal such that $I_\alpha \subseteq I$, $x_\alpha \notin I$. Then

- (1) $(I : x_\alpha)^*$ is an H -ideal.
- (2) $(I : x_\alpha) \cap \mathcal{A} = (I : x_\alpha)^*$.
- (3) If $x_\beta \in (I : x_\alpha)$, then $x_\alpha x_\beta \in I\mathcal{A}$.
- (4) If $x_\alpha \notin (I : x_\alpha)$, then $(I : x_\alpha) = (I : x_\alpha^2)$.

Of these properties, (3) may be seen as the essence of the weak d -sequence. It allows us to control the powers of ideals generated by a weak d -sequence.

PROPOSITION 1.1. Let $\{x_\beta\}$ be a weak d -sequence in R indexed by a poset H . Suppose I is an H -ideal, and suppose α is a minimal element of H . Denote by “—” the map from R to R/I in part (a) and from R to $R/(0 : x_\alpha)$ in part (b).

- (a) $\{\bar{x}_\beta \mid x_\beta \notin I\}$ form a weak d -sequence in R/I .
- (b) $\{\bar{x}_\beta \mid \beta \in H\}$ are a weak d -sequence in $R/(0 : x_\alpha)$.

Proof. First we do (a). Let $H' = H - \{\beta \mid x_\beta \in I\}$, and let \bar{J} be an H' -ideal in R/I . Since I is an H -ideal, if we lift \bar{J} back to J , an ideal of R , we see J is an H -ideal. If $\bar{I}_\beta \subseteq \bar{J}$, then $I_\beta \subseteq J$ and $(\bar{J} : \bar{x}_\beta) = (\bar{J} : x_\beta)$. Since $(J : x_\beta)^*$ is an H -ideal of R , $(\bar{J} : \bar{x}_\beta)^*$ is an H' -ideal of R/I . If $\bar{a} \in \bar{\mathcal{A}} \cap (\bar{J} : \bar{x}_\beta)$ then lifting \bar{a} to $a \in R$ we see $a \in \mathcal{A} \cap (J : x_\beta) \subseteq (J : x_\beta)^*$. Hence $a \in (\bar{J} : \bar{x}_\beta)^*$. If $\bar{x}_\sigma \in (\bar{J} : \bar{x}_\beta)$ then $x_\sigma x_\beta \in J$ and so $x_\sigma x_\beta \in J\mathcal{A}$. Thus $\bar{x}_\sigma \bar{x}_\beta \in \bar{J}\bar{\mathcal{A}}$. Finally, if $\bar{x}_\beta \notin (\bar{J} : \bar{x}_\beta)$ then $x_\beta \notin (J : x_\beta)$ and so $(\bar{J} : \bar{x}_\beta) = (\bar{J} : x_\beta) = (\bar{J} : x_\beta^2) = (\bar{J} : \bar{x}_\beta^2)$.

We now turn to (b). First we prove a lemma.

LEMMA 1.1. *Suppose I is an H -ideal, $I_\alpha \subset I$, $x_\alpha \notin I$ and J is an H -ideal such that $I_\beta \subseteq J + (I : x_\alpha)$, but $x_\beta \notin J + (I : x_\alpha)$. Then*

1. $((I : x_\alpha), J) : x_\beta = (((I : x_\alpha)^*, J) : x_\beta)$.
2. $((I : x_\alpha), J) : x_\beta \cap \mathcal{A} = (((I : x_\alpha)^*, J) : x_\beta)^*$.

Proof. Suppose ax_β is in $(I : x_\alpha) + J$. Then there is a $z \in J$ such that $ax_\beta - z$ is in $(I : x_\alpha) \cap \mathcal{A} = (I : x_\alpha)^*$. Hence $ax_\beta \in (I : x_\alpha)^* + J$, which is an H -ideal which contains I_β . (Notice $I_\beta \subseteq ((I : x_\alpha) + J) \cap \mathcal{A} = ((I : x_\alpha)^* + J)$ follows from 1).

Now let $\bar{R} = R/(0 : x_\alpha)$. Let \bar{I} be an H -ideal in \bar{R} which contains \bar{I}_β but which does not contain \bar{x}_β . Then if $\bar{r} \in (\bar{I} : \bar{x}_\beta)$, lift back to R to obtain $r \in (((0 : x_\alpha), I) : x_\beta)$. By Lemma 1.1, since $I_\beta \subseteq (0 : x_\alpha) + I$, we see $r \in ((0 : x_\alpha)^*, I)$. But then $(\bar{I} : \bar{x}_\beta)^* = (0 : x_\alpha) + ((0 : x_\alpha)^*, I)^*$ which is an H -ideal. Part (2) of Definition 1.1 follows immediately from Lemma 1.1. For (3), suppose $\bar{x}_r \in (\bar{I} : \bar{x}_\beta)$. Then $x_r \in (I, (0 : x_\alpha) : x_\beta) \cap \mathcal{A} = ((I, (0 : x_\alpha)^*) : x_\beta)^*$ and since $(0 : x_\alpha)^*$ is an H -ideal, $I + (0 : x_\alpha)^*$ is an H -ideal and so $x_r x_\beta \in (I + (0 : x_\alpha)^*)(\mathcal{A})$. Thus $\bar{x}_r \bar{x}_\beta \in \bar{I} \mathcal{A}$. Finally, if $\bar{x}_\beta \notin (\bar{I} : \bar{x}_\beta)$, then $x_\beta \notin (((0 : x_\alpha)^*, I) : x_\beta)$ and so $((0 : x_\alpha)^*, I) : x_\beta = (((0 : x_\alpha)^*, I) : x_\beta^2)$. If $\bar{r} \in (\bar{I} : \bar{x}_\beta^2)$, then $rx_\beta^2 \in ((0 : x_\alpha), I) \cap \mathcal{A} = ((0 : x_\alpha)^*, I)$ so that $r \in (((0 : x_\alpha)^*, I) : x_\beta^2) = (((0 : x_\alpha)^*, I) : x_\beta)$ as is needed.

This proposition and Lemma 1.1 are all that is needed to prove the main result (Theorem 2.2). We now give another criterion for a weak d -sequence.

PROPOSITION 1.2. *Suppose R is a commutative ring, H a partially ordered set, and $\{x_\alpha\}$ a collection of elements of R indexed by H . Suppose (1) and (3) of Definition 1.1 hold for the $\{x_\alpha\}$. If I is an H -ideal and $I_\alpha \subset I$, $x_\alpha \notin I$, suppose $(I : x_\alpha) \subseteq (I : x_\beta) + (I : x_\alpha)^*$ whenever $x_\beta \notin I$, and suppose $(I : x_\alpha) = (I : x_\alpha^2)$ for all such ideals. Then the $\{x_\alpha\}$ are a weak d -sequence.*

Proof. Obviously we must show $(I : x_\alpha) \cap \mathcal{A} = (I : x_\alpha)^*$. Suppose $w = \sum_\lambda r_\lambda x_\lambda \in (I : x_\alpha)$. If all the $x_\lambda \in (I : x_\alpha)$ then $w \in (I : x_\alpha)^*$ and we are done. If not, induct on the greatest $\lambda_0 \in A$ such that λ_0 is in A but is not in $(I : x_\alpha)^*$. By assumption, $(I : x_\alpha) \subseteq (I : x_{\lambda_0}) + (I : x_\alpha)^*$ and so $w \in (I : x_{\lambda_0}) + (I : x_\alpha)^*$. Then $w = y + a$, where $y \in (I : x_{\lambda_0})$ and $a \in (I : x_\alpha)^*$, and so $w - a \in (I : x_{\lambda_0})$. Let $\theta = A - \{\lambda_0\}$. Then $r_{\lambda_0} x_{\lambda_0}^2 \in \langle x_\beta \mid \beta \in \theta + (I : x_\alpha)^* \rangle$.

Let J be the least H -ideal containing all the x_β such that $\beta \in \theta$. Then

$$r_{\lambda_0} \in ((J, (I : x_\alpha)^*) : x_{\lambda_0}^2).$$

Hence, $r_{\lambda_0} \in ((J, (I : x_\alpha)^*) : x_{\lambda_0})$ and so $r_{\lambda_0} x_{\lambda_0} \in J + (I : x_\alpha)^*$. Thus we may remove any greatest λ_0 in A from the expression of w as a sum of the x_β . As $|H|$ is finite we finally obtain $w \in (I : x_\alpha)^*$ as required.

We recall the definition of a d -sequence.

DEFINITION [15]. A system of elements x_1, \dots, x_n in a commutative ring R is said to be a d -sequence if:

(1) $(x_1, \dots, x_n) \neq R$ and x_i is not in the ideal generated by the rest of the x_i .

(2) If $i \geq 0$, then for all $k \geq i + 1$, $((x_1, \dots, x_i) : x_{i+1} x_k) = ((x_1, \dots, x_i) : x_k)$. We set $x_0 = 0$.

COROLLARY 1.1. If x_1, \dots, x_n is a d -sequence, then it is a weak d -sequence.

Proof. Linearly order the x_i by $x_i \leq x_j$ if $i \leq j$. Then since x_j is not in $(x_1, \dots, x_i : x_{i+1})$ if $j \geq i + 1$, we see that (1) and (3) of Definition 1.1 are trivial. Thus by Proposition 2.1, it is enough to verify $(I : x_\alpha) \subseteq (I : x_\beta) + (I : x_\alpha)^*$. But if I is an H -ideal containing I_α but not x_α , then I must be of the form $((x_1, \dots, x_i) : x_{i+1})$ and if $x_j \notin (x_1, \dots, x_i)$ then $((x_1, \dots, x_i) : x_{i+1}) \subseteq (x_1, \dots, x_i) : x_{i+1} x_j = ((x_1, \dots, x_i) : x_j)$.

This gives us our first large class of weak d -sequences. We list some examples of d -sequences below; see [15, 17] for the complete details.

1.1. Any R -sequence is a d -sequence.

1.2. If $X = (x_{ij})$ is a generic $n \times (n + 1)$ matrix, then the maximal minors of X form a d -sequence. (We will show below that the maximal minors of a generic $r \times s$ matrix form a weak d -sequence.)

1.3. If R is a Cohen–Macaulay ring and p is a prime ideal of height n generated by $n + 1$ elements (so that p is an almost complete intersection) such that R_p is regular, then p is generated by a d -sequence.

1.4. R is a local Buchsbaum ring (see [24]) if and only if every system of parameters forms a d -sequence.

1.5. Let R be a regular local ring and p a prime such that R/p is Gorenstein. If x_1, \dots, x_k is a maximal R -sequence in p which generate it generically, then $((x_1, \dots, x_k) : p)$ is generated by a d -sequence.

1.6. Any ideal in an integrally closed domain minimally generated by two elements can be generated by two elements which are a d -sequence.

1.7. Let $X = (x_{ij})$ be a generic $r \times s$ matrix and let $I_t(X)$ be the ideal generated by all $t \times t$ minors of X . Then the images of x_{11}, \dots, x_{1s} in $k[x_{ij}]/I_t(X)$ form a d -sequence.

1.8. The defining equations of the ideal $p \subseteq k[x, y, z]$ parametrically given by $k[t^{n_1}, t^{n_2}, t^{n_3}]$ form a d -sequence.

1.9. The defining ideal of the Veronese $V_{2,3}$ is given by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

These form a d -sequence.

1.10. The elements xz , yw , $xw + yz$ defining (up to radical) the ideal $(x, y) \cap (z, w)$ form a d -sequence.

1.11. If a_1, \dots, a_n are not zero divisors in R , then the images of X_1, \dots, X_n in $R[X_1, \dots, X_n]/(a_1X_1 + \dots + a_nX_n)$ form a d -sequence.

1.12. If I is an ideal generated by a d -sequence x_1, \dots, x_n , then the images of the x_i in the first graded piece in both $S_R(I)$ (the symmetric algebra of I) and $gr_R(I)$ (the graded algebra of I) form a d -sequence.

1.13. If $R = k[X, Y, Z]/(X^2 - YZ)$, and I is the ideal in R generated by the images of X and Z , then their images in the first graded piece of $S_R(I)$ form a d -sequence.

1.14. If $R = k[X, Y, Z, W]/(X^nY - Z^nW)$ then the images of X and Z in $S_R(I)$ form a d -sequence, where $I = (X, Z)$.

These are a sample of d -sequences, and hence of weak d -sequences. Example 1.12 can be used repeatedly to give rather complicated examples of d -sequences (see [16]).

Another large class (and the most important class) of weak d -sequences comes from algebras with straightening law, studied by DeConcini *et al.* [9]. We will give the basic definitions below.

Let H be a finite partially ordered set (poset) and let A be a ring with $H \subset A$. We say a monomial $m = \alpha_1 \cdots \alpha_k$ of elements of H is *standard* if $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$. We will write $m \leq n = \beta_1 \cdots \beta_l$ if either $\alpha_1 \cdots \alpha_k$ is an initial subsequence of $\beta_1 \cdots \beta_l$ or if, for the first i with $\alpha_i \neq \beta_i$, we have $\alpha_i < \beta_i$.

