Discrete Mathematics 33 (1981) 149–162

ORIENTED TWO-DIMENSIONAL CIRCUITS

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Received 4 January 1979

The natural generalization of a directed graph is an oriented complex, a fundamental concept in algebraic topology. Our study of such complexes follows combinatorial rather than topological lines; when an n-circuit is defined for oriented complexes as a structure achieved by a certain minimization process, we are able to pose a question not easily answered by topological methods, but one directly accessible by elementary combinatorial techniques. Indeed, having asked ourselves what structure such n-circuits possess, we were able to find an answer, at least when $n=2$.

1. Preliminaries

Let V be a finite set of elements called *vertices*. A *complex* (with vertex set V) is a collection of subsets of V, called *simplexes*, such that every nonempty subset of a simplex is a simplex. If K is a complex and $x \in X$, then the *dimension of the simplex* x is $|x|-1$; the *dimension of the complex* K is the maximum dimension of its simplexes. An n-dimensional complex (simplex) is called an *n-complex* (*n-simplex*) for brevity.

Two simplexes are said to be *incident* if one contains the other. Let $x_1, y_1, x_2, y_2, \ldots, y_{n-1}, x_n$ be an alternating sequence of m-simplexes and n-simplexes. If this sequence has the properties

(a) y_i is incident with x_i and x_{i+1} ($i=1,2,\ldots,n-1$), and

(b) the simplexes in the sequence are all distinct,

then it is called an *(m,n)-path* and is said to *connect* x_1 and x_n. Thus we may define a complex to be *(m,n)-connected* if each pair of its m-simplexes are connected by an (m,n)-path.

These definitions are generalizations of well-known graph-theoretic concepts, see for example [1]. However, this semi-expository paper is concerned primarily with generalizations of concepts from the theory of directed graphs. We give such general definitions now, along with some examples. In the following sections, we develop some basic theory and then prove the main result, a structure theorem for "2-circuits", a concept shortly to be defined.

Let K be a complex and let $x=\{v_0, v_1, \ldots, v_n\}$ be a simplex of K. Two orderings $(v_{i_0}, v_{i_1}, \ldots, v_{i_n})$ and $(v_{j_0}, v_{j_1}, \ldots, v_{j_n})$ of the vertices of x will be called *equivalent* if $i_0, i_1, \ldots, i_n$ is an even permutation of $j_0, j_1, \ldots, j_n$. Each equivalence class of orderings of x under this relation is called an ***orientation*** of x. If $|x| \geq 2$, then x has exactly two orientations: if one of these is denoted by σ, then the other will be denoted by $-\sigma$. If $|x| = 1$, then x has exactly one orientation. Denote by $\vec{K}$ the set of all orientations of the simplexes in K and by $\vec{K}_{(n)}$ the set of all orientations of the simplexes in $K_{(n)}$, the set of all n-simplexes of K.

The members of $\vec{K}_{(n)}$ will be called *oriented n-simplexes.* For any ordering $(v_{i_0}, v_{i_1}, \ldots, v_{i_n})$ of x, we will denote by $v_{i_0}v_{i_1} \cdots v_{i_n}$ the orientation of x to which this ordering belongs. If $\sigma = v_{i_0}v_{i_1} \cdots v_{i_n}$, we define $\|\sigma\| = x$. A mapping $o : K_{(n)} \to \vec{K}_{(n)}$ is called an *n-orientation* of K if $\|o(x)\| = x$ for every $x \in K_{(n)}$; this condition merely insures that o maps x into one of its orientations. Given a complex K and an n-orientation o of K, the pair (K, o) will be called an *n-oriented complex.* When $\dim(K) = 1$ and $n = 1$, (K, o) is just an oriented graph, i.e., a directed graph with no symmetric pairs of arcs.

We now illustrate in Fig. 1(a) the concepts just defined.

Here is a complex K with its vertices labelled v_1 to v_6. The simplex $x = \{v_3, v_4, v_5\}$ has two possible orientations which we may represent by the (inequivalent) orderings $v_3v_4v_5$ and $v_3v_5v_4$. Our convention is to follow common practice in textbooks of elementary algebraic topology by indicating an orientation of a 2-simplex such as x by a circular arrow. In the figure, we have chosen the orientation corresponding to the second of the two orderings since the arrow "visits" these vertices in the sequence $v_3v_5v_4$. Of course, the same orientation could as easily be represented by $v_5v_4v_3$ or $v_4v_3v_5$. Since all the 2-simplexes of K have been oriented, we have a 2-oriented complex.

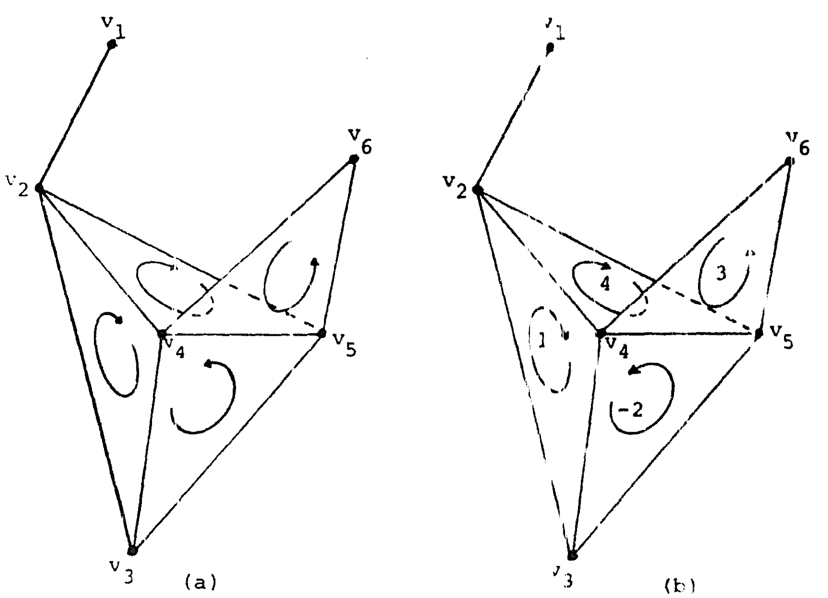

Fig. 1. A 2-oriented complex and some integral chains.

