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FURTHER RESULTS ON GRAPH EQUATIONS FOR LINE GRAPHS AND n-TH POWER GRAPHS

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We present solutions of seven graph equations involving the line graph, complement and n-th power operations. One such equation $L(G)^n = \bar{G}$ generalizes a result of M. Aigner. In addition, some comments are made about graphs satisfying $G^n = \bar{G}$.

1. Introduction

We shall present the solutions of seven graph equations involving the line graph $L(G)$ of a graph G, the complement $\bar{G}$. and the n-th *power*, denoted G^n. The n-th power has the same point set as G, and has two points u and v adjacent if their distance in G is n or less. A *standard form* of such an equation is one in which the maximum number of operations appearing on either side of the equation is as small as possible. The *degree* of an equation is then the maximum number of operations on either side of an equation in standard form. For example, the degree of the equation $G = \overline{L(G)}$ is one, since in standard form it is $L(G) = \bar{G}$, and there is one operation on each side of the equation.

Our goal is to solve all graph equations of degree one and two involving the above mentioned operations. After giving our solutions we indicate which equations remain unsolved.

The equations solved in this paper are:

$$\bar{G} = L(G)^n \tag{1}$$

$$L(\bar{G}) = L(G)^n \tag{2}$$

$$\overline{L(G)} = \bar{G}^n, \quad \text{where } n \geq 2, \tag{3}$$

$$L(G^m) = \bar{G}^n, \quad \text{where } m, n \geq 2, \tag{4}$$

$$L^2(G) = G^n, \quad \text{where } n \geq 2, \tag{5}$$

$$L^2(G) = \overline{G^n}, \quad \text{where } n \geq 2, \quad \text{and} \tag{6}$$

$$L^2(G) = \bar{G}^n, \quad \text{where } n \geq 2. \tag{7}$$

Recall that $L^2(G)$ is the second iterated line graph of G.

Several of these equations can be viewed as generalizations of earlier work. For example, equation (1) is a generalization $L(G) = \bar{G}$, which was solved by Aigner [1]. It is interesting to note that Aigner's solutions are also generalized. There are of course other ways to generalize Aigner's equation, such as (4), but in these cases the solution changes.

When $n = 1$ in equations (3) and (5), we get $L(G) = G$, which was solved by Menon [4] and van Rooij and Wilf [8]. If $n = 1$ in (6) and (7), the resulting equation was solved by Simić [7]. The equation (4) with $m = 1$ and $n \geq 2$ was solved by Akiyama, Kaneko and Simić [2]; when $n = 1$ and $m \geq 2$ in this equation there are no solutions.

Throughout the paper we suppose that a potential solution graph G has p points and q lines, and in general use the notation and terminology of Harary [3]. Occasionally we will not allow our graphs to have isolated points. When this stipulation is made, it shall be stated.

2. The solution of $\bar{G} = L(G)^n$

The corona $G_1 \circ G_2$ of two graphs with order p_1 and p_2 is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i-th point of G_1 to each point in the i-th copy of G_2. This term was coined by Frucht and Harary, see for example [3, p. 167].

In solving equation (1), we restrict our attention to the cases $n \geq 2$, since the case $n = 1$ was solved by Aigner [1]. It is often convenient to write (1) as

$$G = \overline{L(G)^n}. \tag{1'}$$

Preparatory to solving this equation we remind the reader that in a graph G, $d(u, v)$ denotes the distance between the points u and v. We use $d_L(x, y)$ to denote the distance between the lines x and y; this is the same as the distance between the points corresponding to x and y in $L(G)$. It is convenient to abbreviate $\overline{L(G)^n}$ to $H(G)$, or simply H when there can be no confusion. When x is a line of G, we shall use the same symbol to represent the corresponding point of H. The first lemmas require no proof. Lemmas 3 and 4 deal with graphs without isolated points.

Lemma 1. *Let x and y be lines of G. Then x is adjacent to y in H if and only if $d_L(x, y) \geq n + 1$.*

Lemma 2. *Four lines of G, x_0 to x_3, determine C_4 in H if $d_L(x_i, x_{i+1}) \geq n+1$ for $0 \leq i \leq 3$. (Addition is taken modulo 4.)*

Lemma 3. *If G is a solution to* (1), *then G is connected.*

Proof. If G is not connected, then the left hand side of (1) is connected whereas the right hand side is not. □

Lemma 4. *Any solution to* (1) *is unicyclic.*

Proof. We must have $p = q$ and G is connected by Lemma 3. □

We begin the proofs of the important lemmas.