DEFINITION [9]. Let R be a ring, A a commutative R -algebra and H a finite poset. A straightening law on H for A is a set of distinct algebra generators $\{\bar{\alpha} \mid \alpha \in H\}$ for A over R , indexed by H such that any monomial $n = \bar{\alpha}_1 \cdots \bar{\alpha}_k$ in A can be written *uniquely* as an R -linear combination of standard monomials m_i with $m_i \leq n$.

An ideal $I \subset H$ of an algebra A with straightening law is any subset of H such that if $\alpha \in I$ and $\beta \leq \alpha$, then $\beta \in I$. The following propositions have been shown by DeConcini *et al.*:

PROPOSITION 1.A. *Let A be an algebra with straightening law (ASL) on H over R . Suppose I is an ideal of H and denote by \bar{I} the ideal in A*

generated by all $\bar{\alpha}$, $\alpha \in I$. Then \bar{I} is a free R -module with basis consisting of the set of standard monomials beginning with an element of I . Further, A/\bar{I} is an ASL on $H - I$.

PROPOSITION 1B. Let A be an ASL on H over R and suppose $\bar{\alpha}$ corresponds to a minimal element $\alpha \in H$. If $I \subset H$ is the set of all $\beta \in H$, $\beta \not\geq \alpha$, then I is an ideal and $\bar{I} = (0 : \bar{\alpha})$.

There are many examples of algebras with straightening law. We list several below which are found in [7, 9].

1.15. Let $R = k[x_{ij}]$, $X = (x_{ij})$ a generic $r \times s$ matrix. Then R is an ASL on the minors of X with the following order: any minor of X is given by an expression $(j_1, \dots, j_k | i_1, \dots, i_k)$, where $1 \leq j_k < \dots < j_1 \leq r$, $1 \leq i_1 < \dots < i_k \leq s$ and the minor this represents is the $k \times k$ minor determined by the j_1, \dots, j_k th rows and the i_1, \dots, i_k th columns. The partial order is given by

$$(j_1, \dots, j_k | i_1, \dots, i_k) \leq (j'_1, \dots, j'_l | i'_1, \dots, i'_l)$$

if and only if $k \geq l$ and $j_1 \leq j'_1, \dots, j_l \leq j'_l$, $i_1 \leq i'_1, \dots, i_l \leq i'_l$. The straightening law was first given by Doubilet *et al.* [10].

1.16. Let X be a generic $r \times s$ matrix and k be a field. The coordinate ring of the Grassman variety $G_{r,s}$ is an ASL on the maximal minors of X . Any maximal minor can be represented by an r -tuple $[i_1, \dots, i_r]$, where $1 \leq i_1 < \dots < i_r \leq s$ and this is the minor determined by the i_1, \dots, i_r th columns. The straightening law is the standard Plucker relations on the maximal minors.

1.17. If H is a partially ordered set and we let $\{x_\alpha\}$ be a set of indeterminates indexed by $\alpha \in H$. Then the ring $R[x_\alpha]/J$ is an algebra with straightening law on the $\{x_\alpha\}$, ordered by H , where R is any commutative ring and J is the ideal generated by $\{x_\alpha x_\beta \mid \alpha \text{ and } \beta \text{ are incomparable}\}$.

1.18. **THE PFAFFIANS.** If R is a commutative ring and F a finitely generated free R -module, a map $f: F^* \rightarrow F$ is said to be alternating if with respect to some (and therefore every) basis and dual basis of F and F^* the matrix of f is skew symmetric with all diagonal entries zero. If rank F is even and $f: F^* \rightarrow F$ is alternating then $\det(f)$ is a square of a polynomial function of the entries of the matrix for f , called the Pfaffian of F . In general if F has odd rank $2n + 1$, then the determinant of the matrix resulting from f by deleting the i th column and i th row is a square of a polynomial function of the corresponding entries and the ideal generated by these Pfaffians will denote $Pf_{2n}(f)$.

The importance of the Pfaffians comes from the structure theorem of Buchsbaum and Eisenbud.

THEOREM [4]. *Let R be a Noetherian local ring with maximal ideal J .*

(1) *Let $n \geq 3$ be an odd integer and let F be a free R -module of rank n . Let $f : F^* \rightarrow F$ be an alternating map whose image is contained in JF . Suppose $Pf_{n-1}(f)$ has grade 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal, minimally generated by n elements.*

(2) *Every Gorenstein ideal of grade 3 arises as in (1).*

For the remainder of this example we fix a field k , and a generic skew-symmetric matrix with zeros down the diagonal, X .

$$X = \begin{pmatrix} 0 & x_{12} & \cdots & x_{12n} \\ -x_{12} & 0 & & \\ \vdots & & \ddots & \\ -x_{12n} & \cdots & -x_{2n-12n} & 0 \end{pmatrix}.$$

For convenience we will make the convention that $x_{ij} = -x_{ji}$ if $i > j$ and $x_{ij} = 0$ if $i = j$.

In [7] it was proved that the set of all Pfaffians of X (i.e., the square roots of the determinants of any k columns i_1, \dots, i_k and rows i_1, \dots, i_k) gives a straightening law on $k[x_{ij}]$. The partial order is as follows: any Pfaffian is determined by the rows (and corresponding columns) and so we can use $[i_1, \dots, i_{2k}]$ to represent the Pfaffians give by the square root of the minor of the i_1, \dots, i_{2k} columns and rows of X . Say $[i_1, \dots, i_{2k}] \geq [j_1, \dots, j_{2l}]$ if and only if $l \leq k$ and $i_s \leq j_s$ for $s = 1, \dots, 2l$.

The importance of algebras with straightening law for our purposes comes from the following result.

If R is an ASL on H and $\alpha \in H$ we denote by $\bar{\alpha} \in R$ the element of R corresponding to α .

PROPOSITION 1.3. *Let A be an ASL on H over R and suppose \mathcal{A} is an ideal of H such that if $\alpha, \beta \in \mathcal{A}$ are noncomparable and $\bar{\alpha}\bar{\beta} = \sum r_i \bar{\gamma}_i \bar{m}_i$ is the straightening, where $\gamma_i < \alpha$ and $\gamma_i < \beta$, then $\bar{m}_i \in \mathcal{A}$. Then $\{\bar{\alpha} \mid \alpha \in \mathcal{A}\}$ form a weak d -sequence.*

Proof. Let \bar{I} be an H -ideal containing I_α but not $\bar{\alpha}$, and let $I \subset H$ be the set of all $\beta \in H$ such that $\bar{\beta} \in \bar{I}$. Then I is an ideal of H , and by the proposition noted above, $(\bar{I} : \bar{\alpha})$ is an H -ideal of R . Thus (1) of Definition 1.1 is true. For (2), we note the following: if I and $J \subset H$ are ideals, then $\bar{I} \cap \bar{J}$ is an H -ideal generated by those $\alpha \in I \cap J$. For clearly $\overline{I \cap J} \subseteq \bar{I} \cap \bar{J}$. To prove

the converse, suppose $r \in \bar{I} \cap \bar{J}$; since $r \in \bar{I}$, $r = \sum s_i \bar{\alpha}_i \bar{m}_i$ with $\alpha_i \in I$ and all the terms standard (see above). Likewise, $\bar{\beta} \in \bar{J}$ implies $r = \sum t_j \bar{\beta}_j \bar{q}_j$ with $\beta_j \in J$ and this expression is standard. As the standard monomials are a free basis for A over R , we see that each $\bar{\alpha}_i \bar{m}_i$ is some $\bar{\beta}_j \bar{m}_j$ and $s_i = t_j$. This implies either $\alpha_i \leq \beta_j$ or $\beta_j \leq \alpha_i$. Since I and J are ideals of H , either $\alpha_i \in I \cap J$ or $\beta_j \in I \cap J$. This shows $r \in \overline{I \cap J}$. Now $\mathcal{A} \cap (\bar{I} : \bar{\alpha})$ is an intersection of two H -ideals and hence is equal to the ideal generated by all $\bar{\alpha}$, $\alpha \in \mathcal{A} \cap \{\beta \mid \beta \not\leq \alpha\}$ (see above). This is precisely $(\bar{I} : \bar{\alpha})^*$.

If $\bar{\beta} \in (\bar{I} : \bar{\alpha})$ then again by the above proposition, $\beta \not\leq \alpha_j$. Let $\bar{\alpha}\bar{\beta} = \sum r_i \bar{\gamma}_i \bar{m}_i$ be the straightening, where $\gamma_i < \alpha, \beta$. By assumption $\bar{m}_i \in \mathcal{A}$. Since $\bar{\gamma}_i \in I_\alpha \subset \bar{I}$ this shows $\bar{\alpha}\bar{\beta} \in \bar{I}\mathcal{A}$. Finally, if I is an H -ideal, then I is self-radical. Hence $(I : x) = (I : x^2)$ for every $x \in A$.

This proposition provides us with many weak d -sequences; we list below those that will appear in the rest of this paper.

1.19. Let X be a generic $r \times s$ matrix, R a commutative ring. Then the maximal minors of X form a weak d -sequence.

Proof. By 1.13 above $R[x_{ij}]$ is an ASL on all the minors of X , and it is easy to see that the set of all maximal minors is an ideal of the partially order set H of 1.13. It remains to show that the straightening of two noncomparable maximal minors $\bar{\alpha}$ and $\bar{\beta}$ has the form $\sum r_i \bar{\gamma}_i \bar{m}_i$, where $\gamma_i < \alpha, \beta$ and $\bar{m}_i \in \mathcal{A}$ = the ideal generated by all the maximal minors. But the straightening law on these are the Plucher relations which are quadratic, involving one $\gamma_i < \alpha, \beta$ and another maximal minor. Thus, Proposition 1.3 shows these form a weak d -sequence.

1.20. Let X be a generic skew-symmetric matrix as in 1.15. Then the maximal Pfaffians of X form a weak d -sequence.

Proof. This follows at once by examining the straightening law (see [7]).

1.21. Let R be any commutative ring and X a generic $r \times s$ matrix. Let $I_t(X) = I$ be the ideal generated by all $t \times t$ minors of X , and set $A = R[x_{ij}]/I$. If H is the partially ordered set of minors of X minus those of degree $\geq t$, then A is an ASL on H over R (by 1.13 and the propositions). Let \mathcal{A} be the ideal of A generated by all the $(t-1) \times (t-1)$ minors of the first $t-1$ columns. Then those minors form a weak d -sequence.

Proof. These are an ideal of H so by Proposition 1.3 it is enough to check that if $\bar{\alpha}$ and $\bar{\beta}$ are two noncomparable $(t-1) \times (t-1)$ minors of the first $t-1$ columns of X then $\bar{\alpha}\bar{\beta} \in \bar{I}\mathcal{A}$, where $I \subset H$ is an ideal and $I_\alpha \subseteq \bar{I}$. But the straightening law of A is inherited from that of $R[x_{ij}]$ and there, $\bar{\alpha}\bar{\beta}$ straightens modulo I to quadratic terms $\bar{\gamma}\bar{\sigma}$, where $\bar{\gamma}, \bar{\sigma}$ are $(t-1) \times (t-1)$ minors of the first $t-1$ columns and $\gamma < \alpha, \gamma < \beta$.

Remark. If we let \mathcal{J} be the ideal of A generated by all the $(t-1) \times (t-1)$ minors of X , then these form a weak d -sequence.

1.22. In [14] Hochster showed that if $R = K[X, Y, Z, W]$ and p is the prime ideal in R defined parametrically by $k[u^2, u^3, uv, v]$ then $gr_p(R)$ is a Cohen–Macaulay domain even though R/p is not Cohen–Macaulay. The ideal p is defined by four equations, $p = (Y^2 - X^3, XZ - YW, Z^2 - XW^2, YZ - X^2W)$ and is generated by the 2×2 minors of

$$\begin{pmatrix} X & W & Z & Y \\ Y & Z & XW & X^2 \end{pmatrix}.$$

(Two of the 2×2 minors are redundant.) We impose the order given by this representation of the generators as 2×2 minors and show these form a weak d -sequence. Specifically, we set $XZ - YW < X^2W - YZ$ and $X^2W - YZ < XW^2 - Z^2$ and $X^2W - YZ < X^3 - Y^2$. Let us set $a_1 = XZ - YW$, $a_2 = X^2W - YZ$, $a_3 = XW^2 - Z^2$, and $a_4 = X^3 - Y^2$. We compute the relevant ideals:

$$\begin{aligned} (0 : a_1) &= (0), \\ (a_1 : a_2) &= (a_1), \\ (a_1, a_2 : a_3) &= (X, Y), \\ ((a_1, a_2, a_4) : a_3) &= (X, Y), \\ ((a_1, a_2) : a_4) &= (W, Z), \\ ((a_1, a_2, a_3) : a_4) &= (W, Z). \end{aligned}$$

Also,

$$\begin{aligned} (0 : a_1)^* &= (0), \\ (a_1 : a_2)^* &= (a_1), \\ (a_1, a_2 : a_3)^* &= (a_1, a_2, a_4 : a_3)^* = (a_1, a_2, a_4), \\ (a_1, a_2 : a_4)^* &= ((a_1, a_2, a_3) : a_4)^* = (a_1, a_2, a_3). \end{aligned}$$

Each of these is an H -ideal and so (1) of Definition 1.1 is satisfied. For (2), we must check

- (a) $p \cap (0 : a_1) = (0)$,
- (b) $p \cap (a_1 : a_2) = (a_1)$,
- (c) $p \cap (a_1, a_2 : a_3) = p \cap (a_1, a_2, a_4 : a_3) = (a_1, a_2, a_4)$,
- (d) $p \cap (a_1, a_2 : a_4) = p \cap (a_1, a_2, a_3 : a_4) = (a_1, a_2, a_3)$.