The concepts of "integral n-chain", "incidence number" and "boundary" are borrowed from algebraic topology [2, 3, 4]. However, these concepts are used in a way which owes more to discrete optimization theory and less to the field from which they were borrowed.

An *integral n-chain* on K is any function $f:\vec{K}_{(n)}\to Z$ which, when $n\geq 1$, satisfies the condition that $f(\sigma)=-f(-\sigma)$ for every $\sigma\in\vec{K}_{(n)}$. An integral n-chain is thus completely specified when its value on one orientation of each simplex in $K_{(n)}$ is given. Given two integral n-chains f and g on K, f will be called an *integral subchain* of g, written $f\leq g$, if for every $\sigma\in\vec{K}_{(n)}$ either $0\leq f(\sigma)\leq g(\sigma)$ or $0\geq f(\sigma)\geq g(\sigma)$. If $f(\sigma)=g(\sigma)$ for every $\sigma\in\vec{K}_{(n)}$, we write $f=g$. If $f\leq g$ and $f\neq g$, it is reasonable to write $f<g$ and call f a *proper integral subchain* of g. If $f(\sigma)=0$ for every $\sigma\in\vec{K}_{(n)}$, then we write $f=0$ and call f the *zero n-chain* on K.

If $\sigma=v_1\cdots v_n$ is an oriented simplex and if $v_0\notin\{v_1,\ldots,v_n\}$, denote by $v_0\sigma$ the oriented simplex $v_0v_1\cdots v_n$ and observe that $v_0(-\sigma)=-v_0\sigma$. Given two oriented simplexes σ and τ of dimensions $n-1$ and n respectively, we define the *incidence number* $[\sigma,\tau]$ as follows:

$$
\begin{aligned}
&[\sigma,\tau]=0 && \text{if } \|\sigma\|\not\subseteq\|\tau\|,\\
&[\sigma,\tau]=1 && \text{if } \tau=v\sigma,\\
&[\sigma,\tau]=-1 && \text{if } \tau=-v\sigma, \text{ where } v\in\|\tau\|-\|\sigma\|.
\end{aligned}
$$

If $x\in K$, define $N(x,K)$ to be the set $\{v\in V(K)-x: x\cup\{v\}\in K\}$ and define $N(\sigma,K)$ to be $N(\|\sigma\|,K)$. When it is clear what complex K is meant, we will replace $N(x,K)$ and $N(\sigma,K)$ by $N(x)$ and $N(\sigma)$, respectively. If $\sigma\in\vec{K}_{(n-1)}$ and f is an integral n-chain on K we define

$$
\partial f(\sigma)=\begin{cases}\sum_{v\in N(\sigma)} f(v\sigma), & N(\sigma)\neq\emptyset,\\ 0, & N(\sigma)=\emptyset.\end{cases} \tag{1}
$$

The function $\partial f:\vec{K}_{(n-1)}\to Z$ so obtained, is called the *boundary* of f and is clearly an $(n-1)$-chain since, if $n\geq 2$,

$$
\begin{aligned}
\partial f(-\sigma) &= \sum_{v\in N(-\sigma)} f(v(-\sigma)) = \sum_{v\in N(\sigma)} f(-v\sigma)\\
&= -\sum_{v\in N(\sigma)} f(v\sigma) = -\partial f(\sigma).
\end{aligned}
$$

In Fig. 1(b), each of the oriented 2-simplexes of Fig. 1(a) has been given an integer value. By definition, this yields an integral 2-chain f on K. Now let the oriented 1-simplex σ be represented by the directed edge v_4v_5. We will compute the value of ∂f at σ by first observing that there are only three vertices in $N(\sigma)$. These are v_2, v_3 and v_6, for each of these forms a 2-simplex with $\|\sigma\|=\{v_4,v_5\}$. It follows that

$$
\partial f(\sigma)=f(v_2\sigma)+f(v_3\sigma)+f(v_6\sigma).
$$

Now $v_2\sigma = v_2v_4v_5$ and, since this orientation is opposite to the one shown (at which $f=4$), we have $f(v_2\sigma) = -4$. Similarly, $v_3\sigma$ has the orientation opposite to the one at which $f=-2$. Consequently, $f(v_3\sigma) = 2$. Finally, $v_6\sigma$ is the orientation shown, at with $f=3$. Thus $\partial f(\sigma) = -4+2+3 = 1$.

The above definition (i) of boundary coincides with the usual one: in our notation the definition of ∂ given in [2, p. 226] is

$$\partial f(\sigma) = \begin{cases} \sum_{\tau \in o(K_{(n)})} [\sigma, \tau] f(\tau), & K_{(n)} \neq \emptyset, \\ 0, & K_{(n)} = \emptyset, \end{cases} \tag{2}$$

where o is an n-orientation of K. It is readily checked that $f(v\sigma) = [\sigma, \tau] f(\tau)$, where $\tau = \pm v\sigma$.

We shall have occasion to use the definition of ∂ given in both (1) and (2) above.

The next two results about intersection numbers and the boundary ∂ appear as Theorems 6-3 and 6-4 in [2, pp. 224, 227].

Theorem 1. *If f is an integral n-chain on K, then $\partial\partial f = 0$.*

Theorem 2. *If K is a complex, $\rho \in \vec{K}_{(n-2)}$, the oriented simplexes $\sigma, \sigma' \in \vec{K}_{(n-1)}$, $\tau \in \vec{K}_{(n)}$, and if $\|\sigma\|, \|\sigma'\|$ are each incident with both $\|\rho\|, \|\tau\|$, then*

$$[\sigma, \tau][\sigma', \tau] = -[\rho, \sigma][\rho, \sigma'].$$

Strictly speaking, Theorem 2 is not identical to Theorem 6-4 in [2], but is contained in it as a deduction.

The following definition of an integral n-circuit is new, to our knowledge.

Let c be an integral n-chain on K such that $\partial c = 0$. Then c is called an *integral n-cycle* on K. A non-zero integral n-cycle c satisfying the following condition is called *an integral n-circuit* on K:

(c) If c' is an integral non-zero n-cycle on K such that $c' \leqslant c$, then $c' = c$.

Given an n-oriented complex (K, o) and an integral n-chain f on K, we will say that f is *positive* on (K, o) if $f(o(x)) \geqslant 0$ for every $x \in K_{(n)}$. The next theorem shows that we may restrict the minimization condition (c) to those integral non-zero n-cycles which are positive.

