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MINIMAL SURFACES IN SEIFERT FIBER SPACES

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This paper studies the minimal surfaces in Seifert fiber spaces equipped with their natural geometric structures. The minimal surfaces in these 3-manifolds are always either vertical, namely always tangent to fibers, or horizontal, always transverse to fibers. This gives a classification of injective surfaces in these manifolds, previously obtained by Waldhausen for embedded injective surfaces. As usual in this context, equivariant versions of this classification can also be obtained.

AMS Subj. Class. Primary 57; Secondary 53
Seifert fiber space   incompressible surface
minimal surface

Introduction

Recent years have seen a growing number of results tying together 3-dimensional topology and the theory of minimal surfaces. Techniques developed in each field have been successfully applied to solve problems in the other. This paper pursues this trend and examines the interaction of minimal surfaces with the 1-dimensional foliations of 3-manifolds given by a Seifert fiber structure. In a related paper [3] the interaction with codimension one foliations of manifolds is examined.

To use minimal surface techniques to study Seifert fiber spaces and their submanifolds, we need to equip them with some Riemannian metric. These manifolds actually admit certain natural metrics, modelled on one of six geometries. These natural metrics are compatible with the foliations, in that their isometry group will include rotations about the circles, possibly after passing to a double cover, and this allows the nature of the minimal surfaces present to be well understood and in some cases classified. As a consequence, a topological classification of injective surfaces is reached in Corollary 1.2. This classification can be done for surfaces not homotopic to embeddings, as well as for the embedded case where it is well known and due originally to Waldhausen. As is often the case when proving topological results via minimal surface theory, we can get an equivariant version of the results also, if a finite group is acting on the manifold.

A map or a manifold will be assumed smooth throughout this paper. A surface is a *minimal surface* if it is an immersion with zero mean curvature. This is distinguished from a *least area surface* which is one that minimizes area in its homotopy class.

1. One dimensional foliations

In this section we will examine compact 3-manifolds foliated by circles. Such a 3-manifold is a Seifert fiber space [1, 7]. These manifolds have been extensively studied by 3-manifold topologists, recently from a geometric point of view. Like surfaces, they admit natural geometric structures, modelled on one of E^3, S^3, $S^2 \times R$, $H^2 \times R$, $\widetilde{SL}(2, R)$ or Nil.

We will study the minimal surfaces in these 3-manifolds equipped with their natural metrics. In some cases we can classify these. As a consequence we generalize known results on the nature of incompressible surfaces in such manifolds.

Let M be a closed 3-manifold which is a Seifert fiber space and F a closed surface, $F \neq S^2, P^2$. We say that $f: F \to M$ is *injective* if the induced map on the fundamental group is an injection. We abuse notation by referring to the image $f(F)$ as F. If F is an immersed surface in M, we say it is *horizontal* if F is everywhere transverse to the fibers of M, and we say it is *vertical* if F is everywhere tangent to the fibers of M. We now state the main theorem of this section.

Theorem 1.1. *Let M be a Seifert fiber space with a geometric structure. Let F be a closed surface, $F \neq S^2, P^2$, and let $f: F \to M$ be a minimal injective immersion. Then F is either vertical or horizontal.*

Note that the hypothesis of the theorem implies that the geometric structure is not modelled on S^3 or $S^2 \times R$, as the Seifert fiber spaces modelled on these contain no injective surfaces.

Before proceeding we will look at some consequences of this result.

Corollary 1.2. *Let $f: F \to M$ be an injective map. Then f is homotopic to a vertical or horizontal map.*

Proof. It is known that an injective map is homotopic to a map minimizing area in its homotopy class, and that such a map is an immersion [8]. The result now follows from the theorem.