Parts (a) and (b) are trivial. As $((a_1, a_2) : a_3) = ((a_1, a_2, a_4) : a_3) = (X, Y)$, we must show $p \cap (X, Y) = (a_1, a_2, a_4)$. Certainly (a_1, a_2, a_4) is contained in (X, Y) . But, a_3 is not a zero divisor modulo (X, Y) and hence $((a_1, a_2, a_4) : a_3) \cap (a_1, a_2, a_3, a_4) = (a_1, a_2, a_4)$, which is precisely the statement we need. For (d) the same argument holds, noting a_4 is not a zero divisor modulo $((a_1, a_2, a_3) : a_4) = (W, Z)$. For (3) of Definition 1.1 we must show $a_4 a_3 \in (a_1, a_2)p$ as $a_3 a_4 \in (a_1, a_2)$. But this relation follows from the standard Plucher relation on the six 2×2 minors of

$$\begin{pmatrix} X & W & Z & Y \\ Y & Z & XW & X^2 \end{pmatrix}.$$

2. MAIN THEOREM

In this section we prove the main theorem (Theorem 2.2) and derive several corollaries concerned with symbolic powers and the actual computation of depth. Proposition 2.1 is the main technical result, while in Theorem 2.1 we show that weak d -sequences satisfy the conditions of this proposition.

PROPOSITION 2.1. *Suppose R is a commutative ring, I and J two ideals satisfying $(D^1) : I^{k+1} \cap I^k J^m \subseteq I^{k+1}(I, J)^{m-k-1}$. Let $Q = I + J$. Then R/Q^n has a filtration $M_0 = R/Q^n$, $M_n = 0$ such that M_k/M_{k+1} is isomorphic to $I^k/(I^k(I, J^{n-k}))$.*

Proof. Set $M_k = I^k/I^k(I, J)^{n-k}$, where by convention we assume $I^0 = R$. Then $M_0 = R/Q^n$ and

$$M_{n-1} = I^{n-1}/I^{n-1}(Q) = I^{n-1}/(I^n + I^{n-1}J).$$

There is clearly a surjective map from M_k onto $I^k/(I^{k+1} + I^k J^{n-k})$; it is enough to show that the kernel of this map is isomorphic to M_{k+1} .

The kernel is

$$\begin{aligned} & (I^{k+1} + I^k J^{n-k}) / (I^k(I, J)^{n-k}) \\ &= (I^{k+1} + I^k J^{n-k}) / (I^{k+1}(Q)^{n-k-1} + I^k J^{n-k}) \\ &\cong I^{k+1} / (I^{k+1} Q^{n-k-1} + I^{k+1} \cap I^k J^{n-k}). \end{aligned}$$

By assumption, $I^{k+1} \cap I^k J^{n-k} \subseteq I^{k+1} Q^{n-k-1}$ so that this is just

$$I^{k+1} / (I^{k+1} Q^{n-k-1}),$$

as required.

The condition (D') is a natural one as the following propositions show: (See Sally, [24]).

PROPOSITION 2.2. *Let R be a commutative Noetherian local ring, I and J two proper ideals, and $Q = I + J$. Suppose I is generated by elements not in Q^2 . By “ $-$ ” denote the initial forms of elements in $gr_Q(R) = R/Q \oplus Q/Q^2 \oplus \dots$. Then I and J satisfy $I \cap J^m \subseteq I(I, J)^{m-1}$ if and only if*

$$gr_Q(R/I) \simeq gr_Q(R)/\bar{I} \quad (\text{a graded isomorphism}).$$

Proof. $gr_Q(R/I)$ is just

$$R/Q \oplus Q/(Q^2, I) \oplus (Q^2, I)/(Q^3, I) \oplus \dots,$$

which is isomorphic to

$$R/Q \oplus Q/(Q^2, I) \oplus Q^2/(Q^3 + Q^2 \cap I) \oplus \dots.$$

Since \bar{I} is generated by forms of degree 1, $gr_Q(R)/\bar{I}$ is isomorphic to

$$R/Q \oplus Q(Q^2, I) \oplus Q^2/(Q^3 + IQ) \oplus \dots.$$

If these are isomorphic, then

$$Q^n/(Q^{n+1} + IQ^{n-1}) \simeq Q^n/(Q^{n+1} + Q^n \cap I)$$

for every n . Since there is a surjective map from $Q^n/(Q^{n+1} + IQ^{n-1})$ to $Q^n/(Q^{n+1} + Q^n \cap I)$, and these are finitely generated modules over a commutative Noetherian ring, this map must be an isomorphism in this case. This shows $Q^n \cap I \subseteq Q^{n+1} + IQ^{n-1}$ for every n . But then,

$$Q^n \cap I \subseteq Q^{n+1} \cap I + IQ^{n-1} \subseteq Q^{n+2} + IQ^n + IQ^{n-1} = Q^{n+2} + IQ^{n-1}.$$

By induction,

$$Q^n \cap I \subseteq Q^{n-1}I + Q^{n+k} \quad \text{for all } k \geq 1.$$

Since R is local, we obtain

$$Q^n \cap I = Q^{n-1}I. \tag{*}$$

We claim $(*)$ holds for every n if and only if I and J satisfy

$$I \cap J^m \subseteq IQ^{m-1}.$$

For suppose $(*)$. Then $I \cap J^m \subseteq I \cap Q^m = IQ^{m-1}$. Conversely, if $I \cap J^m \subseteq IQ^{m-1}$ then $I \cap Q^m = IQ^{m-1} + I^m \cap I \subseteq IQ^{m-1}$, as required.

This shows that if $gr_Q(R/I)$ and $gr_Q(R)/\bar{I}$ are isomorphic, then

$$I \cap J^m \subseteq IQ^{m-1}.$$

Conversely if this holds, then (*) holds and so

$$Q^n/(Q^{n+1} + IQ^{n-1}) = Q^n/(Q^{n+1} + Q^n \cap I)$$

and $gr_Q(R)/\bar{I}$ is isomorphic to $gr_Q(R/I)$.

PROPOSITION 2.3 [24]. *Suppose I is generated by an R -sequence x_1, \dots, x_d of elements not in Q^2 . Let the notation be as in Proposition 2.2. Then I and J satisfy (D') if and only if*

$$gr_Q(R/I) \simeq gr_Q(R)/\bar{I}.$$

Proof. By Proposition 2.2, $gr_Q(R/I) \simeq gr_Q(R)/\bar{I}$ if and only if

$$I \cap J^m \subseteq IQ^{m-1}.$$

We must show $I^k \cap I^{k-1}J^m$ is contained in I^kQ^{m-1} . For this it is enough to show

$$I^k \cap Q^n = Q^{n-k}I^k$$

for all $n \geq k$.

We do this by induction on k . For $k = 1$, this is (*) of Proposition 2.2. So assume $k > 1$. Then if $a \in I^k \cap Q^n$, $a \in I^{k-1} \cap Q^n = Q^{n-k+1}I^{k-1}$ by induction. But I^{k-1}/I^k is a free R/I module; let $\bar{y}_1, \dots, \bar{y}_s$ be a basis for it over R/I .

Then modulo I^k , $a = \sum_{i=1}^s t_i y_i$, where the t_i are in Q^{n-k+1} . Since $a \in I^k$, we see that $\sum_{i=1}^s t_i y_i \in I^k$ and by the choice of the y_i , this implies

$$t_i \in I \cap Q^{n-k+1} = IQ^{n-k}.$$

Then, $a \in I^{k-1} \cdot IQ^{n-k} = I^kQ^{n-k}$ as required.

We have shown $gr_Q(R/I) \simeq gr_Q(R)/\bar{I}$ if and only if $I^k \cap Q^n = Q^{n-k}I^k$ for all $k \geq 1$ and all $n \geq k$. This shows I and J satisfy (D') since

$$I^k \cap I^{k-1}J^m \subseteq I^k \cap Q^{m+k-1} = I^kQ^{m-1}.$$

Conversely if I and J satisfy (D') then by Proposition 2.2,

$$gr_Q(R/I) \simeq gr_Q(R)/\bar{I}.$$

DEFINITION 2.1. Let J be an ideal of R , H a partially ordered set and $\{x_\alpha\} \subseteq R$ a set of elements indexed by H . We will say the $\{x_\alpha\}$ form a *weak d -sequence modulo J* if

- (a) the images of the x_α in R/J form a weak d -sequence and
- (b) if \bar{I} is an H -ideal in R/J containing \bar{I}_α but not \bar{x}_α and $\bar{x}_\beta \in (\bar{I} : \bar{x}_\alpha)$, then lifting back to R ,

$$x_\alpha x_\beta \in (I, J)\mathcal{A}.$$

This is stronger than assuming (a), as from (a) we only obtain

$$x_\alpha x_\beta \in I\mathcal{A} + J.$$

THEOREM 2.1. Let R be a commutative ring, H a partially ordered set, $\{x_\alpha\} \subseteq R$ a set of elements indexed by H which form a weak d -sequence modulo J for some ideal J . Let $\mathcal{A} = \langle x_\alpha \mid \alpha \in H \rangle$. Then

$$J \cap \mathcal{A}^k \subseteq \mathcal{A}^{k-1}J + \mathcal{A}^{k-2}J^2. \tag{*}$$

Proof. We induct on the number of elements of H . If this is one, then $\mathcal{A} = (x)$. There are two cases:

Case 1. $x \notin (J : x)$. Then $(J : x) = (J : x^2)$ and so $x^{kr} \in J$ implies $xr \in J$. Then $x^{kr} \in x^{k-1}J$. Thus

$$\mathcal{A}^k \cap J \subseteq \mathcal{A}^{k-1}J.$$

Case 2. If $x \in (J : x)$ then $x^2 \in J\mathcal{A} = Jx$. Hence $(x^k) \cap J \subseteq x^{k-2}(Jx) = \mathcal{A}^{k-1}J$.

We may suppose H has more than one element. Let α be a minimal element of H . Set $x = x_\alpha$ and let I be the ideal generated by the rest of the x_β , $\beta \neq \alpha$. We require the following observation.

LEMMA 2.1. Suppose $\{x_\beta\}$ is a weak d -sequence modulo J and I_1 is an H -ideal. Write $\mathcal{A} = I_1 + I_2$, where $I_2 = \langle x_\beta \mid x_\beta \notin I_1 \rangle$. Then $(J, I_1) \cap \mathcal{A}^m \subseteq I_1\mathcal{A}^{m-1} + I_2^m \cap (J, I_1)$.

Proof. Immediate.

Now as $w \in \mathcal{A}^m \cap J$ we may write $w = xw' + z$, where $z \in I^m$ and $xw' + z \in J$. Then

$$z \in I^m \cap (J, x). \tag{2}$$

However, as in Proposition 1.1, it is easy to see that since x is minimal, $\{x_\beta \mid \beta \neq \alpha\}$ are a weak d -sequence modulo (J, x) and $\{x_\beta\}$ are a weak d -sequence modulo $(J : x)$. We may use the induction on Eq. (2) to conclude

$$z \in I^{m-1}(J, x) + I^{m-2}(J, x)^2$$

and so

$$z \in \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2 + I^{m-1}x + I^{m-2}x^2.$$

Thus $w = z + xw'$ (where $w' \in \mathcal{A}^{m-1}$) is in

$$J \cap (x\mathcal{A}^{m-1} + \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2).$$

Hence

$$w \in (x\mathcal{A}^{m-1} \cap J) + \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2.$$

Write $w = xa + c$, where $a \in \mathcal{A}^{m-1}$ and $c \in \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2$. Then $xa \in J$ implies $a \in (J : x)$. Thus $a \in (J : x) \cap \mathcal{A}^{m-1} \subseteq (J : x) \cap \mathcal{A} = (J : x)^*$ as the $\{x_\beta\}$ are a weak d -sequence in R/J . Set $I_1 = (J : x)^*$. We have $a \in I_1 \cap \mathcal{A}^{m-1} \subseteq I_1\mathcal{A}^{m-2} + I_2^{m-2} \cap (J, I_1)$ by Lemma 1, where $I_2 = \{x_\beta \mid x_\beta \notin I_1\}$. We observe:

LEMMA 2.2. *Let $R, H, x, \{x_\beta\}$, and J be as above. If I is an H -ideal of R , and $H_1 = \{\beta \mid x_\beta \notin I\}$, then $\{x_\beta \mid \beta \in H_1\}$ are a weak d -sequence modulo (J, I) .*

Proof. The proof is exactly as in Proposition 1.1.