Theorem 3. *Let (K, o) be an n-oriented complex and c a non-zero integral n-cycle on K which is positive on (K, o). Then c is an integral n-circuit on K if and only if it satisfies the condition:*

(c′) *If c' is an integral non-zero n-cycle on K which is positive on (K, o) and if $c' \leqslant c$, then $c' = c$.*

Proof. If c is an integral n-circuit on K, then condition (c) above implies condition (c′) here.

Let c be a non-zero integral n-cycle on K which is positive on (K, o) and which satisfies condition (c′). Let c' be any non-zero n-cycle on K such that $c' \leqslant c$. If $x \in K_{(n)}$, then $c'(o(x)) \geqslant 0$ since $c' \leqslant c$. Hence c' is also positive on (K, o) and, by condition (c′), $c' = c$.

Obviously, there can be an infinite number of integral n-chains, including integral n-cycles, on some n-complexes. However, this is not true for integral n-circuits, as we now show.

Theorem 4. *There are only a finite number of distinct integral n-circuits on any complex K.*

Proof. This observation depends on the fact that for no two distinct integral n-circuits c and c' on K can we have $c \leqslant c'$.

Assume that there exist an infinite number of distinct integral n-circuits on K. These are obviously denumerable and can be listed as $c_1, c_2, c_3, \ldots$. For any infinite subsequence $s = i_1, i_2, i_3, \ldots$ of $1, 2, 3 \ldots$, and any oriented n-simplex $\sigma \in \vec{K}$, denote by $c_s(\sigma)$ the sequence $c_{i_1}(\sigma), c_{i_2}(\sigma), c_{i_3}(\sigma), \ldots$. Let $K_{(n)} = \{x_1, x_2, \ldots, x_\alpha\}$, where α is the number of n-simplexes in K and for each n-simplex x_i in this set, choose an orientation σ_i. Now select an infinite subsequence s_1 of $1, 2, 3, \ldots$ such that $c_{s_1}(\sigma_1)$ is monotonic, an infinite subsequence s_2 of s_1 such that $c_{s_2}(\sigma_2)$ is monotonic, and so on, ending with an infinite subsequence s_α of $s_{\alpha-1}$ such that $c_{s_\alpha}(\sigma_\alpha)$ is monotonic. A subsequence of a monotonic sequence is of course monotonic. It follows that $c_{s_\alpha}(\sigma_i)$ is monotonic for each $i = 1, 2, \ldots, \alpha$. If $c_{s_\alpha}(\sigma_i)$ is monotonic increasing (decreasing), then there is an integer k_i such that all terms of $c_{s_\alpha}(\sigma_i)$ beyond the k_ith are positive (negative) or else equal a constant quantity. Let k be the maximum of all integers k_i and observe that if c_{i_1} and c_{i_2} are two integral n-circuits such that $i_1, i_2 \in s_\alpha$, $k < i_1 < i_2$, then $c_{i_1} \leqslant c_{i_2}$. This contradiction yields the result. □

The statement that for some n-complexes there exist an infinite number of integral n-cycles is true even for $n = 1$. Here, our terminology conflicts slightly with that of graph theory [1] where a "cycle" means, in this context, a 1-cycle in which all the coefficients are unity. Of the latter kind of cycle, there are obviously only a finite number. However, in the next section, these unit coefficients play an important role in describing the structure of integral n-circuits.

2. Structure of integral 2-circuits

When $n = 1$, an integral n-circuit turns out to mean exactly the same thing as "circuit". In this case, the coefficients turn out to be unity. When $n > 1$, this is not always the case and a definition becomes necessary to distinguish such simple integral n-circuits.

Let K be a complex and f an n-chain on K. Then f is called *primitive* if $|f(\sigma)| \le 1$ for every $\sigma \in \vec{K}_{(n)}$. It is not difficult to show that if c is an integral 1-circuit on K, then c is primitive. This fact is due both to the minimality condition in the definition of c and to the especially simple structure of 1-complexes. We show in the next theorem how far from being primitive certain integral 2-circuits can be. We then show how the structure of a non-primitive integral 2-circuit can be related to a set of primitive ones.

Theorem 5. *Given any positive integer m, there exist a 2-complex K and an integral 2-circuit c on K such that* $|c(\sigma)| \ge m$ *for every* $\sigma \in \vec{K}_{(2)}$.

Proof. First a complex L_k is constructed, then a 2-orientation o of L_k is defined. A complex K is constructed from two complexes like L_k and the required oriented 2-circuit is then defined.

Let k be a positive integer and let L_k be the 2-complex having $6k$ vertices and the following 2-simplexes:

$$\{u_1, v_1, w_1\}, \quad \{u_2, v_{3k-1}, w_{3k-1}\};$$

$$\left.\begin{array}{l}\{u_2, v_{3j-1}, v_1\}, \quad \{v_1, v_{3j-1}, w_{3j-2}\}, \quad \{w_{3j-2}, w_{3j-1}, u_1\},\\ \{v_{3j-1}, w_{3j-1}, w_{3j-2}\}, \quad \{w_{3j-1}, u_2, u_1\}\end{array}\right\} j = 1, 2, \ldots, k;$$

$$\left.\begin{array}{l}\{u_2, u_{3j}, v_{3j-1}\}, \quad \{v_{3j-1}, v_{3j}, w_{3j-1}\}, \quad \{w_{3j-1}, w_{3j}, u_2\},\\ \{u_{3j}, v_{3j}, v_{3j-1}\}, \quad \{v_{3j}, w_{3j}, w_{3j-1}\}, \quad \{w_{3j}, u_{3j}, u_2\}\end{array}\right\} j = 1, 2, \ldots, k-1;$$

$$\left.\begin{array}{l}\{u_{3j}, u_1, v_{3j}\}, \quad \{v_{3j}, v_1 w_{3j}\}, \quad \{w_{3j}, w_{3j+1}, u_{3j}\},\\ \{u_1, v_1, v_{3j}\}, \quad \{v_1, w_{3j+1}, w_{3j}\}, \quad \{w_{3j+1}, u_1, u_{3j}\}\end{array}\right\} j = 1, 2, \ldots, k-1.$$

The complex L_k can also be seen as the complex obtained from the complex L'_k in Fig. 2 below by identifying certain pairs of vertices. Complex L'_k has the form of a triangular tube with the ends closed off and having k triangular openings (u_1, v_1, u_2), (u_4, v_4, u_5), and so on. These openings are all put together by identifying the vertices u_{3j+1} with u_1, u_{3j+2} with u_2 and v_{3j+1} with v_1 ($j = 1, 2, \ldots, k-1$).