Lemma 5. *If G is a cycle, then G is a solution to* (1) *if and only if $G = C_{2n+3}$.*

Proof. If $G = C_{2n+3}$, then (1) holds. Conversely, suppose that G is a solution. Since the radius of G must be at least $n+1$ (or H has isolated points), $g \geq 2n+2$. If $g = 2n+2$, then $H = (n+1)K_2$ and this does not give a solution. And if $g \geq 2n+4$, the minimum degree $\delta(H) \geq 3$, and here also we get no solution. Thus $g = 2n+3$ as required. □

For the next lemma we suppose G is a unicyclic solution to G, but is not a cycle. The points of the unique cycle in G are denoted by $v_0, \ldots, v_{g-1}$, and the tree, possibly trivial, attached to the cycle at v_i is called T_i. We let

$$h(T_i) = \max_{u \in V(T_i)} \{d(v_i, u)\},$$

the *height* of T_i. A point u of T_i is called a *k-point* if $d(v_i, u) = k$, and similarly a line e of T_i is called an *k-line* if the points of e are at distances $k-1$ and k from v_i.

Lemma 6. *If G is not a cycle, then G is a solution to* (1) *if and only if $G = C_{2n+1} \circ K_1$.*

Proof. The proof is divided into cases according to the girth g, which by Lemma 4 is the length of the only cycle in G.

Case A, $g = 4$

Let $e_i = \{v_i, v_{i+1}\}$, $i = 0$ to 3, be the lines of the 4-cycle. We first show that $h(T_i) \leq n+1$, for all i. If this is not the case, let x and y denote an $n+1$-line and an $n+2$-line of T_i for which $h(T_i) \geq n+2$. Then the sets of lines $\{x, e_{i-1}, y, e_{i-2}\}$ and $\{x, e_i, y, e_{i+1}\}$ both determine 4-cycles in H, which is impossible if H is to be unicyclic. So $h(T_i) \leq n+1$, for all i.

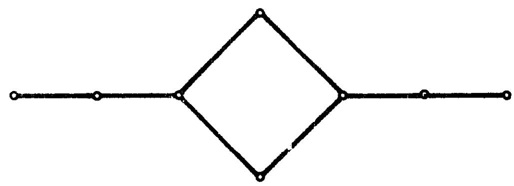

Fig. 1.

Next consider subcases, (a) $h(T_i)=n+1$ for some i, and (b) $h(T_i)\leqslant n$ for all i. In subcase (a) we lose no generality in supposing $h(T_0)=n+1$. This implies $\deg v_i=2$ for $i=1,2,3$. For otherwise H would contain at least two 4-cycles; these can be located in a manner analogous to that above. Let x be a 1-line of T_0, on the path from v_0 to an $n+1$-point. In order that x not be isolated in H, T_0 must contain at least two line disjoint paths of length $n+1$, starting at v_0. This implies the existence of two $n+1$-lines, y_1 and y_2 in T_0. But then the lines y_1, y_2 and e_2 determine a 3-cycle in H, so in subcase (a) there are no solutions.

In subcase (b) we are given that $h(T_i)\leqslant n$ for each i. We claim that either $h(T_0)=h(T_2)=n$ or $h(T_1)=h(T_3)=n$. If not, we suppose, without loss of generality, that $h(T_0)$, $h(T_1)<n$. But then the line $e_3=\{v_2, v_3\}$ is isolated in H, so the claim is proved, and in complete generality we take $h(T_0)=h(T_2)=n$. Let u_0 and u_2 be n-points of T_0 and T_2. Let y_1 to y_n and z_1 to z_n be the lines of the paths from v_0 to u_0 and from v_2 to u_2. Then, if $n\geqslant 3$ we find at least two 4-cycles in H in the usual manner. Two such 4-cycles are given by the lines y_n, z_n, y_{n-1}, z_{n-1} and y_n, z_n, y_{n-1}, z_{n-2}. So we are left to consider $n=2$, when $\deg v_1=\deg v_3=2$ lest H contain a 3-cycle. In addition $\deg v_0=\deg v_2=3$ or H contains two 4-cycles. It follows that $h(T_0)=h(T_2)=2$ or many 4-cycles are found in H. So G is the graph in Fig. 1, and this graph is easily seen not to be a solution. So subcase (b) offers no solutions and if G is a solution to (1) we now have $g\neq 4$.

Case B, $g\geqslant 2n+3$

In this case we get $\delta(H)\geqslant 2$, which is impossible since G has endpoints.