Applying this above, we see as $m-1 < m$ that by induction

$$I_2^{m-1} \cap (J, I_1) \subseteq I_2^{m-2}(J, I_1) + I_2^{m-3}(J, I_1)^2.$$

Hence w is in $\mathcal{A}^{m-2}J^2 + \mathcal{A}^{m-1}J + xI_1\mathcal{A}^{m-2} + xI_2^{m-2}J + xI_2^{m-2}I_1 + xI_2^{m-3}J^2 + xI_2^{m-3}I_1^2$. Now either $I_1 = J$ or $xI_1 = x(J : x)^* \subseteq J\mathcal{A}$ by assumption. In either case w is in $\mathcal{A}^{m-2}J^2 + \mathcal{A}^{m-1}J$ as required.

If the $\{x_\alpha\}$ form a d -sequence modulo J then

$$\mathcal{A}^m \cap J \subseteq \mathcal{A}^{m-1}J.$$

The weak d -sequence has the property that

$$\mathcal{A}^m \cap J \subseteq \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2.$$

To apply the filtration of Proposition 2.1, we basically need

$$\mathcal{A}^m \cap J \subseteq \mathcal{A}^{m-1}J + \mathcal{A}^{m-2}J^2 + \dots + J^m. \quad (\text{D}')$$

However, we know of no workable definition for a sequence of elements to satisfy (D') except for the above two. Clearly (D') is closely tied to the analytic properties of I and J .

COROLLARY 2.1. *Suppose $\{x_\beta\}$ are a weak d -sequence in R , with partially ordered set H , and $\alpha \in H$ is minimal. Set $x_\alpha = x$, and let J be the ideal generated by the rest of the x_β . Then*

$$(x^{k+1}) \cap x^k J^m \subseteq x^{k+1}(x, J)^{m-1}.$$

Proof. Suppose $x^{k+1}a = x^k b$, where $b \in J^m$. Then $x^k(xa - b) = 0$. If $x^2 = 0$ then there is nothing to show; if not then $(0 : x) = (0 : x^2)$ and so $b \in ((0 : x), x) \cap J^m$. But by Proposition 1.1 the $\{x_\beta \mid \beta \neq \alpha\}$ form a weak d -sequence modulo $((0 : x)^*, x)$ and $b \in ((0 : x)^*, x)$ as $b \in \mathcal{A}$. By Theorem 2.1,

$$b \in J^{m-1}((0 : x), x) + J^{m-2}((0 : x), x)^2$$

and so

$$x^k b \in J^{m-1} x^{k+1} + J^{m-2} x^{k+2} \subseteq x^{k+1}(x, J)^{m-1}.$$

Note if $k = 0$ then

$$(x) \cap J^m \subseteq (x)J^{m-1} + x^2 J^{m-2}$$

by Theorem 2.1.

DEFINITION 2.2. An ideal J is said to be *related* to a weak d -sequence $\{x_\alpha\}$ if $J = ((I : x_\alpha), \mathcal{A})$, where I is an H -ideal containing I_α but not x_α . We also include \mathcal{A} as a related ideal. A prime ideal P is said to be a *related prime* to the weak d -sequence $\{x_\alpha\}$ if P is associated to some related ideal.

THEOREM 2.2. *Let R be a commutative ring and $\{x_\beta\}$ a weak d -sequence in R on the partially ordered set H . Let \mathcal{A} be the ideal generated by all the $\{x_\beta\}$. Then for all $n \geq 1$, R/\mathcal{A}^n has a filtration $M_0 = R/\mathcal{A}^n \supseteq M_1 \supseteq \dots \supseteq M_d = (0)$ such that $M_k/M_{k+1} \simeq R/J$ where J is some related ideal of $\{x_\alpha\}$.*

Proof. Induct on the number of elements of H . If H has only one element, then $\mathcal{A} = (x)$. If $x^2 = 0$, then clearly there is nothing to prove. If $x^2 \neq 0$, then $(0 : x) = (0 : x^2)$ and hence

$$(x)/(x^2) \simeq (x^2)/(x^3) \simeq (x^n)/(x^{n+1}) \quad \text{for every } n \geq 1.$$

Then there are exact sequences

$$0 \rightarrow (x)/(x^2) \rightarrow R/(x^n) \rightarrow R/(x^{n-1}) \rightarrow 0.$$

By induction on n , we may assume $R/(x^{n-1})$ has the required filtration. As $(x)/(x^2) \simeq R/(x, (0 : x))$ this shows $R/(x^n)$ also does.

We may suppose H has more than one element. Let α be a minimal element of H and set $x = x_\alpha$. By Corollary 2.1, (x) and J satisfy the condition (D') of Proposition 2.1. Here J is the ideal generated by all $\{x_\beta \mid \beta \neq \alpha\}$. By Proposition 2.1, we may conclude R/\mathcal{A}^n has a filtration whose factors are isomorphic to

$$(x^k)/(Rx^{k+1} + x^kJ^{n-k}).$$

Map R onto this by multiplication by x^k ; again we may assume $x^2 \neq 0$ and so $(0 : x) = (0 : x^2)$. This shows

$$R/(x, (0 : x), J^{n-k}) \simeq (x^k)/(Rx^{k+1} + x^kJ^{n-k}).$$

Set $S = R/(x, (0 : x))$. Then the images of $\{x_\beta \mid \beta \neq \alpha\}$ form a weak d -sequence in S by Proposition 1.1. We may thus apply the induction to conclude S/\bar{J}^{n-k} have filtrations whose factors are of the form S/\bar{I} , where \bar{I} is a related ideal of $\{x_\beta \mid \beta \neq \alpha\}$. Lifting this back to R , we see I is of the form $((0 : x), x, I_1 : x_\beta, \mathcal{A})$, where I_1 is an $(H - \{\alpha\})$ -ideal containing I_β but not x_β .

By Lemma 1.1, since (x, I_1) is an H -ideal (α is minimal), $((0 : x), x, I_1 : x_\beta) = (((0 : x)^*, x, I_1) : x_\beta)$ and so I is in fact a related ideal of $\{x_\beta\}$. Since $S/\bar{I} \simeq R/I$ we have shown the conclusion of Theorem 2.2.

COROLLARY 2.2. *Suppose R, H, \mathcal{A} , and $\{x_\alpha\}$ are as in Theorem 2.2. Suppose in addition that \mathcal{A} is prime and R is Noetherian. If*

$$\mathcal{A}_p^{(n)} = \mathcal{A}_p^n \quad \text{for all related primes } p,$$

then

$$\mathcal{A}^{(n)} = \mathcal{A}^n.$$

In particular, if \mathcal{A}_p is generated by a regular sequence for all such p , then

$$\mathcal{A}^{(n)} = \mathcal{A}^n \quad \text{for every } n \geq 1.$$

Before we prove this we note two standard lemmas.

LEMMA 2.3 (see [19]). *Suppose R is a local Noetherian ring and*

$$0 \rightarrow K \rightarrow N \rightarrow L \rightarrow 0$$

is an exact sequence of finitely generated R -modules. Then

- (a) $\text{depth } K \geq \text{depth } N = \text{depth } L$,
- (b) $\text{depth } N \geq \text{depth } K = \text{depth } L + 1$, or
- (c) $\text{depth } L > \text{depth } K = \text{depth } N$.

In particular if $\text{depth } K = \text{depth } L$, then

$$\text{depth } K = \text{depth } L = \text{depth } N.$$

Also if $\text{depth } K \geq \text{depth } L$ then $\text{depth } N \geq \text{depth } L$.

Proof. Well known and easy from any of the homological characterizations of depth.

LEMMA 2.4. Suppose R is a Noetherian local ring and M is a finitely generated module with a filtration $\{M_k\}_{k=0}^n$. If

$$t = \min_{0 < k < n-1} \text{depth } (M_k/M_{k+1})$$

then $\text{depth } M \geq t$.

Proof. Induct on the length of the filtration. We may assume $\text{depth } M_1 \geq t$. We have an exact sequence,

$$0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$$

and $\text{depth } M_1 \geq t$, $\text{depth } M/M_1 \geq t$. By Lemma 2.3, $\text{depth } M \geq t$.

We now prove Corollary 2.2. Suppose $\mathcal{A}^{(n)} \neq \mathcal{A}^n$. Then there is an associated prime Q of \mathcal{A}^n which is not equal to \mathcal{A} . Localize at Q . By Theorem 2.2, $(R/\mathcal{A}^n)_Q$ has a filtration whose factors are isomorphic to $(R/J)_Q$, where J is a related ideal of $\{x_\alpha\}$. If $Q \not\supseteq J$, just leave these terms out. ($(R/J)_Q = 0$ in this case.)

Since $\mathcal{A}_Q^{(n)} = \mathcal{A}_Q^n$ by assumption if Q is a related prime, we see Q is not a related prime so that $\text{depth}(R/J)_Q \geq 1$ if $(R/J)_Q \neq 0$. By Lemma 2.4, $\text{depth}(R/\mathcal{A}^n)_Q \geq 1$, which contradicts the assumption.

COROLLARY 2.3. Let R be a Noetherian ring, $\{x_\alpha\}$ a weak d -sequence on a partially ordered set H and \mathcal{A} the ideal generated by all the x_α . Let Q be a prime which contains \mathcal{A} . Then for every $n \geq 1$,

$$\text{depth}(R/\mathcal{A}^n)_Q \geq \min \text{depth}(R/J)_Q,$$

where the right side runs through all related ideals $J \subseteq Q$.

Proof. Immediate from Theorem 2.2 and Lemma 2.4.

In general, the exact depths of R/\mathcal{A}^n can be quite difficult to compute. In the last section we explicitly compute one example, and observe some more subtle changes in depth. However, one observation can be made.

COROLLARY 2.4. *Let R, H, \mathcal{A} , and $\{x_\alpha\}$ be as in Corollary 2.3. Let Q be a prime and set $t = \{\min \text{depth}(R/J)_Q \mid J \subseteq Q, J \text{ related}\}$. If α is minimal in H and $x = x_\alpha$ is not a zero divisor, then if $\text{depth}(R/\mathcal{A}^n)_Q = t$ for some n , $\text{depth}(R/\mathcal{A}^m)_Q = t$ for all $m \geq n$.*

Proof. Clearly it is enough to show this for $m = n + 1$.

From Proposition 2.1 applied to (x) and J , where J is the ideal generated by the rest of the x_β , we see there is an exact sequence,

$$0 \rightarrow ((x)/x(\mathcal{A}^n))_Q \rightarrow (R/\mathcal{A}^{n+1})_Q \rightarrow (R/(x, J^{n+1}))_Q \rightarrow 0.$$

Since x is not a zero divisor, $(x)/x\mathcal{A}^n$ is isomorphic to R/\mathcal{A}^n . By Corollary 2.3, $\text{depth}(R/(x, J^{n+1}))_Q \geq t$. (Here we are implicitly using Proposition 1.1 and Lemma 1.1 as we have done above.) Apply Lemma 2.3. The possibilities are:

(a) $t = \text{depth}(R/\mathcal{A}^n)_Q \geq \text{depth}(R/\mathcal{A}^{n+1})_Q = \text{depth}(R/(x, J^{n+1}))_Q \geq t$, which forces $\text{depth}(R/\mathcal{A}^{n+1})_Q = t$.

(b) $\text{depth}(R/\mathcal{A}^{n+1})_Q \geq \text{depth}(R/\mathcal{A}^n)_Q = t = \text{depth}(R/(x, J^{n+1}))_Q + 1 \geq t + 1$ which cannot happen.

(c) $\text{depth}(R/(x, J^{n+1}))_Q > \text{depth}(R/\mathcal{A}^{n+1})_Q = \text{depth}(R/\mathcal{A}^n)_Q = t$, which is the desired conclusion.

COROLLARY 2.5. *Suppose R is a Noetherian domain and $\{x_\alpha\}$ a weak d -sequence, \mathcal{A} the ideal generated by the $\{x_\alpha\}$. Then*

$$\text{Ass}(R/\mathcal{A}) \subseteq \text{Ass}(R/\mathcal{A}^2) \subseteq \dots$$

Proof. By the above exact sequence, we have if $x = x_\alpha$, where α is minimal, that $0 \rightarrow x/x\mathcal{A}^n \rightarrow R/\mathcal{A}^{n+1}$. But $x/x\mathcal{A}^n \simeq R/\mathcal{A}^n$ as R is a domain and so $0 \rightarrow R/\mathcal{A}^n \rightarrow R/\mathcal{A}^{n+1}$, which gives the required result.

3. SYMBOLIC POWERS AND DEPTHS

In this section we use the results from the previous two sections to explicitly calculate the depths of powers of ideals generated by weak d -

sequences and prove the symbolic powers are equal to the ordinary powers for several of the examples. Since most of this has already been done in [15, 17] for d -sequences, we concentrate upon the later examples of Section 1.