Now define the 2-orientation o of L_k so that the following oriented 2-simplexes

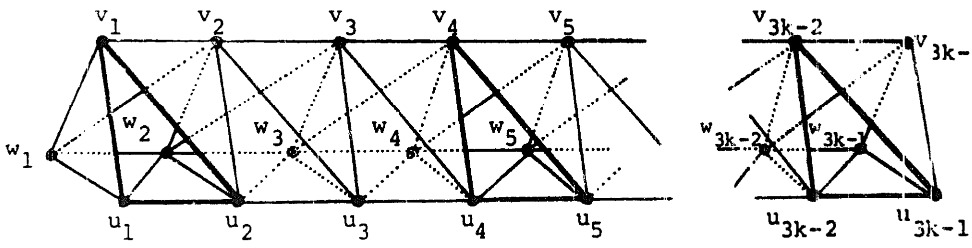

Fig. 2. The 2-complex L'_k.

lie in $o(L_{k(2)})$:

$$\begin{aligned}
&w_1v_1u_1 \quad u_2v_{3k-1}w_{3k-1};\\
&\left.\begin{aligned}&v_{3j-1}u_2v_1, \quad v_1w_{3j-2}v_{3j-1}, \quad w_{3j-2}u_1w_{3j-1},\\ &w_{3j-1}v_{3j-1}w_{3j-2}, \quad u_2w_{3j-1}u_1\end{aligned}\right\} j=1,2,\dots,k;\\
&\left.\begin{aligned}&u_2v_{3j-1}u_{3j}, \quad v_{3j-1}w_{3j-1}v_{3j}, \quad w_{3j-1}u_2w_{3j},\\ &v_{3j}u_{3j}v_{3j-1}, \quad w_{3j}v_{3j}w_{3j-1}, \quad u_{3j}w_{3j}u_2\end{aligned}\right\} j=1,2,\dots,k-1;\\
&\left.\begin{aligned}&u_{3j}v_{3j}u_1, \quad v_{3j}w_{3j}v_1, \quad w_{3j}u_{3j}w_{3j+1},\\ &v_1u_1v_{3j}, \quad w_{3j+1}v_1w_{3j}, \quad u_1w_{3j+1}u_{3j}\end{aligned}\right\} j=1,2,\dots,k=1.
\end{aligned}$$

Denote by S_k the set of 1-simplexes consisting of $\{u_1, v_1\}\{u_2, v_1\}, \{u_1, u_2\}$

Let f be the integral 2-chain on L_k such that $f(o(y)) = r$ for every 2-simplex y of L_k. If σ is any oriented 1-simplex of $\vec{L}_k$ such that $\|\sigma\| \notin S_k$, then it is clear, either from examining the list of 2-simplexes in L_k or considering Fig. 2 and visualizing the identification, that $\|\sigma\|$ is incident with exactly two 2-simplexes in L_k. If these 2-simplexes are denoted by y_1 and y_2, then by the definition of o above $[\sigma, o(y_1)] = -[\sigma, o(y_2)]$. This implies that

$$\begin{aligned}
(\partial f)(\sigma) &= \sum_{y \in L_{k(2)}} [\sigma, o(y)]f(o(y))\\
&= [\sigma, o(y_1)]f(o(y_1)) + [\sigma, o(y_2)]f(o(y_2)) = 0.
\end{aligned}$$

Now if $\sigma \in \vec{L}_{k(1)}$ and $\|\sigma\| \in S_k$, then it is easily shown that $\partial f(\sigma) = \pm kr$, the sign of $\partial f(\sigma)$ depending on which orientation of $\|\sigma\|$ is taken. For example, if $\sigma = v_1u_1$, there are k 2-simplexes incident with $\|\sigma\|$, namely $\{u_1, v_1, w_1\}$ and $\{u_1, v_1, v_{3j}\}$, $j = 1, 2, \dots, k-1$. By the definition of o, for each of these 2-simplexes y, $o(y)$ can be written tv_1u_1, where t is one of w_1, v_{3j}, $j = 1, 2, \dots, k-1$. Therefore

$$\begin{aligned}
\partial f(\sigma) &= \sum_{y \in L_{k(2)}} [\sigma, o(y)]f(o(y))\\
&= \sum_t [v_1u_1, tv_1u_1]r = kr.
\end{aligned}$$

On the other hand, if $\sigma = u_1v_1$ we find that $\partial f(\sigma) = -kr$.

Let g be any integral 2-chain on L_k, positive on (L_k, o) and satisfying $\partial g(\sigma) = 0$ for each $\sigma \in \vec{L}_{k(1)}$ such that $\|\sigma\| \notin S_k$. Given any two 2-simplexes y and y' in L_k, there is a (2, 1)-path connecting them, no interior 1-simplex of which lies in S_k. Let a corresponding (2, 1)-path sequence be $y = y_1, x_1, y_2, x_2, \dots, x_m, y_{m+1} = y'$. Now

$$\partial g(\sigma) = [\sigma, o(y_i)]g(o(y_i)) + [\sigma, o(y_{i+1})]g(o(y_{i+1})) \quad \text{if } \|\sigma\| = x_i,$$

for $i = 1, 2, \dots, m$. By the definition of o above, $[\sigma, o(y_i)] = -[\sigma, o(y_{i+1})]$. Since $\partial g(\sigma) = 0$ by assumption, $g(o(y_i)) = g(o(y_{i+1}))$ for each $i = 1, 2, \dots, m$. Therefore, $g(o(y)) = g(o(y'))$ and it follows that g has the same value, say q, at every oriented 2-simplex $o(y)$, $y \in L_{k(2)}$. We may now replace f and r above by g and q respectively, and obtain $\partial g(\sigma) = \pm kq$ if $\|\sigma\| \in S_k$.