For further cases, we of course know $g\neq 4$, so the following results hold:

(i) $h(T_i)\leqslant n+2-\frac{1}{2}g$, if g is even, and

(ii) $h(T_i)\leqslant n+3-\frac{1}{2}(g+1)$.

To show (i), let g be even and $k=n+2-\frac{1}{2}g$. If $h(T_i)\geqslant k+1$, let x and y be k- and $k+1$-lines of T_i. Let z_1 and z_2 be two lines of the cycle which are as far as possible from v_i. Then x, y, z_1 and z_2 determine a 4-cycle in H, a contradiction, proving (i). Inequality (ii) is proved similarly.

Case C, $g=2n+2$

From (i) we have $h(T_i)\leqslant 1$ for all i. Suppose T_0 is nontrivial. Then T_n, T_{n+1} and T_{n-2} are trivial, or else H contains a 4-cycle. Another 4-cycle results unless $\deg v_i\leqslant 3$ for all i. If two adjacent points of the cycle have degree three, then there are two endpoints of G at distance three, all endpoints of H correspond to lines of the cycle in G. Clearly it is impossible for two such lines to be at distance three in H and be endpoints of H.

But then for each i such that $\deg v_i = 3$, the lines $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ have degree one in H. Thus H has twice as many endpoints as G, but $G \cong H$ so $g \neq 2n+2$.

Case D, $5 \leq g \leq 2n$

We split this case into two subcases, (a) $h(T_i) \leq n+2-\{\frac{1}{2}g\}$ for all i, and (b) $h(T_i) = n+3-\{\frac{1}{2}g\}$, some i. By (i), in subcase (b), g is odd. In (a), if some T_j has height less than $n+2-\{\frac{1}{2}g\}$, then a line on the cycle at greatest possible distance from v_j will be isolated in H. So $h(T_i) = n+2-\{\frac{1}{2}g\}$ for all i. So for some i there is a point w_i on T_i at distance $n+2-\{\frac{1}{2}g\}$ from v_i. Then let e_i be a line incident with w_i for each i. The lines $e_0, e_1, e_{\{g/2\}-1}$ and $e_{\{g/2\}}$ determine a 4-cycle in H, a contradiction.

In subcase (b) $h(T_i) = n+3-\{\frac{1}{2}g\}$ for some i. Suppose $h(T_0) = n+3-\{\frac{1}{2}g\}$, and let w_0 be an endpoint of T_0. If $h(T_k) \geq n+2-\{\frac{1}{2}g\}$ for $k = \{\frac{1}{2}g\}-1$, $\{\frac{1}{2}g\}$ or $\{\frac{1}{2}g\}+1$, then a 4-cycle is determined in the usual way. But if not $\{v_0, v_1\}$ is isolated in H. This dispenses with subcase (b).

Case E, $g = 2n+1$

By (i) and (ii) either

(iii) $h(T_i) = n+3-\{\frac{1}{2}g\} = 2$ for some i, or

(iv) $h(T_i) = n+2-\{\frac{1}{2}g\} \leq 1$, for all i.

If we suppose (iii) holds we can show no solutions are obtained using the same techniques as in the last case. Thus we suppose (iv) holds. If $\deg v_i \geq 4$, then H contains a 4-cycle and if $\deg v_i = 2$, then we get an isolated point. So $\deg v_i = 3$, for all i and $G = C_{2n+1} \circ K_1$ which is a solution, as is easily verified.

Case F, $g = 3$

If $g = 3$, then by (iii) we know $h(T_i) \leq n+1$, for all i. Again we consider whether

(v) $h(T_i) = n+1$ for some i, or

(vi) $h(T_i) \leq n$ for all i.

If we have (v), then suppose $h(T_0) = n+1$. Since T_0 contains at most one $n+1$-line (or H would have several 4-cycles), either T_1 or T_2 has height n or greater, suppose $h(T_1) \geq n$. But an $n+1$-line of T_0, an n-line of T_0, an n-line of T_1, and an $n-1$-line of T_1 determine a 4-cycle in H. Thus $g \neq 3$.

This last case exhausts the possibilities and $G = C_{2n+1} \circ K_1$ is the only solution.

Lemmas 5 and 6 together constitute the following theorem.

Theorem 1. *The graphs C_{2n+3} and $C_{2n+1} \circ K_1$ are the only graphs without isolated points satisfying $\bar{G} = L(G)^n$, $n \geq 2$.*

The solutions to (1) obtained by Aigner for $n=1$ were C_5 and $C_3 \circ K_1$. Thus Theorem 1 holds for $n \geq 1$.