3.1. The maximal minors. Let $X = (x_{ij})$ be an $r \times s$ generic matrix over a base ring R which we assume to be Cohen–Macaulay and Noetherian.

We need to review some more material on algebras with straightening law. Again this will be found in [9].

Let A be an ASL on H over R . H is said to be *wonderful* if, after adjoining least and greatest elements, H has the following property: if $\beta_1, \beta_2 \leq \sigma$ are covers of the same element α , then there is a $\gamma \leq \sigma$ which is a common cover of β_1 and β_2 . Here, β is said to be a *cover* of α if $\alpha < \beta$ and there is no σ such that $\alpha < \sigma < \beta$. Then, the following are shown by DeConcini *et al.*

PROPOSITION 3A [9]. *Suppose A is an ASL on H over R and*

- (1) *A is graded and all the $\bar{\alpha}$ are forms,*
- (2) *R is Cohen–Macaulay (i.e., R is locally Cohen–Macaulay),*
- (3) *H is wonderful.*

Then A is locally Cohen–Macaulay.

PROPOSITION 3B [9]. *Let H be wonderful, A an ASL on H over R . If I is an ideal of H such that for every two minimal elements α_1, α_2 of $H - I$ with $\alpha_1, \alpha_2 < \gamma$ there is a common cover $\beta < \gamma$ for α_1, α_2 , then $H - I$ is wonderful.*

THEOREM 3.1. *Let X be as above and set $A = k[x_{ij}]_{(x_i)}$, k a field. Let \mathcal{A} be the ideal generated by all the maximal minors of X . It is well known that \mathcal{A} is prime. Then $\mathcal{A}^{(n)} = \mathcal{A}^n$ for every $n \geq 1$, and*

$$\text{depth } A/\mathcal{A}^n \geq r^2 - 1 \quad \text{for all } n.$$

If $\text{depth } A/\mathcal{A}^n = r^2 - 1$ for any n , then $\text{depth } A/\mathcal{A}^m = r^2 - 1$ for every $m \geq n$.

Proof. First, in 1.17 it was established that the maximal minors of X form a weak d -sequence on H , the poset of maximal minors. In addition, by Propositions 1A and 1B every related ideal of this weak d -sequence is of the form \bar{I} , where $I \subseteq H'$ is an ideal of the poset H' of all minors of X . It is easy to check using Proposition 1B that every related ideal looks like \mathcal{A} plus the ideal generated by all minors $(t_k, \dots, t_1 \mid j_1, \dots, j_k)$ with $j_1 \leq i_1, \dots, j_k \leq i_k$ and

strict inequality in at least one entry. Here, $[i_1, \dots, i_r]$ is some fixed maximal minor. Thus the related ideals are of the form \bar{I} , where $I \subset H$ and

$$I = \{(r, \dots, 1 \mid j_1, \dots, j_r) \mid 1 \leq j_1 < \dots < j_r \leq s\} \\ \cup \{(t_k, \dots, t_1 \mid j_1, \dots, j_k) \mid j_1 \geq i_1, \text{ or } \dots, \text{ or } j_k \geq i_k \\ \text{for some fixed } 1 \leq i_1 < \dots < i_r \leq s\}.$$

$H - I$ has a unique minimal element, $(r - 1, \dots, 1 \mid i_1, \dots, i_{r-1})$ and so by Proposition 3.B, A/\bar{I} is Cohen–Macaulay. As \bar{I} was any related ideal of the maximal minors, we see every related ideal is perfect.

However, the maximal minors have a unique greatest element, $a = [s - r + 1, \dots, s]$ (here the notation is as in Section 1). In addition, if I is any H -ideal then a is not a zero divisor modulo I . This follows at once from the following lemma.

LEMMA 3C [9]. *Let A be an ASL or H over R with greatest element $a \in H$. Then \bar{a} is not a zero-divisor.*

Proof. Easy exercise using the definition of algebra with straightening law.

Let I be an H -ideal $\neq \mathcal{A}$ containing I_α but not x_α . Then $a \notin (I : x_\alpha)$ since $a \in (I : x_\alpha)$ means $ax_\alpha \in I$. By the remarks above, a is not a zero divisor modulo I so that this would imply $x_\alpha \in I$, which is not the case. But in this case by Lemma 1.1, if we let J be the ideal generated by all the x_β except for a , we see

$$(I : x_\alpha) \subseteq ((I : x_\alpha), J) \subseteq (I : x_\alpha), J : a \\ \subseteq ((I : x_\alpha)^*, J : a) = (J : a).$$

Hence $((I : x_\alpha), \mathcal{A}) \subseteq ((J : a), a)$. We have thus shown every related ideal is contained in $((J : a), a)$. Since all these are perfect ideals we see that

$$\min\{\text{depth } A/J \mid J \text{ related to } \{x_\alpha\}\} = \text{depth } A/((J : a), a).$$

Let us compute this ideal explicitly. As $j \leq s - r$, by elementary linear algebra, $x_{ij}a$ is in the ideal J generated by the rest of the maximal minors. Hence, $J \subseteq \langle x_{ij} \mid j \leq s - r \rangle \subseteq (J : a)$. As a is not a zero divisor modulo the middle ideal, we see $(J : a) = \langle x_{ij} \mid j \leq s - r \rangle$. As there are $r(s - r)/x_{ij}$ in this ideal we see

$$\text{depth } A/((J : a), a) \\ = \dim A - \text{height}((J : a), a) \\ = rs - (r(s - r) + 1) = r^2 - 1.$$

It now follows from Corollary 2.3 that $\text{depth } A/\mathcal{A}^n \geq r^2 - 1$ for every n . From Corollary 2.4 the last statement of Theorem 3.1 follows. It remains to show $\mathcal{A}^{(n)} = \mathcal{A}^n$ for all n . For this we use Corollary 2.2. Set $(J : a, a) = Q$. It is well known that the related ideals of the weak d -sequence are all prime (see [11, 8]). Consequently, if \mathcal{A}_Q is a complete intersection then Corollary 2.3 will imply $\mathcal{A}^{(n)} = \mathcal{A}^n$ for every n . For this, since \mathcal{A} is regular it suffices to show $(A/\mathcal{A})_Q$ is regular. But the singular locus of \mathcal{A} is determined by all the $(r - 1) \times (r - 1)$ minors and clearly Q does not contain all of these. (For instance, $[s - r + 2, \dots, s] \notin Q$). Hence $(A/\mathcal{A})_Q$ is regular and so \mathcal{A}_Q is a complete intersection. Thus $\mathcal{A}^{(n)} = \mathcal{A}^n$ for all n . The proof of Theorem 3.1 is now complete.

Buchsbaum has conjectured that if \mathcal{A} denotes the maximal minors of a generic $r \times s$ matrix, then $\text{depth } R/\mathcal{A}^n$ should be independent on s if $n \geq 2$. Robbiano [21] has conjectured that these depths are independent on n if $n \geq 2$. We give a partial answer to these conjectures.

COROLLARY 3.1. *Let A, X, \mathcal{A} be as in Theorem 3.1; then for $n \gg 0$,*

$$\text{depth } A/\mathcal{A}^n = r^2 - 1.$$

Consequently the depth of these for all but finitely many n is independent of s and becomes constant.

Proof. Cowsik and Nori [6] showed that the analytic spread, $l(\mathcal{A})$, of \mathcal{A} is $r(s - r) + 1$. Burch proved,

PROPOSITION 3D [5]. *Let R be a Noetherian local ring and I an ideal. By $l(I)$ denote the analytic spread of I . Then*

$$\dim R - \inf \text{depth } R/I^n \geq l(I).$$

(See [20] for information concerning analytic spread.)

Now in Theorem 3.1 we showed

$$\inf \text{depth } A/\mathcal{A}^n \geq r^2 - 1.$$

From Proposition 3D, we see

$$\dim A - l(\mathcal{A}) \geq \inf \text{depth } A/\mathcal{A}^n$$

and so $rs - (r(s - r) + 1) \geq \inf \text{depth } A/\mathcal{A}^n$. This shows that

$$\inf \text{depth } A/\mathcal{A}^n = r^2 - 1.$$

By Theorem 3.1 once $\text{depth } A/\mathcal{A}^n = r^2 - 1$, this is true for all $m \geq n$.

To prove both the conjectures of Buchsbaum and Robbiano, it is enough to show $\text{depth } A/\mathcal{A}^2 = r^2 - 1$.

In fact an explicit A/\mathcal{A}^2 -sequence of length $r^2 - 1$ can be given (see Section 6); the only work that remains is to show it is a maximal A/\mathcal{A}^2 -sequence. We do this in Section 6 for the case of the $2 \times n$ minors. This recovers a result of Robbiano [21], who showed this case. We will see in Example 3.3 that there is something strange going on which should make the calculation somewhat nontrivial.

3.2. The Pfaffians. In 1.16 and 1.18 we discussed the Pfaffians and showed that the maximal order Pfaffians of a generic skew-symmetric matrix of odd rank form a weak d -sequence.

THEOREM 3.2 (see [1]). *Let k be a field, X a generic $(2n + 1) \times (2n + 1)$ skew-symmetric matrix with zeros down the diagonal. If $\alpha = [i_1, \dots, i_{2n}]$, where $1 \leq i_1 < \dots < i_{2n} \leq 2n + 1$, let x_α denote the Pfaffian determined by the i_1, \dots, i_{2n} columns and rows of X . Let \mathcal{A} be the ideal the x_α generate. Then \mathcal{A} is a prime ideal and $\mathcal{A}^{(n)} = \mathcal{A}^n$ for every n . Further, if $A = (k[x_{ij}])_{(x_{ij})}$ then*

$$\inf_m \text{depth } A/\mathcal{A}^m = (2n + 1)(n - 1)$$

and if $\text{depth } A/\mathcal{A}^m = (2n + 1)(n - 1)$, then $\text{depth } A/\mathcal{A}^k = (2n + 1)(n - 1)$ for $k \geq m$.

Proof. We proceed as in 3.1, first calculating the related ideals. The maximal Pfaffians are in fact linearly ordered; let f_i be the Pfaffian $[1, \dots, 2n - 1, 2n - i + 2, \dots, 2n + 1]$. Then $f_1 < f_2 < \dots < f_{2n+1}$. We must calculate $((f_1, \dots, f_i) : f_{i+1})$. By using Proposition 1B, it is easy to see that this ideal is precisely the set of Pfaffians $[i_1, \dots, i_{2k}]$ of X which are not $\geq f_{i+1}$. By the partial order of 1.16, this is precisely the ideal generated by all $[i_1, \dots, i_{2k}]$ such that $i_1 = 1, \dots, i_{2n-i} = 2n - i, i_{2n-i+1} = 2n - i + 1$. Hence the related ideals are these plus (f_1, \dots, f_{2n+1}) . These are all of the form \bar{I} , where I is an ideal of H , the poset of all Pfaffians of X . Using Proposition 1B we can compute what the related ideal looks like; if $I_j \subseteq H$ is defined by

$$I_j = \{[i_1, \dots, i_{2n}] \mid 1 \leq i_1 < \dots < i_{2n} \leq 2n + 1\} \\ \cup \{[i_1, \dots, i_{2k}] \mid i_1 = 1, \dots, i_j = j \text{ for some fixed } j, 1 \leq j \leq 2n\},$$

then the \bar{I}_j are precisely the set of related ideals. In this case $H - I_j$ has a unique minimal element, namely, $[1, \dots, j - 1, j + 1, \dots, 2n - 1]$. Thus Proposition 3B shows $H - I$ is wonderful and Proposition 3A allows us to conclude A/I_j is Cohen–Macaulay. As f_{2n+1} is a greatest element of the subset of maximal Pfaffians, just as in 3.1, we conclude every related ideal is

Thus, since $\dim A = 2n(2n + 1)/2 = n(2n + 1)$, we see

$$(n - 1)(2n + 1) = n(2n + 1) - (2n + 1) \geq \inf \text{depth } A/\mathcal{A}^m.$$

This finishes the proof of Theorem 3.2.

3.3 In this example we consider the $n \times n$ minors of an $(n + 1) \times (n + 1)$ generic matrix $X = (x_{ij})$. If \mathcal{A} is the ideal generated by them, then \mathcal{A} is a Gorenstein ideal [18]. This ideal *does not* satisfy the condition of Proposition 1.3. Indeed if $\bar{\alpha}$ and $\bar{\beta}$ are two noncomparable $n \times n$ minors of X , then their straightening looks like

$$\bar{\alpha}\bar{\beta} = \text{quadratic terms in other } n \times n \text{ minors} + \Delta \text{ (linear terms),}$$

where Δ is the determinant of X .

Evidently, the latter term is not in $I_{\alpha}\mathcal{A}$, as is needed. Indeed it is known [6] that \mathcal{A} is not well behaved in the sense that $\text{depth } A/\mathcal{A}^n = 0$ for some n , $A = k[x_{ij}]_{(x_{ij})}$, k a field.