We are now ready to construct complex K and circuit c of the theorem. Let s, t be distinct primes with $s>t>m$. Let K be the 2-complex consisting of two subcomplexes L_s, L_t and satisfying the following condition: the vertices of L_s and L_t, corresponding to u_j, v_j, w_j, denoted by u_{sj}, v_{sj}, w_{sj} and u_{tj}, v_{tj}, w_{tj} respectively, are all distinct with the following exceptions: $u_{s1}=u_{t1}, u_{s2}=v_{t1}, v_{s1}=u_{t2}$. There should be no confusion in speaking of the 2-orientation o of K for which $o|_{L_s}$ and $o|_{L_t}$ are just the 2-orientation given to L_k above when $k=s$ and $k=t$ respectively.

Define the integral 2-chain c on K by the equation

$$c(o(y))=\begin{cases} t, & y\in L_{s(2)},\\ s, & y\in L_{t(2)}.\end{cases}$$

Denote by c_s and c_t the restriction of c to $\vec{L}_{s(2)}$ and $\vec{L}_{t(2)}$ respectively and observe, with c_s in place of f and s in place of k above, that

$$\partial c_s(\sigma)=0, \qquad \sigma\in \vec{L}_{s(1)}, \quad \|\sigma\|\notin S_s \tag{3}$$

and

$$\partial c_s(\sigma)=\pm st, \qquad \sigma\in \vec{L}_{s(1)}, \quad \|\sigma\|\in S_s. \tag{4}$$

Obviously, equations (3) and (4) hold with s and t interchanged. In equation (3), $\|\sigma\|$ is incident with the same 2-simplexes in K as in L_s and it follows in this case that $\partial c(\sigma)=0$. In equation (4), the sign of $\partial c_s(\sigma)$ is determined exactly as for $\partial f(\sigma)$. For example, with $\sigma=v_{s1}u_{s1}, \partial c_s(\sigma)=\partial c_s(v_{s1}u_{s1})=st$. We also have $\sigma=u_{t2}u_{t1}$ and since $\partial c_t(u_{t2}u_{t1})=-st$, it follows that $\partial c(\sigma)=\partial c_s(\sigma)+\partial c_t(\sigma)=0$. The same equation is readily verified for any choice of σ such that $\|\sigma\|\in S_s=S_t$. Therefore, c is an integral 2-cycle on K. Moreover, c is positive on (K, o).

Suppose now that c' is any non-null integral 2-cycle on K, which is positive on (K, o) and satisfies $c'\leq c$. Let $c_s'=c'|_{L_s}$, $c_t'=c'|_{L_t}$. Replacing the integral 2-chain g introduced above by c_s', we observe that c_s' has the same value, say $q_s\geq 0$, at all oriented 2-simplexes $o(y), y\in K_s$. On the other hand, $\partial c_s'(\sigma)=\pm sq_s, \|\sigma\|\in S_s$. Specifically, $\partial c_s'(v_{s1}u_{s1})=sq_s$. Similarly, $\partial c_t'(u_{t2}u_{t1})=-tq_t$, where q_t is the constant value of c_t' on $o(L_{t(2)})$. Now $v_{s1}u_{s1}=u_{t2}u_{t1}$ and, since c' is an integral 2-cycle on K,

$$\begin{aligned} 0=\partial c_c'(v_{s1}u_{s1})&=\partial c_s'(v_{s1}u_{s1})+\partial c_t'(u_{t2}u_{t1})\\ &=sq_s-tq_t.\end{aligned}$$

Therefore, $sq_s=tq_t$, which, since s and t are distinct primes, implies that $q_s=rt$ and $q_t=rs$ for some non-negative integer r. Therefore, $c'(o(y))=c_s'(o(y))=rt$ for every $y\in L_{s(2)}$ and $c'(o(y))=c_t'(o(y))=rs$ for every $y\in L_{t(2)}$. Since $c'\leq c$ and $c'\neq 0$, it follows that $r=1$ and $c'=c$. As c' was arbitrarily chosen and positive on (K, o), c is an integral 2-circuit on (K, o) by Theorem 3.

Clearly, $c(o, y))\geq m$ for every $y\in K_{(2)}$ and the theorem follows $\square$

Although the emphasis of this section is primarily on the structure of integral 2-circuits, it is appropriate to indicate briefly how Theorem 5 may be generalized.

This is done inductively as follows. Suppose that K is an n-complex and c an integral n-circuit on K such that $|c(\sigma)| \geq m$ for every $\sigma \in \vec{K}_{(n)}$. Denote by DK the complex

$$K \cup \{x \cup \{s\} : x \in K\} \cup \{x \cup \{t\} : x \in K\},$$

where s and t are two vertices not in $V(K)$. If we define an integral $(n+1)$-chain on DK

$$d(s\sigma) = c(\sigma), \quad d(t\sigma) = -c(\sigma),$$

$\sigma \in \vec{K}_{(n)}$, then it is easy to prove that d is an integral n-circuit on DK. Obviously $|d(\tau)| \geq m$ for every $\tau \in \overrightarrow{DK}_{(n+1)}$. Thus the truth of Theorem 5 with 2 replaced by n, $n \geq 2$, implies its truth with 2 replaced by $n+1$.

Let K, K' be complexes with $\phi : V(K') \to V(K)$ a mapping such that if $x' \in K'$, then $\phi(x') \in K$. Then ϕ is called a *simplical map* [2, p. 249]. If for any simplex $x' \in K'$, $\dim(\phi(x')) < \dim(x')$, then ϕ is said to *collapse* x'. The mapping ϕ is extended to oriented simplexes $\sigma' = v_0'v_1' \cdots v_n'$, where $\|\sigma'\|$ is not collapsed by ϕ, as follows:

$$\phi(v_0'v_1' \cdots v_n') = \phi(v_0')\phi(v_1') \cdots \phi(v_n').$$

If $\sigma \in \vec{K}_{(n)}$, let $\phi^{-1}(\sigma)$ denote the set of all $\sigma' \in \vec{K}'_{(n)}$ such that $\phi(\sigma') = \sigma$. If f' is an n-chain on K', let $\phi f'$ be the n-chain f on K such that for every $\sigma \in \vec{K}_{(n)}$,

$$f(\sigma) = \begin{cases} \sum_{\sigma' \in \phi^{-1}(\sigma)} f'(\sigma') & \text{if } \phi^{-1}(\sigma) \text{ is non-empty,} \\ 0, & \text{if } \phi^{-1}(\sigma) \text{ is empty.} \end{cases}$$

Now suppose that $\phi : V(K') \to V(K)$ is a simplicial map and that $\phi(K') = K$. Denote by $V_\phi(K')$ the set of vertices of K' which are fixed under ϕ. The number $|V(K') - V_\phi(K')|$ will be called the *order* of ϕ. If $\phi(v) \in V_\phi(K')$ for every $v \in V(K')$, then ϕ will be called a *contraction* of K'. From this definition, it is easily proved that the composition of two contractions is a contraction.