3. Solutions of other equations

Theorem 2. *For any $n \geq 2$, $G = K_{1,3}$ is the only solution to $L(\bar{G}) = L(G)^n$ without isolated points.*

Proof. It suffices to consider connected graphs, for if G is disconnected then $\bar{G}$ is connected.

Suppose G is a solution to (2). Since $|V(L(\bar{G}))| = |E(\bar{G})| - q$ and $|V(L(G)^n)| = q$, we have $q = \frac{1}{4}p(p-1)$.

Let $(d_1, \ldots, d_p)$ and $(d'_1, \ldots, d'_p)$ be the degree sequences of G and $\bar{G}$. Note that $d'_i = p-1-d_i$. And as $|E(G)| = |E(\bar{G})| = q = \frac{1}{4}p(p-1)$, we have $\sum d_i = \sum d'_i = 2q = \frac{1}{2}p(p-1)$. Let D_1 and D'_1 denote the degree sums of $L(G)$ and $L(\bar{G})$. Then $D_1 = \sum d_i(d_i - 1) = \sum d_i^2 - 2q$ and $D'_1 = \sum d'_i\,(d'_i - 1) = \sum d_i'^2 - 2q$, and so $D_1 = D'_1$.

Next let D_n denote the degree sum of $L(G)^n$ and suppose $L(G)$ is not complete. Then $D_n > D_1$ for $n \geq 2$. But then $D_n > D'_1$, a contradiction which implies that $L(G)$ is complete, and so $G = K_{1,m}$, if anything.

If $m > 3$, then $L(G)^n$ is complete but $L(\bar{G})$ is not, thus G is not a solution. If $m < 3$ we again find no solution. But by direct verification $K_{1,3}$ is found to be a solution, thus the only solution. □

It is now convenient to define a family $\mathcal{F}$ of graphs which will be useful in solving the next equation. Let $G \in \mathcal{F}$ if and only if G is unicyclic and either the diameter $d(\bar{G}) \geq 3$ or $\bar{G}$ is disconnected. We now list the graphs in $\mathcal{F}$.

If the girth g of G exceeds four then $d(\bar{G}) = 2$. Thus the unique cycle in G has length 3 or 4. If $g = 4$, then no point can be at distance $d \geq 2$ from the nearest point on the cycle. Similarly if two opposite points on the cycle have degrees exceeding two then $d(\bar{G}) \leq 2$. So if $g = 4$ it is easy to see that only the graph in Fig. 2 are in $\mathcal{F}$. Note that the distance from u to v in $\bar{G}$ is three, and that there can be any number of pendant lines at points u and v.

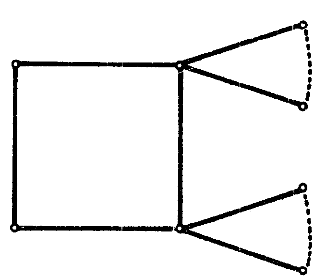

Fig. 2.

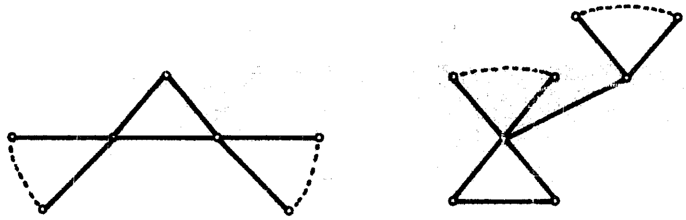

Fig. 3.

A similar analysis for the case $g=3$ reveals that only graphs of the types shown in Fig. 3 are in $\mathcal{F}$. Observe that in each case the number of pendent lines is arbitrary.

We solve the third equation.

Theorem 3. *The graphs C_3 and C_4 are the only solutions to the equation $\overline{L(G)}=\bar{G}^n$, $n\geq 2$.*

Proof. If G is not connected then $\bar{G}^n$ is complete, thus G cannot be a solution.

If G is connected, then G must be unicyclic to be a solution. This follows because the graphs on either side of the equation must have the same number of points, and this means that G has an equal number of points and lines.

Now if $G\notin\mathcal{F}$ and G is unicyclic, then $\bar{G}^n=K_p$ but $\overline{L(G)}=K_p$ is impossible. On the other hand if $G\in\mathcal{F}$ we can consider each type of graph in $\mathcal{F}$ separately and verify that C_3 and C_4 are the only solutions. □

Theorem 4. *The equation $L(G^m)=\bar{G}^n$ has no solutions, $m, n\geq 2$.*

Proof. If G is disconnected then $\bar{G}^n$ is connected, and so $L(G^m)$ must be connected. The only possible form of G is the following:

$$G=H\cup xK_1,$$

where H is a connected graph and x is a positive integer. In such a graph G there is a point v whose degree is $p-1$ in $\bar{G}^n$. However it is easy to verify that the maximum degree of $L(G^m)$ is less than $p-1$, a contradiction. Thus, there are no disconnected solutions.