However, (Δ) is equal to \bar{I} , where I is the ideal of the poset H of all minors of X given by $\{(n + 1, \dots, 1 \mid 1, \dots, n + 1)\}$. Hence, $A = k[x_{ij}]/(\Delta)$ is an algebra with straightening law, and by the remarks above concerning the straightening relation of two $n \times n$ minors, we see that the condition of Proposition 1.3 is satisfied and we may conclude that the set of $n \times n$ minors of X , read in A , form a weak d -sequence. Let \mathcal{A} be the ideal they generate.

THEOREM 3.3. *Let A, \mathcal{A} be as above. Then*

$$\inf \text{depth } A/\mathcal{A}^m \geq n^2 - 1$$

and if

$$\text{depth } A/\mathcal{A}^m = n^2 - 1 \quad \text{for some } m,$$

then $\text{depth } A/\mathcal{A}^k = n^2 - 1$ for all $k \geq m$. In addition $\mathcal{A}^{(m)} = \mathcal{A}^m$ for all m .

Proof. For these statements, we proceed as in the last two theorems. Firstly the poset H' of $n \times n$ minors has a maximal element, $a = (n + 1, \dots, 2 \mid 2, \dots, n + 1)$ and so as above, every related ideal of the $n \times n$ minors is contained in $((J : a), a)$, where J is the ideal generated by all the other $n \times n$ minors. Hence to obtain the first two statements of Theorem 3.3, we need only show that the related ideals are perfect, and that $\text{depth } A/((J : a), a) = n^2 - 1$; the statements then follow from Corollaries 2.3 and 2.4.

As before, by using Proposition 1B it is easy to check that the related ideals are all of the form \bar{I} , where $I \subseteq H$ is given by

$$\begin{aligned}
 I = & \{(j_n, \dots, j_1 \mid i_1, \dots, i_n) \mid 1 \leq i_1 < \dots < i_n \leq n + 1 \\
 & 1 \leq j_1 < \dots < j_n \leq n + 1\} \\
 & \cup \{(j_k, \dots, j_1 \mid i_1, \dots, i_k) \mid (j_k, \dots, j_1 \mid i_1, \dots, i_k) \not\supseteq (b_n, \dots, b_1 \mid a_1, \dots, a_n), \\
 & \text{for some fixed } n \times n \text{ minor } (b_n, \dots, b_1 \mid a_1, \dots, a_n)\}.
 \end{aligned}$$

$H - I$ has a unique minimal element $(b_{n-1}, \dots, b_1 \mid a_n, \dots, a_{n-1})$ and so since H is wonderful, so is $H - I$, and hence (using Proposition 3A) A/\bar{I} is Cohen–Macaulay. Thus all the related ideals are perfect. By the above computation applied to $b_n = n + 1, \dots, b_1 = 2, a_1 = 2, \dots, a_n = n + 1$, we see $((J : a), a) = (x_{11}, \dots, x_{1n+1}, x_{21}, \dots, x_{n+1,1}, a)$ and so $\text{depth } A/((J : a), a) = (n + 1)^2 - (2n + 2) = n^2 + 2n + 1 - (2n + 2) = n^2 - 1$.

To show $\mathcal{A}^{(m)} = \mathcal{A}^m$ for all m , induct on the rank of X . By Corollary 2.2 it is enough to show $\mathcal{A}_q^{(m)} = \mathcal{A}_q^m$ for every related prime of the $n \times n$ minors. However, by Lemma 3C we observe that $x_{n+1, n+1}$ is not a zero divisor modulo any ideal of the form \bar{I} , where I is an ideal of H , and hence $x_{n+1, n+1}$ is not a zero divisor modulo any related ideal. Thus if q is a related prime, $x_{n+1, n+1} \notin q$ and in A_q we may transform X to a matrix

$$\begin{pmatrix} & & & 0 \\ & X' & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

and the $(n - 1) \times (n - 1)$ minors of X' generate \mathcal{A}_q . But the entries of X' are algebraically independent and so by the induction $\mathcal{A}_q^{(m)} = \mathcal{A}_q^m$ as required. Therefore $\mathcal{A}^{(m)} = \mathcal{A}^m$ for all m .

3.4. This example will show that the explicit calculations of $\text{depth } A/\mathcal{A}^n$ can in general be difficult, even when \mathcal{A} is generated by a “nice” weak d -sequence, i.e., one that comes from an algebra with straightening law in a wonderful poset H . We let $X = (x_{ij})$ be a generic $r \times s$, $r \leq s$ matrix over a field K , and fix $t + 1 \leq r$. Set $A = K[x_{ij}]/I_{t+1}(X)$, where $I_t(X)$ is the ideal generated by all $t \times t$ minors of X . A inherits the straightening law of $K[x_{ij}]$ on the minors of X , and accordingly we let H be the poset on all the minors of X of degree $\leq t$ (see 1.19). In 1.19 we proved the $t \times t$ minors of the first t columns of X (we use the same notation for the elements of A) form a weak d -sequence. Let \mathcal{A} be the ideal they generate.

THEOREM 3.4. *Let $B = A_{(x_{ij})}$. Then*

$$\inf \text{depth } B/\mathcal{A}^m = st - 1.$$

If $\text{depth } B/\mathcal{A}^m = st - 1$ for some m , $\text{depth } B/\mathcal{A}^k = st - 1$ for all $k \geq m$. \mathcal{A} is a prime ideal and $\mathcal{A}^{(m)} = \mathcal{A}^m$ for every m .

Proof. We simply note the salient points as the verifications are precisely the same as those in Theorem 3.1, 3.2, and 3.3. First, every related ideal is of the form \bar{I} , where $I \subseteq H$ is an ideal and $H - I$ has a unique minimal element. Thus B/J is Cohen–Macaulay for every related ideal J . As the $t \times t$ minors of the first t columns have a maximal element, namely, $a = (r, \dots, r - t + 1 \mid 1, \dots, t)$, every related ideal is contained in $((I : a), a)$, where I is the ideal in A generated by all the other $t \times t$ minors of the first t columns. We compute this ideal using Proposition 1B. To do so we must compute the set of all minors $(j_k, \dots, j_1 \mid i_1, \dots, i_k)$ of X which are not $\geq a$. Clearly this is the set of all minors such that $j_1 < r - t + 1$ and so $((I : a), a) = (x_{ij} \mid i < r - t + 1) + (a)$. Hence $\text{height } ((I : a), a) = s(r - t) + 1$, where the height is computed in $K[x_{ij}]$. Thus $\dim A/((I : a), a) = rs - (s(r - t) + 1) = st - 1$. These observations show (using Corollaries 2.3 and 2.4)

$$\inf \text{depth } B/\mathcal{A}^m \geq st - 1 \quad (1)$$

and if $\text{depth } B/\mathcal{A}^m = st - 1$, then $\text{depth } B/\mathcal{A}^k = st - 1$ for all $k \geq m$.

To obtain equality in (1) we use Proposition 3D. By specializing all the $x_{ij} = 0$ for $j > t$ the analytic spread of \mathcal{A} is $t(r - t) + 1$ [6]. The analytic spread can only go up so that in B ,

$$l(\mathcal{A}) \geq t(r - t) + 1.$$

By Proposition 3D,

$$\dim B - l(\mathcal{A}) \geq \inf \text{depth } B/\mathcal{A}^m$$

and thus

$$\inf \text{depth } B/\mathcal{A}^m \leq \dim B - (t(r - t) + 1).$$

Now, $\text{height } I_{t+1}(X) = (s - t)(r - t)$ and so $\dim A = \dim B = rs - (s - t)(r - t) = rt + st - t^2$. Thus,

$$\inf \text{depth } B/\mathcal{A}^m \leq rt + st - t^2 - (t(r - t) + 1)$$

$= st - 1$. This shows equality is obtained.

It is well known that \mathcal{A} is prime (see [15]). To show $\mathcal{A}^{(m)} = \mathcal{A}^m$ for all m it is enough by Corollary 2.2 to show $\mathcal{A}_q^{(m)} = \mathcal{A}_q^m$ for every related prime q . We induct on the size of X to show $\mathcal{A}^{(m)} = \mathcal{A}^m$. We claim $x_{rt} \notin q$. For if $x_{rt} \in q$, certainly $x_{rt} \in q + (x_{ij} \mid j > t)$. However, modulo $(x_{ij} \mid j > t)$, q becomes equal to the respective related ideal of the $t \times t$ minors of the remaining t -columns and x_{rt} is not in this ideal (see 3.1; x_{rt} becomes a

greatest element in this ASL and we may apply Lemma 3C as we did in 3.1). Thus over A_q , the matrix X can be written

$$\begin{pmatrix} & t & & & \\ & 0 & & & \\ x'_{ij} & \vdots & x'_{ij} & & \\ & 0 & & & \\ 0 \dots 0 & 1 & 0 \dots 0 & & \end{pmatrix}.$$

If X' = the remaining $(r - 1) \times (s - 1)$ matrix then $A_q \simeq C_{q'}$, where $C = K[x'_{ij}]/I_t(X')$ and q' is the appropriate prime. In addition if I is the ideal in C generated by all $(t - 1) \times (t - 1)$ minors of the first $t - 1$ columns of X' , then $I_q = \mathcal{A}_q$. Since by induction we may assume $I^{(m)} = I^m$ for all m , we find $I_q^{(m)} = \mathcal{A}_q^{(m)} = I_{q'}^m = \mathcal{A}_{q'}^m$, which gives the required result.

This example is fairly mysterious. It behaves much like the maximal minors of a generic $r \times s$ matrix. (See Theorem 3.1.) However, if one tries to actually compute $\text{depth } B/\mathcal{A}^m$ in this case, one will run into trouble. Recall that Robbiano has conjectured that $\text{depth } A/I^2 = \text{depth } A/I^3 = \dots = \text{depth } A/I^n = \dots$ if $A = K[x_{ij}]_{(x_{ij})}$, $X = (x_{ij})$ is an $r \times s$ matrix and I is the ideal of maximal minors. In Theorem 3.1 we showed all but finitely many of these depths were equal to $r^2 - 1$. Robbiano [21] has shown all these depths are 3 in the case $r = 2$; a result we will recover rather simply in the last section. We have noted it is enough to show $\text{depth } A/I^2 = r^2 - 1$.

Now consider the example above. One might expect that $\text{depth } B/\mathcal{A}^2 = \text{depth } B/\mathcal{A}^3 = \dots$. However, *this cannot be the case*. For Hochster (unpublished) has noted that the canonical module for B is isomorphic to \mathcal{A}^{s-r} . In particular $\text{depth } B/\mathcal{A}^{s-r} \geq \dim B - 1 = st + rt - t^2 - 1$, which is $> st - 1$ unless $r = t$, in which case \mathcal{A} is generated by one element. This author does not as yet understand the difference between these two examples.

3.5. We now turn to a simpler example, 1.22, studied by Hochster in [14]. Recall we showed that the ideal $p \subseteq K[X, Y, Z, W]$, K a field, such that $K[X, Y, Z, W]/p \simeq K[u^2, u^3, uv, v]$, is generated by a weak d -sequence, namely, $a_1 = XZ - YW$, $a_2 = X^2W - YZ$, $a_3 = XW^2 - Z^2$, $a_4 = X^3 - Y^2$, with the partial order $a_1 < a_2$; $a_2 < a_3$ and $a_2 < a_4$ ($a_3 \not\ll a_4$). Set $A = (K[X, Y, Z, W]/p)_{(X, Y, Z, W)}$.

PROPOSITION 3.5. *Let A, p be as above. Then*

$$\text{depth } A/p^2 = \text{depth } A/p^3 = \dots = 1,$$

and

$$p^{(n)} = p^n \quad \text{for every } n.$$

Proof. The first statement will follow if we can show $\text{inf depth } A/p^n \geq 1$ and $\text{depth } A/p^2 = 1$. For this we simply calculate $\text{depth } A/J$, where J runs through the related ideals. All these are listed in 1.20, and a quick check shows

$$1 = \min\{\text{depth } A/J \mid J \text{ related to } \{a_i\}\}.$$

Hence Corollaries 2.3 and 2.4 show

$$\text{inf depth } A/p^n \geq 1$$

and if equality ever occurs then it continues to occur. So for the first statement it is enough to show $\text{depth } A/p^2 = 1$. $X - W$ is not a zero divisor modulo any related ideal and so cannot be a zero divisor modulo p^2 . Hence it is enough to show that the maximal ideal (X, Y, Z, W) is associated to $(p^2, X - W)$. Consider $I = (p^2, X - W : (Y - Z)^2)$. We find

$$\begin{aligned} (p, X - W) &= (X - W, X^3 - YZ, X^3 - Y^2, X^3 - Z^2, W(Z - Y)) \\ &= (X - W, (Z - Y)W, (Z - Y)Y, (Z - Y)Z, X^3 - Y^2). \end{aligned}$$

But then, $W^2(Y - Z)^2 \in (p^2, X - W)$, $Y^2(Z - Y)^2 \in (p^2, X - W)$, $Z^2(Y - Z)^2 \in (p^2, X - W)$ and so since $Y^2 \in I$ and $(X^3 - Y^2)^2 \in I$ we obtain $X^6 \in I$. Thus I is primary and so m is associated to $(p^2, X - W)$ provided $(Y - Z)^2 \notin (p^2, X - W)$. But specialize X and W to 0, and notice $(Y - Z)^2 \notin (Z - Y)^2(Y, Z)^2$.