In order to display the structure of integral 2-circuits, two lemmas are needed.

Lemma 6a. *If $\phi : V(K') \to V(K)$ is a simplical map and c' is an integral n-cycle on K', then $c = \phi(c')$ is an integral n-cycle on K.*

Proof. This is Lemma 6-15 in [2, p. 251]. □

Let K be a complex, f an integral n-chain on K. Then f will be called *univalent over* K if $|f(\tau)| = |f(\tau')|$ for every two oriented n-simplexes τ, τ' such that $\|\tau\| \cap \|\tau'\| \neq \emptyset$ and $f(\tau), f(\tau') \neq 0$.

Lemma 6b. *Let K be a complex, c an integral n-circuit on K. If c is univalent over K, then c is primitive.*

Proof. Denote by C the smallest complex whose n-simplexes lie in $\{\|\sigma\|: c(\sigma) \neq 0\}$. If C is not connected, it may be written as the disjoint union $C = C_1 + C_2$ of two 2-complexes. Define c_i to be the integral 2-chain on K such that

$$c_i(\tau) = \begin{cases} c(\tau), & \tau \in \vec{C}_{i(2)} \\ 0, & \tau \notin \vec{C}_{i(2)} \end{cases}, \quad i = 1, 2.$$

Observe that c_1 and c_2 are non-zero integral 2-cycles on K such that $c_1 \leq c$, $i = 1, 2$. Since c is an integral 2-circuit on K, $c = c_i$, $i = 1, 2$. But obviously $c = c_1 + c_2$, implying that one of c_1 or c_2 is zero and contradicting the fact that both are non-zero.

Since every simplex of C lies in an n-simplex and since C is connected, C is $(0, n)$-connected. Now let $u, v \in V(C)$. There is a $(0, n)$-path $\{u\} = x_0, y_0, x_1, y_1, \ldots, y_{n-1}, x_n = \{v\}$ in C. Denote by k_i the positive integer such that $|c(\tau)| = k_i$ whenever $x_i \subseteq \|\tau\|$. For each $i = 0, 1, \ldots, n-1$, $x_1, x_{i+1} \subseteq y_i$ and for any orientation τ_i of y_i, $|c(\tau_i)| = k_i, k_{i+1}$. It follows that $k_0 = k_1 = \cdots = k_n$. Therefore, $|c(\tau)|$ is a constant, say k, over all $\tau \in \vec{C}_n$. Thus $(1/k)c$ is a non-zero integral n-cycle for which $(1/k)c \leq c$. Since c is an integral n-circuit, $k = 1$ and c is, therefore, primitive. □

As seen in Theorem 5 there are non-primitive integral 2-circuits. Such integral 2-circuits are now related to primitive ones through the notion of contraction.

Theorem 6. *Let c be an integral 2-circuit on a complex K. There is a complex K', a primitive integral 2-circuit c' on K' and a contraction ϕ of K' such that*

(i) *$K = \phi(K')$ and*

(ii) *$c = \phi c'$.*

Proof. Let d be an arbitrary integral 2-circuit on a complex L. Define the quantity

$$p(d) = \sum_{\tau \in L_{(2)}} \big| |d(\tau)| - 1 \big| \cdot |d(\tau)|$$

and observe that d is primitive if and only if $p(d) = 0$. Otherwise $p(d) > 0$. The quantity $p(d)$ thus measures the "non-primitiveness" of d. The proof proceeds by induction on $p(c)$.

If $p(c) = 0$, there is nothing to prove since f may be taken as the identity map (a contraction of order 0) on $K' = K$ and c is already primitive.

Let $p(c) = k > 0$ and assume the theorem holds for all integral 2-circuits d on K such that $p(d) < k$. Since $p(c) > 0$, c is non-primitive and, by Lemma 6b, non-univalent. Let v be a vertex of K for which there exist two oriented 2-simplexes τ, τ' such that $v \in \|\tau\|, \|\tau'\|$ and $c(\tau), c(\tau') \neq 0$ while $|c(\tau)| \neq |c(\tau')|$. Denote by m the maximum non-zero value of $|c(\tau)|$ over all oriented 2-simplexes τ such that

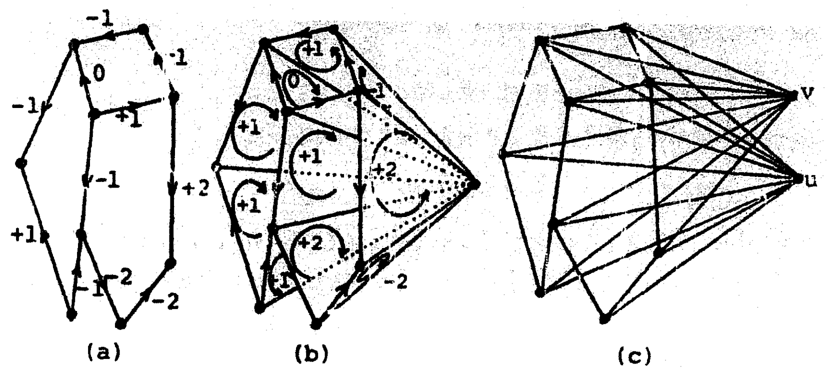

Fig. 3. The basic operation.

$v \in \|\tau\|$. Define a 1-chain c_v on $K_v = \mathrm{lk}(v, K)$, the link of v in K, by

$$c_v(\sigma) = c(v\sigma), \quad \sigma \in \vec{K}_{v(1)}.$$

To better visualize the formation of c_v, Fig. 3(a) illustrates K_v with the values of c_v at orientations of its 1-simplexes. Fig. 3(b) displays a portion of $\vec{K}$ in the neighborhood of v with the values of c at orientations of its 2-simplexes.