Corollary 1.3. *Let $f: F \to M$ be a 2-sided injective embedding. Then f is isotopic to a vertical or horizontal embedding.*

Proof. As in the previous corollary f is homotopic to a least area immersion which is vertical or horizontal. This map is either embedded or a 2-1 cover of an embedding by Theorem 2.1 of [2]. In the latter case, we take instead the boundary of a regular neighborhood of the minimal immersion, which can clearly be chosen to be horizontal or vertical if the minimal immersion is horizontal or vertical. This surface is isotopic to the original map f as two homotopic and embedded 2-sided incompressible surfaces in a Seifert fiber space are isotopic [9].

Note: This corollary is well-known and previously obtained by topological methods.

We can get an equivariant version of the above theorems using the following result of Meeks and Scott.

Theorem 1.4. (Meeks–Scott.) *Let M be a closed Seifert Fiber space with metric modelled on one of E^3, $H^2 \times R$, $PSL(2, R)$ or Nil and let G be a finite group acting on M. Then after conjugation by a diffeomorphism, G acts on M by isometries. If the geometric structure is not modelled on E^3, then the action can be taken to preserve the fibers of the Seifert fibration.*

We now state the equivariant version of the above corollaries.

Theorem 1.5. *Let $f: F \to M$ be an injective map and let G be a finite group acting on M. Then f is homotopic to a map $f': F \to M$ such that $g \cdot f'(F)$ is either horizontal or vertical for each $g \in G$. If f is embedded and 2-sided then f' can also be taken to be embedded and $g \cdot f'(F)$ is isotopic to $g \cdot f(F)$.*

Proof. By the Meeks–Scott theorem we can assume that G acts as isometries on the natural geometric structure on M. Then $g \cdot f'(F)$ is a minimal immersion if $f'(F)$ is. Picking f' to be the minimal immersion homotopic to f, the result follows. The result in the embedded case is similar to Corollary 1.3. If f collapses to double cover a 1-sided embedded surface when it is homotoped to a minimal embedding, then the boundary of a small regular neighborhood of this minimal immersion serves as $f'(F)$.

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. *Case* 1. F is 2-sided in M and M is an S^1 bundle over an orientable surface, admitting an S^1 action where the action is isometries preserving each fiber of the Seifert fibration.

We let M_F be the covering space of M with fundamental group corresponding to F. That is, if $p: M_F \to M$ is the covering map, then $p_\# \pi_1(M_F) = f_\# \pi_1(F)$. f lifts to a minimal immersion $f_1: F \to M_F$. Clearly f is vertical or horizontal if f_1 is vertical or horizontal respectively, as these are local properties. M_F is foliated by either circles or lines which cover the fibers of M and it is relative to these that we speak of f_1 being vertical or horizontal. To proceed we need some lemmas on the local behavior of minimal surfaces. The first states that a minimal surface can not lie locally on one side of a second minimal surface which it meets.

Lemma 1.6. (Maximal principle for minimal surfaces.) *Let $f_i : D \to M$, $i = 1, 2$, be two minimal immersions of the 2-disk into a Riemannian 3-manifold M, such that $f_1(0) = f_2(0)$. Then either $f_1(D)$ coincides with $f_2(D)$ in a neighborhood of $f_1(0)$ or $f_1(D)$ meets both sides of $f_2(D)$ in any neighborhood of $f_1(0)$, similarly to the intersection of $x^3 = 0$ and $x^3 = \mathrm{Re}(x^1 + ix^2)^n$, $n \geq 2$, in R^3, up to a C^1 diffeomorphism.*

Proof. See [2] or [5].

Lemma 1.7. *Let M be a geometric Seifert fiber space, let $p \in M$ and let V be a vector in TM_p not tangent to the fiber through p. Then there is a vertical minimal surface W in a neighborhood of p with $V \in TW_p$.*

Proof. This is a local property that is being claimed so it suffices to prove it in each of E^3, $H^2 \times R$, $\widetilde{SL(2, R)}$, Nil, S^3 and $S^2 \times R$. For $E^3 = E^2 \times E^1$ and for $H^2 \times R$ we simply pick the union of all the fibers over a geodesic, which is a totally geodesic surface. Since a geodesic can be found in the base having any tangent vector at a given point, the result follows for these product geometries. S^3 and $S^2 \times R$ similarly have totally geodesic vertical surfaces whose tangent planes contain any given vector at a point. These are the great 2-spheres and products of equatorial circles with R.