We have shown the first statement. The second follows as no related prime contains (X, Y, Z, W) . But if we invert u^2, u^3, uv , or v in $A/p = k[u^2, u^3, uv, v]_{(u^2, u^3, uv, v)}$ we obtain a regular ring. Hence p becomes a complete intersection upon inverting X, Y, Z , or W and now Corollary 2.2 shows $p^{(n)} = p^n$ for all n .

3.6. Let p be the ideal in $k[X, Y, Z, W]$ defining the projection of the twisted quartic, t^4, t^3u, tu^3 , and u^4 . It is defined by four equations, $XW - YZ$, $Y^3 - X^2Z$, $Z^3 - W^2Y$, $Z^2X - Y^2W$, or alternatively by the 2×2 minors of

$$\begin{pmatrix} X & Z & Y^2 & WY \\ Y & W & XZ & Z^2 \end{pmatrix}.$$

We order the generators as in the partial order of the 2×2 minors. Thus if we set

$$\begin{aligned} a_1 &= XW - YZ, \\ a_2 &= X^2Z - Y^3, \\ a_3 &= XZ^2 - WY^2, \\ a_4 &= Z^3 - W^2Y. \end{aligned}$$

We will place a linear order $a_1 < a_2 < a_3 < a_4$. (Compare with 1.20.)

We prove these form a weak d -sequence. Computing the relevant ideals, we find

$$\begin{aligned}(0 : a_1) &= (0), \\ (a_1 : a_2) &= (a_1), \\ ((a_1, a_2) : a_3) &= (X, Y), \\ (a_1, a_2, a_3 : a_4) &= (X, Y).\end{aligned}$$

The quadratic equation on the generic 2×2 minors, $\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 0$, shows

$$-YZa_1^2 - a_2a_4 + a_3^2 = 0.$$

Hence $a_3 \in ((a_1, a_2) : a_3)$. We note that $((a_1, a_2) : a_3)^* = (a_1, a_2, a_3 : a_4)^* = (X, Y) \cap \mathcal{A} = (a_1, a_2, a_3)$ is clear. This observation together with the quadratic relation above makes it clear these form a weak d -sequence. (Notice a_4 is not a zero divisor modulo (X, Y) .)

PROPOSITION 3.6. *Let p be the prime above. Then $p^{(n)} = p^n$ for all n .*

Proof. We need only check this at the related primes, which are p and (X, Y, Z) . But if we invert W , then $(k[X, Y, Z, W]/p)_w \simeq k[(t')^4, (t')^3, (t')^2]$, where $t' = t/u$. This is defined by two equations, $Y^2 = XZ$ and $Z^3 = Y^2$. Hence p_w becomes a complete intersection, which together with Corollary 2.2 gives the desired conclusion.

3.7. In this example we show that the defining ideal of the Veronese $V_{2,4}$ is generated by a weak d -sequence. This ideal \mathcal{A} is defined by the vanishing of the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

However, the natural order on the 2×2 minors *does not* make them into a weak d -sequence. Instead we consider the matrix

$$X = \begin{pmatrix} x_0 & x_3 & x_2 & x_1 \\ x_1 & x_4 & x_3 & x_2 \end{pmatrix}$$

and now put on the natural order. Hence, setting

$$a_{12} = x_0x_4 - x_1x_3,$$

$$a_{13} = x_0x_3 - x_1x_2,$$

$$a_{23} = x_3^2 - x_2x_4,$$

$$a_{14} = x_0x_2 - x_1^2,$$

$$a_{24} = x_3x_2 - x_4x_1,$$

$$a_{34} = x_2^2 - x_3x_1.$$

We order them by the order on the 2×2 minors discussed in 1.13. Let \mathcal{A} be the ideal they generate. We calculate the relevant ideals.

$$(0 : a_{12}) = (0),$$

$$(a_{12} : a_{13}) = (a_{12}),$$

$$(a_{12}, a_{13} : a_{23}) = (x_0, x_1),$$

$$(a_{12}, a_{13} : a_{14}) = (a_{12}, a_{13}, a_{23}), \text{ and this is a prime,}$$

$$(a_{12}, a_{13}, a_{14}, a_{23} : a_{24}) = (x_0, x_1, a_{23}),$$

$$((a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) : a_{34}) = (x_0, x_1, x_3, x_4).$$

Then,

$$(0 : a_{12})^* = (0 : a_{12}) \cap \mathcal{A} = (0),$$

$$(a_{12} : a_{13})^* = (a_{12}) \cap \mathcal{A} = (a_{12}),$$

$$(a_{12}, a_{13} : a_{23})^* = (a_{12}, a_{13}, a_{14} : a_{23})^* = (a_{12}, a_{13}, a_{14}) \\ = (a_{12}, a_{13} : a_{23}) \cap \mathcal{A} = (a_{12}, a_{13}, a_{14} : a_{23}) \cap \mathcal{A},$$

$$(a_{12}, a_{13} : a_{14})^* = (a_{12}, a_{13}, a_{23} : a_{14})^* = (a_{12}, a_{13}, a_{23}) \\ = (a_{12}, a_{13} : a_{14}) \cap \mathcal{A}.$$

$$(a_{12}, a_{13}, a_{14}, a_{23} : a_{24})^* = (a_{12}, a_{13}, a_{14}, a_{23}) \\ = (a_{12}, a_{13}, a_{14}, a_{23} : a_{24}) \cap \mathcal{A},$$

$$((a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) : a_{34})^* = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) \\ = ((a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) : a_{34}) \cap \mathcal{A}.$$

This verifies conditions (1) and (2) of Definition 1.1. For (3), the only case with which we need concern ourselves is that of $(a_{12}, a_{13} : a_{23})$; $a_{14}a_{23} \in (a_{12}, a_{13})$, but the relation is given by the standard quadratic relation on the 2×2 minors of X and hence (3) is satisfied. Finally, (4) is easy to check. (All the ideals above are prime.)

Now it is known that \mathcal{A} is a prime, but $\mathcal{A}^{(n)} \neq \mathcal{A}^n$ and $\text{depth } A/\mathcal{A}^n = 0$ for some n , where $A = k[x_{ij}]_{(x_{ij})}$, k a field. We find one related ideal is

$$\begin{aligned} & ((a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) : a_{34}), a_{34}) \\ & = (x_0, x_1, x_3, x_4, x_2^2) \end{aligned}$$

and

$$\text{depth } A/(x_0, x_1, x_3, x_4, x_2^2) = 0.$$

It is probable that the Veronese in general form a weak d -sequence; however, it is hardly worth proving this as the depths will in general be zero. Perhaps the important point is that this is reflected in the linear relations.

4. BEHAVIOR UNDER RING EXTENSION

In this section we let R be a commutative Noetherian ring, $\{x_\alpha\}$ a weak d -sequence in R on a poset H and $f : R \rightarrow S$ a homomorphism into a commutative Noetherian S . We let \mathcal{A} be the ideal generated by all the x_α .

THEOREM 4.1. *Let $R, \{x_\alpha\}$ be as above and suppose $S = R/Q$ for some ideal Q of R . Suppose*

- (a) $Q \cap J = QJ$ for every related ideal of $\{x_\alpha\}$ and for every H -ideal.
- (b) If I is an H -ideal containing I_α but not x_α then

$$Q \cap (I : x_\alpha, x_\alpha) = Q((I : x_\alpha), x_\alpha).$$

- (c) If $x_\beta \notin (I : x_\alpha)$ then $x_\beta \notin (I : x_\alpha) + Q$.

Then if we denote the map $R \rightarrow R/Q$ by “—” the $\{\bar{x}_\alpha\}$ form a weak d -sequence in S .

Proof. We note that (a) and (b) could also read $\text{Tor}_1^R(R/J, S) = 0$ and $\text{Tor}_1^R(R/((I : x_\alpha), x_\alpha), S) = 0$, respectively. As we will have to prove the next remarks in this general case we separate the next point as a lemma.

LEMMA 4.1. *Suppose $f : R \rightarrow S$ is a ring homomorphism and*

- (a) $\text{Tor}_1^R(R/J, S) = 0$ for all related ideals and all H -ideals.
- (b) If I, x_α are as in (b) above, then

$$\text{Tor}_1^R(R/((I : x_\alpha), x_\alpha), S) = 0.$$

Then

- (1) $f((I : x_\alpha)) = (IS : f(x_\alpha))$.
- (2) $f(\mathcal{A} \cap (I : x_\alpha)) = f(\mathcal{A}) \cap (f(I) : f(x_\alpha))$.
- (3) If $x_\alpha \notin (I : x_\alpha)$ then $f(x_\alpha) \notin (f(I) : f(x_\alpha))$ and $(f(I) : f(x_\alpha))^2 = (f(I) : f(x_\alpha))$.

Proof. We set $y_\alpha = f(x_\alpha)$ and write $f(J) = JS$, for J an ideal. We evaluate $(I : x_\alpha)S$. Since $R/(I : x_\alpha) \simeq (I, x_\alpha)/I$, there is an exact sequence,

$$0 \rightarrow R/(I : x_\alpha) \rightarrow R/I \rightarrow R/(I, x_\alpha) \rightarrow 0.$$

Since (I, x_α) is an H -ideal, (a) shows that

$$0 \rightarrow S/(I : x_\alpha)S \rightarrow S/IS \rightarrow S/(I, x_\alpha)S \rightarrow 0$$

is exact and this implies $S/(I : x_\alpha)S \simeq (I, x_\alpha)S/IS$, which is in turn isomorphic to $S/(IS : y_\alpha)$. Thus

$$(I : x_\alpha)S = (IS : y_\alpha). \quad (1)$$

Now consider $0 \rightarrow R/\mathcal{A} \cap (I : x_\alpha) \rightarrow R/\mathcal{A} \oplus R/(I : x_\alpha) \rightarrow R/(\mathcal{A}, (I : x_\alpha)) \rightarrow 0$. Since $(\mathcal{A}, (I : x_\alpha))$ is a related ideal, (a) shows that $0 \rightarrow S/(\mathcal{A} \cap (I : x_\alpha))S \rightarrow S/\mathcal{A}S \oplus S/(I : x_\alpha)S \rightarrow S/(\mathcal{A}, (I : x_\alpha))S \rightarrow 0$ is exact. Since $(I : x_\alpha)S = (IS : y_\alpha)$ we see that $0 \rightarrow S/(\mathcal{A} \cap (I : x_\alpha))S \rightarrow S/\mathcal{A}S \oplus S/(IS : y_\alpha) \rightarrow S/(\mathcal{A}S, (IS : y_\alpha)) \rightarrow 0$ is exact and so

$$(\mathcal{A} \cap (I : x_\alpha))S = \mathcal{A}S \cap (IS : y_\alpha). \quad (2)$$

Finally if $x_\alpha \notin (I : x_\alpha)$ consider the exact sequence

$$0 \rightarrow R/(I : x_\alpha) \xrightarrow{x_\alpha} R/(I : x_\alpha) \rightarrow R/((I : x_\alpha), x_\alpha) \rightarrow 0.$$

By (b), we obtain an exact sequence

$$0 \rightarrow S/(I : x_\alpha)S \xrightarrow{y_\alpha} S/(I : x_\alpha)S \rightarrow S/((I : x_\alpha)S, y_\alpha) \rightarrow 0,$$

which together with (1) shows \bar{y}_α is not a zero divisor modulo $(IS : y_\alpha)$.

We return to the proof of Theorem 4.1. We claim $(\bar{I} : x_\alpha)^* = (\bar{I} : \bar{x}_\alpha)^*$. Clearly $(I : x_\alpha)^* \subseteq (\bar{I} : \bar{x}_\alpha)^*$ so suppose $x_\alpha x_\beta \in (I, Q)$. If $x_\alpha x_\beta = i + s$, where $s \in Q$, then $s \in Q \cap (I, x_\alpha) = Q(I, x_\alpha)$ and so $s = s_1 i_1 + s_2 x_\alpha$, where $i_1 \in I$ and $s_1, s_2 \in Q$. Then $x_\alpha(x_\beta - s_2) \in I$, and so $x_\beta - s_2 \in (I : x_\alpha)$, which implies by (c) that $x_\beta \in (I : x_\alpha)$, and so $x_\beta \in (I : x_\alpha)^*$. Thus (1) of Definition 1.1 follows immediately. Part (2) of Definition 1.1 follows from Lemma 4.1(b) and the equality above. Part (4) follows from Lemma 4.1(c). Now suppose

$\bar{x}_\beta \in (\bar{I} : \bar{x}_\alpha)$. By above this implies $x_\beta \in (I : x_\alpha)$ and so $x_\alpha x_\beta \in I \setminus \mathcal{I}$ and $\bar{x}_\alpha \bar{x}_\beta \in \bar{I} \setminus \mathcal{I}$. This proves Theorem 4.1.