We now observe that:

(d) c_v is an integral 1-cycle on K_v, and that:

(e) for any non-zero integral 1-cycle f on a complex and for any oriented 1-simplex σ such that $f(\sigma) \neq 0$, there is a primitive integral 1-circuit g on the complex such that $g(\sigma) \neq 0$ and $g \leq f$.

Statement (e) is obvious. To establish statement (d), let ρ be an oriented 0-simplex, i.e., a vertex, say $\rho = w$, in $\vec{K}_v$. Then

$$\begin{aligned}
\partial c_v(\rho) &= \sum_{u \in N(\rho, K_v)} c_v(u\rho), && \text{by the definition of } \partial \\
&= \sum_{u \in N(\rho, K_v)} c(vuw), && \text{by the definition of } c_v \\
&= -\sum_{u \in N(vw, K)} c(uvw), && \text{since } vuw = -uvw \text{ and } N(\rho, K_v) = N(vw, K) \\
&= -\partial c(vw) = 0.
\end{aligned}$$

Therefore, c is an integral 1-cycle on K_v. Since $m \geq 1$, c_v is non-zero and, by statement (e), there exists a primitive integral 1-circuit g on K_v such that $g \leq c_v$ and such that $g(\sigma_1) \neq 0$, where $c_v(\sigma_1) = c(v\sigma_1) = m$.

Now let u be a vertex not in $V(K)$ and define $L = K \cup C(u, K_v)$ as illustrated in Fig. 3(c). Here only the 1-simplexes have been displayed: the 2-simplexes are easily inferred from Fig. 3(b). Let ψ be the mapping $\psi: V(L) \to V(L)$ defined by $\psi(u) = v$, ψ being the identity elsewhere on L. Obviously ψ is a contraction and $K = \psi(L)$. In what follows denote by S_v and S_u the sets of oriented 2-simplexes containing v and u respectively.

Define an integral 2-chain d on L as follows:

$$d(\tau)=\begin{cases} c(\tau) & \text{if } u, v\notin\|\tau\|,\\ c(\tau)-g(\sigma) & \text{if } v\in\|\tau\| \text{ and } \tau=v\sigma,\\ g(\sigma) & \text{if } u\in\|\tau\| \text{ and } \tau=u\sigma.\end{cases}$$

We now show that d is an integral 2-circuit on L. First, it must be shown that d is an integral 2-cycle on L. This is done by examining four cases for σ in the equation:

$$\partial d(\sigma)=\sum_{w\in N(\sigma,L)} d(w\sigma),\quad \sigma\in\vec{L}_{(1)}. \tag{5}$$

Case 1: $u\in\|\sigma\|$.

Here $N(\sigma, L)=N(\rho, K_v)$, where $\sigma=\pm u\rho$. Suppose that $\sigma=u\rho$ and that $\|\rho\|=\{t\}$. Now $w\sigma=wut=utw$ and, by (5)

$$\partial d(\sigma)=\sum_{w\in N(\rho,K_v)} g(tw)=\sum_{w\in N(\rho,K_v)} g(w\rho)=-\partial g(\rho)=0.$$

Similarly, $\partial d(\sigma)=0$ if $\sigma=-u\rho$.

Case 2: $v\in\|\sigma\|$.

Here $N(\sigma, L)=N(\rho, K_v)$, where $\sigma=\pm v\rho$. Suppose that $\sigma=v\rho$ and that $\|\rho\|=\{s\}$. Now $w\sigma=wvs=-vws$, $d(vws)=c(vws)-g(ws)=c_v(ws)-g(ws)$ and, by (5)

$$\begin{aligned}\partial d(\sigma)&=-\sum_{w\in N(\rho,K_v)} c_v(ws)+\sum_{w\in N(\rho,K_v)} g(ws)\\ &=-\partial c_v(\rho)+\partial g(\rho)=0.\end{aligned}$$

Similarly, $\partial d(\sigma)=0$ if $\delta=-v\rho$.

Case 3: $\sigma\in\vec{K}_v$.

Let $S=N(\sigma, L)-\{u, v\}$ and observe that

$$\begin{aligned}\partial d(\sigma)&=\sum_{w\in S} d(w\sigma)+d(v\sigma)+d(u\sigma)\\ &=\sum_{w\in S} c(w\sigma)+c(v\sigma)-g(\sigma)+g(\sigma)\\ &=\sum_{w\in N(\sigma,L)} c(w\sigma)=\partial c(\sigma)=0.\end{aligned}$$

Case 4: $u, v\notin\|\phi\|$, $\sigma\notin\vec{K}_v$.

In this case, $d(w\sigma)=c(w\sigma)$ by the definition of d and (vi) becomes $\partial c(\sigma)$ directly.

In order to show that d is an integral n-circuit on L, let d' be a non-zero integral n-cycle on L such that $d'\leq d$. We will show first that $\psi d'=c$, then that $d'=d$.

Let $\tau\in\vec{K}_{(2)}$ and suppose that $c(\tau)\geq 0$. Since $d'\leq d\leq c$, it follows that $d'(\tau)\leq d(\tau)\leq c(\tau)$. If $\tau\notin S_v$, then $\psi d'(\tau)=d'(\tau)\leq c(\tau)$. If $\tau\in S_v$, then $\tau=v\sigma$ for some $\sigma\in\vec{K}_{v(1)}$ and $\psi d'(\tau)=d'(u\sigma)+d'(v\sigma)$. Since $c_v(\sigma)\geq 0$, it follows from the definition of d that $d(u\sigma)\geq 0$, whence $d'(u\sigma)\leq d(u\sigma)$. Therefore, $d'(u\sigma)+d'(v\sigma)\leq d(u\sigma)+d(v\sigma)=c(v\sigma)$. We have thus shown that when $c(\tau)\geq 0$, $\psi d'(\tau)\leq c(\tau)$. By a similar argument, the same inequality holds when $c(\tau)\leq 0$. To see that $\psi d'\neq 0$, let $\tau\in\vec{K}_{(2)}$ such that $d'(\tau)\neq 0$. If $\tau\notin S_v$, then $\psi d'(\tau)=d'(\tau)\neq 0$. If $\tau\in S_v$, then $\tau=v\sigma$ for some $\sigma\in\vec{K}_{v(1)}$ and $\psi d'(\tau)=d'(u\sigma)+d'(v\sigma)$. By the definition of d and the inequality $d'\leq d$, d' has the same sign at both $u\sigma$ and $v\sigma$. Hence, $\psi d'(\tau)\neq 0$.