For $\widetilde{SL(2, R)}$ and Nil, we first note that a vertical minimal surface exists. We can see this by looking at a compact manifold which is a circle bundle over a surface with one of these two structures and admitting an S^1 action. Consider a least area surface homotopic to a vertical surface over an embedded circle. It follows from [2] that this least area surface is equivariant under the S^1 action, and thus must be a vertical surface, in fact a vertical torus. This lifts to a vertical minimal plane in $\widetilde{SL(2, R)}$ and Nil respectively. We now note that the isometry group of each of these includes a screw motion about any vertical line, and thus there is a vertical minimal surface spanning any vertical tangent plane through a given point. The result follows. It is easy to see that these vertical minimal surfaces consist of all the fibers above a geodesic in the associated orbit space.

Lemma 1.8. *Let M be one of E^3, $H^2 \times R$, $\widetilde{SL(2, R)}$ or Nil, F a minimal immersion in M which is tangent to a fiber at $p \in M$ and $A: \mathbb{R} \times M \to M$ an action of $\mathbb{R}$ on M as isometries preserving each fiber. Then $A(t) \cdot F \cap F \neq \emptyset$ for t in some open neighborhood of 0.*

Proof. Let G be the vertical minimal surface through p which is tangent to F at p. G exists by Lemma 1.7. If F is vertical the lemma is trivial. If not then $G \cap F$ is transverse in a deleted neighborhood of p, and looks locally like the graph of $x^3 = \mathrm{Re}(x^1 + x^2)^n$, $n \geqslant 2$ by Lemma 1.6. The local picture for $G \cap A(t) \cdot F$ at

$A(t) \cdot (p) = q$ is similar. Let r and s be two lines in the intersection which cross transversely. Then for small values of t, $A(t) \cdot r \cap s \neq \emptyset$. But $A(t) \cdot r \subset A(t) \cdot F \cap G$ and it follows that $F \cap A(t) \cdot F$ is non-empty. See Fig. 1.

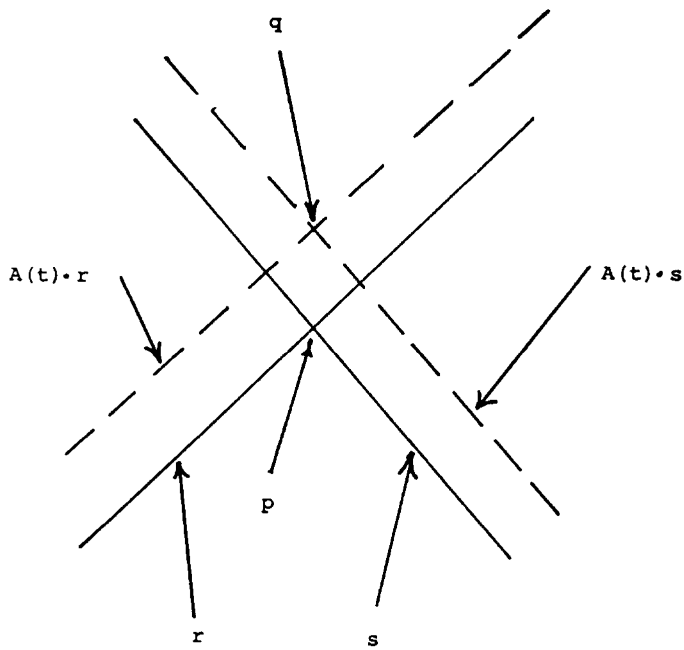

Fig. 1. Local picture of $G \cap F$ on G.