THEOREM 4.2. *Suppose $R, \{x_\alpha\}$ are as above and $f : R \rightarrow S$ is a ring injection. Suppose (a) and (b) of Lemma 4.1 hold and $IS \cap R = I$ for all H -ideals. Then the $\{x_\alpha\}$ form a weak d -sequence in S . In particular if S is a faithfully flat ring extension of R , the conclusion holds.*

Proof. Again we must show $(I : x_\alpha)^* = (IS : x_\alpha)^*$ if I is an H -ideal containing I_α but not x_α . Again it is clear $(I : x_\alpha)^* \subseteq (IS : x_\alpha)^*$. Suppose $x_\beta \in (IS : x_\alpha)$. Then $x_\alpha x_\beta \in IS \cap R = I$ and so $x_\beta \in (I : x_\alpha)^*$. Now the proof is exactly the same as that for Theorem 4.1.

Finally, if S is faithfully flat, then $\text{Tor}_i^R(M, S) = 0$ for all $i \geq 1$ and all finitely generated modules M , and $IS \cap R = I$ (see [3, 4.3]).

COROLLARY 4.1. *Suppose R, x_α are as above and $(z_1, \dots, z_k) = Q$ are elements in R such that their images form R/J -sequences for every related ideal J and every H -ideal J , and all J of the form $((I : x_\alpha), x_\alpha)$, where I is as in Theorem 4.1. Then the images of the $\{x_\alpha\}$ form a weak d -sequence in R/Q . Further, if we denote by “ $-$ ” the map $R \rightarrow R/Q$ then $(\bar{I} : \bar{x}_\alpha) = \overline{(I : x_\alpha)}$.*

Proof. The last statement will follow from Lemma 4.1 if we can show that (a) and (b) of Theorem 4.1 hold. It is enough to show that if the images of z_1, \dots, z_k in R/J form an R/J -sequence, then

$$(z_1, \dots, z_k) \cap J = (z_1, \dots, z_k)J.$$

This is well known.

These results can be effectively used to understand several of the “almost generic” examples.

4.3. We apply Corollary 4.1 to the case of the 2×2 minors of a generic $2 \times n$ matrix $X = (x_{ij})$. Let $A = k[x_{ij}]$, where k is a field and $B = k[x_{ij}]_{(x_{ij})}$. We note a proposition on algebras with straightening law.

DEFINITION. Let A be an algebra with straightening law on H . If $\alpha \in H$, we denote by $\text{ht}(\bar{\alpha})$ the maximal length of any chain descending from $\bar{\alpha}$ (not counting $\bar{\alpha}$ itself). $\max\{\text{ht}(\bar{\alpha}) \mid \alpha \in H\}$ will be called the $\dim H$.

PROPOSITION 4A. *Let A be an algebra with straightening law on a wonderful poset H over R . Let*

$$y_i = \sum_{\text{ht}(\bar{\alpha})=i} \pm \bar{\alpha}$$

Then y_0, \dots, y_d form an A -sequence ($d = \dim H$).

Returning to our example we let H be the poset on the minors of X . If $d = \dim H$, then $y_d = x_{2n}$, $y_{d-1} = x_{2n-1} + x_{1n}$, and $y_{d-2} = x_{2n-2} + x_{1n-1}$. If $I \subseteq H$ is an ideal of H then A/\bar{I} is an ASL and clearly unless \bar{I} contains one of y_{d-2} , y_{d-1} , or y_d , then these form a B/\bar{I} -sequence. (As B is local we may rearrange the A/\bar{I} -sequence given by Proposition 4A applied to A/\bar{I} .)

Now as has been shown (using Proposition 2A) every H -ideal, every related ideal, and every ideal of the form $((I : x_\alpha), x_\alpha)$ (where I is an H -ideal containing I but not x_α) is of the form \bar{I} , where $I \subseteq H$ is an ideal. Also, every one of these is contained in $((J : a), a)$, where $a =$ the 2×2 minor $[n-1, n]$ and J is the ideal generated by the rest of the maximal minors. This ideal is equal to $(x_{ij} | j < n-1) + (a)$. With these preliminaries, we obtain the following result: Let

$$\begin{pmatrix} y_{11}, \dots, y_{1n-2} & y_{2n-2} & y_{2n-1} \\ y_{21}, \dots, y_{2n-2} & y_{2n-1} & y_{2n} \end{pmatrix},$$

$$A' = k[y_{ij}], \quad B' = k[y_{ij}]_{(y_{ij})}.$$

Then the 2×2 minors of Y form a weak d -sequence, with the same ordering as that in the generic case. If \mathcal{A} denotes the ideal generated by all these, then \mathcal{A} is prime and $\mathcal{A}^{(n)} = \mathcal{A}^n$ for all n and

$$\inf \text{depth } B'/\mathcal{A}^m = 1.$$

Proof. Define a homomorphism f from A to A' by sending the entries of X to the respective entries of Y . The kernel is generated by $x_{2n-1} - x_{1n}$ and $x_{2n-2} - x_{1n-1}$, and by the remarks above, these are an A/J -sequence for every H -ideal J , every related ideal J and every $J = ((I : x_\alpha), x_\alpha)$, where I is an H -ideal containing I but not x . By Theorem 3.1, if we denote by \mathcal{A}' the ideal generated by all the maximal minors of X ,

$$\inf \text{depth } B/(\mathcal{A}')^m = 3.$$

By the filtration (Theorem 2.1) we see that $x_{2n-1} - x_{1n}$, $x_{2n-2} - x_{1n-1}$ are a $B/(\mathcal{A}')^m$ sequence for all m and this implies

$$\inf \text{depth } B'/\mathcal{A}^m = 1,$$

as $B' = B/(x_{2n-1} - x_{1n}, x_{2n-2} - x_{1n-1})$.

Next we show \mathcal{A} is prime. By Proposition 4A and the remarks after it, we see the image of x_{2n} , that is to say, y_{2n} , is not a zero divisor in B' . (In fact, it is not a zero divisor module \mathcal{A}^m for every m .) Thus if $\mathcal{A}_{y_{2n}}$ is prime, then \mathcal{A} must be prime. Set $y = y_{2n}$. In A'_y we may change Y to the matrix

$$\begin{pmatrix} (y_{11} - uy_{21}, y_{12} - uy_{22}, \dots, y_{1n-3} - uy_{2n-3}, y_{1n-2} - u^3y_{2n} & y_{2n-2} - u^2y_{2n} & 0) \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

where $u = y_{2n-1}/y_{2n}$, and \mathcal{A}_y is equal to the ideal generated by the top row of this matrix. This is clearly prime and hence so is \mathcal{A} . Also \mathcal{A}_y becomes a complete intersection and so $\mathcal{A}_y^{(m)} = \mathcal{A}_y^m$ for all m . As y is not a zero divisor modulo any related ideal of the 2×2 minors of y , we see by Corollary 2.2 that $\mathcal{A}^{(m)} = \mathcal{A}^m$ for all m .

Notice if $n = 3$ then the ideal defined is the defining ideal of the Veronese $V_{2,3}$.

We can push this analysis one step further and prove the result promised in Section 3. See Robbiano [21].

THEOREM 4.4 [21]. *Let $X = (x_{ij})$ be a generic $2 \times n$ matrix and let \mathcal{A} be the ideal generated by the 2×2 minors of X in $A = k[x_{ij}]_{(x_{ij})}$, where k is in a field. Then*

$$\text{depth } A/\mathcal{A}^2 = \text{depth } A/\mathcal{A}^3 = \dots = 3.$$

Proof. By the remarks above, $x_{2n} - x_{1n}, x_{2n-2} - x_{1n-1}, x_{2n}$ form an A/\mathcal{A}^m -sequence for all m . By Corollary 2.4 we need only show $\text{depth } A/\mathcal{A}^2 = 3$ and for this it will suffice to show the three elements above form a maximal A/\mathcal{A}^2 -sequence.

Let $B = A/(x_{2n-1} - x_{1n}, x_{2n-2} - x_{1n}, x_{2n})$ and denote by y_{ij} the image of x_{ij} in B . Then $B \simeq k[y_{ij}]$ and the image of \mathcal{A} is the ideal I generated by the 2×2 minors if

$$\begin{pmatrix} y_{11} \cdots y_{1n-2} y_{2n-2} y_{2n-1} \\ y_{21} \cdots y_{2n-2} y_{2n-1} & 0 \end{pmatrix}.$$

Hence it is enough to show that (y_{ij}) , the maximal ideal of B , is associated to I^2 . We claim if

$$J = (I^2 : y_{2n-1}^3)$$

then J is primary for (y_{ij}) . Let $y = y_{2n-1}$ for convenience. Note that $y^3 \notin I^2$ but $y^4 = (y_{2n-1}^2)^2 \in I^2$. Hence $y \in J$. Now

$$y_{2j} y_{2n-1} \in I$$

so that

$$y_{2j}^2 y_{2n-1}^2 \in I^2;$$

hence $y_{2j}^2 \in J$. Finally, $y_{1j} y_{2n-1} - y_{2j} y_{2n-2} \in I$ and hence,

$$y_{2n-1}^2 (y_{1j} y_{2n-1} - y_{2j} y_{2n-2}) \in I^2.$$

But this is equal to

$$y^3 y_{1j} - y^2 y_{2j} y_{2n-2}$$

and $y^2 y_{2j} y_{2n-2} = (y y_{2j})(y y_{2n-2}) \in I^2$. Hence $y^3 y_{ij} \in I^2$ and so $y_{ij} \in J$ as required.

REFERENCES

1. S. ABEASIS AND A. DEL FRA, Young Diagrams and Ideals of Pfaffians, preprint, Universita Degli Studi di Roma, 1978–1979.
2. M. AUSLANDER AND D. BUCHSBAUM, Codimension and multiplicity, *Ann. of Math.* **68** (1958), 625–657.
3. M. BRODMANN, The asymptotic nature of analytic spread, to appear.
4. D. BUCHSBAUM AND D. EISENBUD, Algebra structures for finite free resolutions and some structure theorems for ideals of codimension three, *Amer. J. Math.* **99**, No. 3 (1977), 447–485.
5. L. BURCH, Codimension and analytic spread, *Proc. Cambridge Philos. Soc.* **72**(1972), 369–373.
6. R. COWSIK AND NORI, On the fibers of blowing up, preprint.
7. C. DECONCINI AND C. PROCESI, A characteristic-free approach to invariant theory, *Advances in Math.* **21** (1976), 333–354.
8. C. DECONCINI, D. EISENBUD, AND C. PROCESI, “Young Diagrams and Determinantal Varieties,” to appear.
9. C. DECONCINI, D. EISENBUD, AND C. PROCESI, Algebras with straightening law, to appear.
10. DOUBILLET, G. C. ROTA, AND STEIN, Foundations of combinatorics. IX. combinatorial methods in invariant theory, *Studies in Appl. Math.* **53** (1974), 185–216.
11. J. EAGON AND M. HOCHSTER, Cohen–Macaulay rings, invariant theory and generic perfection of determinantal loci, *Amer. J. Math.* **93** (1971), 1020–1058.
12. M. FIORENTINI, On relative regular sequences, *J. Algebra* **18** (1971), 384–389.
13. J. HERZOG, Ein Cohen–Macaulay Kriterium mit Anwendungen auf den Konormalen module und den Differential module, preprint.
14. M. HOCHSTER, Criteria for equality of ordinary and symbolic powers of primes, *Math. Z.* **133** (1973), 53–65.
15. C. HUNEKE, The theory of d -sequences and powers of ideals, *Adv. in Math.*, in press.
16. C. HUNEKE, On the symmetric and Rees algebras of an ideal generated by a d -sequence, *J. Algebra* **62** (1980), 268–275.
17. C. HUNEKE, Symbolic powers of prime and special graded algebras, *Comm. Algebra*, in press.
18. R. E. KUNZ, Cohen–Macaulay rings and ideal theory in rings of invariants of algebraic groups, *Trans. Amer. Math. Soc.* **194** (1974), 115–129.
19. H. MATSUMURA, “Commutative Algebra,” Benjamin, New York, 1970.
20. D. G. NORTHCOTT AND D. REES, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.
21. L. ROBBIANO, A property of $IP^1 \times IP^n$, *Comm. Algebra* (1978).
22. L. ROBBIANO AND G. VALLA, On normal flatness and normal torsion freeness, *J. Algebra* **43** (1976), 222–229.
23. L. ROBBIANO AND G. VALLA, Primary powers of a prime ideal, *Pacific J. Math.* **63**, No. 1. (1976), 491–498.
24. J. SALLY, Super-regular sequences, to appear.
25. J. STUCKROD AND W. VOGEL, Towards a theory of Buchsbaum singularities, *Amer. J. Math.* **100** (1978), 727–746.