Assume that G is connected, then G is unicyclic. Since $m\geq 2$, the only possible solutions are P_3 and K_3, as they are the only connected graphs G for which G^m is unicyclic, $m\geq 2$. But neither is a solution. □

The next three equations involve the second iterated line graph $L^2(G)$. It will be necessary to determine the graphs G for which $L^2(G)$ and G have the same number of points.

Fig. 4.

Lemma 7. *If $L^2(G)$ and G have the same order p and if G is connected then G is a cycle or a subdivision of $K_2 \circ 2K_1$ (see Fig. 4).*

Proof. Let p_0 be the order of $L^2(G)$. If G has more than one cycle then $p_0 > p$, and if G is unicyclic but not a cycle, then again $p_0 > p$. The same inequality holds for trees with a point of degree greater than three and for trees with more than two points of degree three. On the other hand if G is a tree with fewer than two points of degree three and no points of higher degree $p_0 < p$, then it is easily verified that graphs of the types mentioned in the statement of the lemma satisfy $p_0 = p$. □

Theorem 5. *For $n \geq 2$, the only connected solution to $L^2(G) = G^n$ is K_3.*

Proof. It is immediate that K_3 is the only cycle which is a solution. We now show that no subdivision of $K_2 \circ 2K_1$ is a solution.

Since $n \geq 2$, G^n contains at least two copies of K_4 as subgraphs. But $L^2(G)$ contains at most one copy of K_4 and this maximum occurs only when the two points of degree 3 are adjacent. Thus no subdivision of $K_2 \circ 2K_1$ is a solution. □

The next two theorems are rather easy, so that proofs are omitted.

Theorem 6. *For $n \geq 2$, the only solution to $L^2(G) = \overline{G^n}$ is C_{2n+3}.*

Theorem 7. *There are no solutions to the equation $L^2(G) = \bar{G}^n$, $n \geq 2$.*

4. Remarks

We note that equations (5), (6) and (7) can be easily generalized, replacing $L^2(G)$ by $L^m(G)$. Using the following lemma it is easy to verify that the solutions are the same.

Lemma 8. *Let G be a connected graph with p points, and let $L^m(G)$ have p_m points. Then*

(i) *for $m > 3$, $p_m = p$ only when G is a cycle,*

(ii) *$p_3 = p$, when G is a cycle or a graph of the type shown in Fig. 5.*

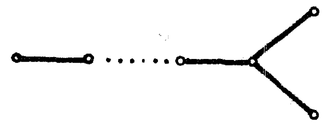

Fig. 5.

Note that the graph equation $L(\bar{G}) = \overline{G^n}$ is essentially the same as the equation (3), as is easily verified by replacing $\bar{G}$ by H.

5. An unsolved equation

Among first degree equations only $G^n = \bar{G}$ is not solved. Of course for $n = 1$ this reduces to the problem of finding all self-complementary graphs, which was done independently by Ringel [5] and Sachs [6].

Proposition. *The equation $G^n = \bar{G}$ has solutions of all orders $p \geq 2n+3$.*

Proof. Let $G_0 = C_{2n+3}$, which is clearly a solution. Let u and v be two points of G_0 at distance two. Define G_k by adding k points to G and joining each to u and v (see Fig. 6). Clearly G_k is a solution for all k. □

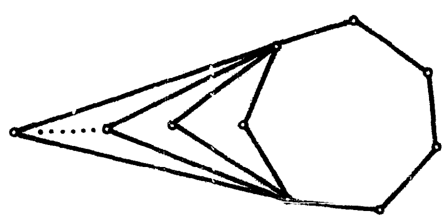

Fig. 6.

We have not yet found all solutions to $G^2 = \bar{G}$, but have found several solutions, see Fig. 7.

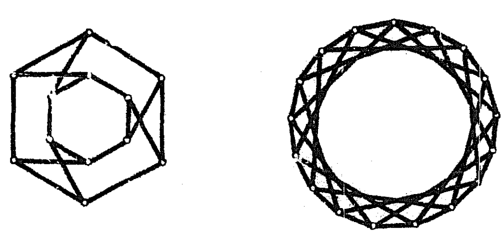

Fig. 7.

Acknowledgement

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