We now return to the proof of the theorem. We are considering a minimal immersion $f_1 : F \to M_F$. M_F admits either a circle action or an $\mathbb{R}$-action where the action is by isometries preserving each fiber. If M_F admits an $\mathbb{R}$-action, $A: R \times M \to M$, then the compactness of F implies that $A(t) \cdot f_1(F) \cap f_1(F) = \emptyset$ for t sufficiently large. If f_1 is not horizontal, then $A(t) \cdot f_1(F) \cap f_1(F) \neq \emptyset$ for sufficiently small positive t by Lemma 1.8. Thus there exists $t_0 = \{\sup t | A(t) \cdot f_1(F) \cap f_1(F)\} \neq \emptyset$. But $A(t_0) \cdot f_1(F)$ is a minimal surface meeting $f_1(F)$ and lying on one side of it and this contradicts the maximal principle, Lemma 1.6. Thus f_1 is horizontal if M_F admits an $\mathbb{R}$-action of this type.

If M_F admits an S^1-action preserving the fibers then F must be a torus or a Klein bottle, as the center of its fundamental group is nontrivial. So $f(F)$ is an immersed minimal torus or Klein bottle in M. Our assumptions on M imply that it's a torus. The next lemma will describe the nature of such tori.

Lemma 1.9. *Let M be a circle bundle over a surface, modelled on one of $H^2 \times R$, E^3, $\widetilde{PSL}(2, R)$, Nil. A minimal torus T in M is either horizontal or the vertical torus above a geodesic in B, where B is the natural orbit surface associated to the Seifert fiber space.*

Proof. Let $G = f_{\#}\pi_1(T)$ where $f: T \to M$ is a minimal map. If $G \cap \{\text{center}(\pi_1(M))\}$ is trivial then f lifts to a homotopy equivalence $f_T: T \to M_T$ and M_T admits an $\mathbb{R}$-action, so the previous argument shows f is horizontal. If M_T admits an S^1-action preserving the fibers, then there is a natural projection $p_T: M_T \to B_T$ where B_T is an annulus. B_T inherits a hyperbolic structure if M is modelled on $H^2 \times R$ or $\widetilde{PSL}(2, R)$, and a flat metric if M is modelled on E^3 or Nil. Let K be the compact set $p_T(f_T(T)) \subset B_T$. If K is a simple closed geodesic in B_T then the conclusion follows. If B_T is hyperbolic and K is not the unique simple geodesic γ, then there is a point p in K at maximal distance from γ. There is a geodesic δ in B_T through p, whose closest point of approach to γ is at p. The fibers above δ in M_T form a minimal surface meeting $f_T(T)$ in a manner contradicting the maximal principle.

If B_T is flat, a similar argument shows that K is precisely one geodesic and the Lemma is proved.

Returning to the proof of the theorem, we see that in the case of an S^1-action on $M_F, f(F)$ lifts to a vertical torus in M_F. Since verticalness is a local property, f is a vertical map itself. This concludes the proof of Case 1.

To prove the general case, we use again the observation that verticalness and horizontalness are local properties. Given any injective map $f_1: F_1 \to M_1$ where M is any Seifert fiber space, there is a finite cover M_2 of M_1, a finite cover F_2 of F_1 and a map $f_2: F_2 \to M_2$ covering $f_1: F_1 \to M_1$ such that $f_2: F_2 \to M_2$ falls into Case 1. Since a lift of a minimal surface is minimal, f_2 and thus f_1 is either vertical or horizontal.

Note. The theorem is false without the assumption of injectivity. For an example, consider a lens space containing a geometrically incompressible surface. Such a surface can always be isotoped to a minimal surface [6] but in general can not be homotoped to a vertical or horizontal surface, when the lens space is fibered over the 2-sphere with two critical fibers. A vertical surface would have to be a torus or klein bottle and a horizontal surface would have to be orientable.

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