By Lemma 6(a), $\psi d'$ is an integral n-cycle on K. We have shown, moreover, that $\psi d'$ is non-zero and that $\psi d'\leq c$. Since c is an integral n-circuit, $\psi d'=c$. Finally, we will now show that $d'=d$, proving that d is an integral 2-circuit.

Let $\tau\in\vec{L}_{(2)}$ and suppose that $\tau\notin S_u, S_v$. Then $d(\tau)=c(\tau)$ while $d'(\tau)=\psi d'(\tau)=c(\tau)$. Therefore $d'(\tau)=d(\tau)$. Suppose next that τ lies in one of S_u, S_v so that τ is one of $u\sigma$, $v\sigma$, where $\sigma\in\vec{K}_{v(1)}$. Suppose further that $c_v(\sigma)\geq 0$. By the definition of d and by the inequalities $c_v(\sigma)\geq 0$, $d'\leq d$, it follows that $d'(v\sigma)\leq d(v\sigma)$ and $d'(u\sigma)\leq d(u\sigma)$. Therefore,

$$
\begin{aligned}
c(v\sigma)=\psi d'(v\sigma)&=d'(v\sigma)+d'(u\sigma)\\
&\leq d(v\sigma)+d(u\sigma)=c(v\sigma),
\end{aligned}
$$

which combined with the foregoing two inequalities, implies that $d'(v\sigma)=d(v\sigma)$ and $d'(u\sigma)=d(u\sigma)$, whence $d'(\tau)=d(\tau)$. We would reach the same conclusion if we initially assumed that $c_v(\sigma)\leq 0$. Since $d'(\tau)=d(\tau)$ for every $\tau\in\vec{L}_{(2)}$ it follows that $d'=d$ and we have proved that d is an integral 2-circuit.

We now compare $p(d)$ with $p(c)$. For this purpose it will be convenient to regard c as an integral 2-circuit on L by setting $c(\tau)=0$ for every $\tau\in\vec{L}_{(2)}-\vec{K}_{(2)}$. Thus

$$
p(c)=p(d)=\sum_{\tau\in\vec{L}_{(2)}}(\big||c(\tau)|-1\big|\cdot|c(\tau)|-\big||d(\tau)|-1\big|\cdot|d(\tau)|). \tag{6}
$$

Since $d(\tau)=c(\tau)$ whenever $\tau\notin S_u\cup S_v$, the terms of (6) corresponding to such oriented 2-simplexes all vanish. The remaining terms can be recombined and each oriented 2-simplex τ rewritten as $\tau=u\sigma$ or $\tau=v\sigma$, where $\sigma\in\vec{K}_{v(1)}$ and

$$
p(c)-p(d)=\sum_{\sigma\in\vec{K}_{v(1)}} r(\sigma),
$$

where

$$
\begin{aligned}
r(\sigma)=(\big||c(v\sigma)|-1\big|\cdot|c(v\sigma)|&-\big||d(v\sigma)|-1\big|\cdot|d(v\sigma)|\\
&-\big||d(u\sigma)|-1\big|\cdot|d(u\sigma)|).
\end{aligned}
$$

The last summation can be rewritten

$$
p(c)-p(d)=2\sum_{\sigma\in T} r(\sigma),
$$

where T is the set of oriented 1-simplexes $\sigma\in\vec{K}_v$ at which $c(v\sigma)$ (and, therefore,

$d(v\sigma)$ and $d(u\sigma)$) is non-negative. Each of the terms $r(\sigma)$ now becomes

$$\begin{aligned} r(\sigma) &= (c(v\sigma)-1)c(v\sigma)-(d(v\sigma)-1)d(v\sigma)-(d(u\sigma)-1)d(u\sigma) \\ &= 2g(\sigma)(c(v\sigma)-g(\sigma)). \end{aligned}$$

Since $g \leqslant c$, $c(v\sigma) \geqslant 0$ and $g(\sigma) \geqslant 0$, we have $r(\sigma) \geqslant 0$. Therefore, $p(c) \geqslant p(d)$. Recalling that σ_1 is an oriented 1-simplex for which $c(v\sigma_1) = m$ and that $m > 1$, we have

$$\begin{aligned} r(\sigma_1) &= 2g(\sigma_1)(c(v\sigma_1)-g(\sigma_1)) \\ &= 2(m-1) > 0. \end{aligned}$$

Therefore, $p(c) > p(d)$ and by the induction hypothesis, there exists a complex K', a primitive integral 2-circuit c' on K' and a contraction ϕ of K' such that

$$L = \phi(k') \quad \text{and} \quad d = \phi c'.$$

Clearly, $c = \psi d$ and, therefore, $c = \psi\phi c'$. Moreover, $K = \psi\phi(K')$ and the theorem follows at once from the fact that the composition of two contractions is a contraction. □

3. Summary and conclusions

We de not know whether a higher-dimensional form of Theorem 6 exists. The argument in its present form is "blocked" in higher dimensions at the generalization of Statement (e) in the proof of Theorem 6. It will be recognized moreover that the following statement, if it were true, would generalize the essential part of Statement (e).

(e′) For any non-zero integral n-cycle f on K there is a primitive non-zero integral n-circuit g on K such that $g \leqslant f$.

However, Theorem 5 provides an integral 2-circuit c (non-zero by the definition of integral n-circuit) the minimum magnitude of whose values exceeds an arbitrarily prescribed number. Clearly no primitive non-zero integral 2-circuit can be an integral sub-chain of c, which shows (e′) to be false. Perhaps a generalization of Theorem 6 is possible, however, if one attempts a second induction in place of Statement (e), using the theorem in a lower dimension to "expand" K_v to the point where the required primitive integral $(n-1)$-circuit exists, then to "expand" K by the argument used above.

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