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ON THE SPRINGER RESOLUTION OF THE MINIMAL UNIPOTENT CONJUGACY CLASS

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1. Introduction

Let G be a simple complex Lie group, U the subvariety of unipotent elements in G, and $\mathscr{B}$ the variety of all Borel subgroups of G. It was shown by T. Springer [12] that the projection

$$\pi : V=\{(u,B)\in U\times \mathscr{B}: u\in B\}\to U$$

resolves the singularities of U. The fibre $\mathscr{B}_u=\pi^{-1}(u)$ depends (up to isomorphism) only on the conjugacy class of an element $u\in U$. The second projection $V\to\mathscr{B}$ identifies this fibre with the subvariety of $\mathscr{B}$ of elements fixed under u, which acts on $\mathscr{B}$ by conjugation.

The projection, π, is an isomorphism over the conjugacy class, C_{reg}, of regular unipotent elements, the unique class whose closure is the whole variety U. On the other hand, $\mathscr{B}_1$ is the fibre of maximal dimension, and is isomorphic to the variety $\mathscr{B}$.

For general G, the only other fibre whose structure has been known is the fibre $\mathscr{B}_u$ over a subregular unipotent element u. The conjugacy class, C_{sreg}, of these elements is the class which 'goes next after' C_{reg}, i.e. $U=C_{\mathrm{reg}}\cup\bar{C}_{\mathrm{sreg}}$. It was shown by J. Tits and R. Steinberg (cf. [14, p. 147]) that $\mathscr{B}_u$ is isomorphic to a union of nonsingular rational curves, which form a configuration described by the Dynkin diagram of a simple group G' of type A, D or E; $G=G'$ if G is one of these types. If $\mathscr{B}$ is identified with the quotient variety G/B_0 for a fixed Borel subgroup B_0 of G, then each irreducible component, C, of $\mathscr{B}_u$ can be identified with the subvariety P_i/B, where P_i is one of the $r=\mathrm{rk}(G)$ minimal parabolic subgroups of G, containing B_0.

In general, one can associate a graph Γ_u to any fibre, $\mathscr{B}_u$, as follows (cf. [7, §6.3]): The set of vertices of Γ_u is the set of irreducible components of $\mathscr{B}_u$; two vertices are joined by an edge if the corresponding components of $\mathscr{B}_u$ intersect along a subvariety of codimension 1 in each of them. Finally, the graph, Γ_u, is labelled by associating to each vertex, v, of Γ_u a subset, I_v, of the set, S, of simple roots, determined as follows. For each simple root α_i, let P_i be the corresponding minimal parabolic subgroup of G containing B_0. Then, $\alpha_i \in I_v$ when the projection $G/B \to G/P_i$ induces a structure of a $\mathbb{P}^1$-bundle on C_v, the component of $\mathscr{B}_u$ corresponding to v.

It is known [12] that $\mathscr{B}_u$ is connected, and that all irreducible components have the same dimension [10; 11, Ch. 2, 1.12]. Of course, if dim $\mathscr{B}_u > 1$, then the graph, Γ_u, gives only a part of the information about the structure of $\mathscr{B}_u$.

In this paper, we compute the graph, Γ_u, for a unipotent element u belonging to the *minimal conjugacy class*, the unique class whose closure does not contain any other conjugacy class, except the class $\{1\}$; see Table 3. If G is of type A, D or E, then Γ_u turns out to be 'dual' to the graph, $\Gamma_{u'}$, of the resolution of a subregular element; they coincide as unlabelled graphs, and the labelling sets, I_v, of the former graph are complimentary, with respect to S, to the corresponding sets, I'_v, of the latter graph.

In [7, §1], D. Kazhdan and G. Lusztig associate to the Weyl group, W, of G a set of graphs (left cells). In the next section, we recall the construction of these graphs, which we call the KL-graphs. As was noticed in [7, §6.3], in the case G is of type A_n, $n \leq 5$, every KL-graph is isomorphic to a graph, Γ_u, for some $u \in G$, and vice versa; this may be true for all n.

We show that for every G of type other than G_2 (resp. of type A, D, E) the graph, Γ_u, of a subregular element u (resp. of an element u in the minimal class) can be found among the KL-graphs. Also, if G is of type A, D or E, then the KL-graph corresponding to a unipotent element from the minimal conjugacy class is the 'dual' to the KL-graph corresponding to a subregular element; see Proposition 2.3.

There is a representation theoretic motivation of the connection between the KL-graphs and the resolution graphs. From this point of view, it is not surprising that the case G_2 is exceptional, already, for the subregular class; the representations of the corresponding Weyl group may arise from other W-graphs, in the sense of [7, §1], not necessarily KL-graphs. Also, the failure of the duality between the subregular and minimal classes in the cases B, C, F and G is not very surprising.

The previous remark was pointed out to us by N. Spaltenstein, as well as the fact that there exists another unipotent class which may play the role of the minimal conjugacy class in the cases B, C and F. This is a minimal special unipotent class; see [8, 11]. When G is of type A, D, or E, this special class is just the minimal class, and when $G = G_2$ it is the subregular class. However, for G of type B, C or F, we do not know the resolution graph of this special class.

2. The Kazhdan–Lusztig graphs (KL-graphs)

2.1. Let T be a maximal torus in a Borel subgroup $B_0 \subset G$, R the root system of T, $S=\{\alpha_1,\dots,\alpha_r\}$ the simple roots in R determined by B_0, $W=N(T)/T$ the Weyl group, and for $\alpha\in R$, $s_\alpha\in W$ is the corresponding reflection. As usual, we write s_i in place of s_{α_i}, to denote a simple reflection, and $l(w)$, $w\in W$, is the length of w, i.e. the number of simple reflections in a reduced decomposition of w. For $w, w'\in W$ we write $w'<w$ (the Bruhat order) if there exist elements $w_1=w, w_2,\dots,w_n=w'$ such that $w_i=s_{\gamma_i}w_{i-1}$, $i=2,\dots,n$ where the γ_i are positive roots and $w_{i-1}^{-1}(\gamma_i)<0$; see [4; Ch. III]. According to [1; 15] it also can be defined as $w'<w$ if and only if one can get a reduced expression for w' by deleting some factors in a reduced expression for w.

2.2. For any $w', w\in W$ such that $w'<w$, a certain polynomial $P_{w',w}(t)\in\mathbb{Z}[t^{1/2}]$ is defined in [7, 1.1] (the Kazhdan–Lusztig polynomial). Its degree is at most $\frac{1}{2}(l(w)-l(w')-1)$. Let $\mu(w',w)$ be the coefficient of $P_{w',w}(t)$ at $t^{(l(w)-l(w')-1)/2}$. Following [7, §1], we write $w'\prec w$ if $l(w)-l(w')$ is odd, $w'<w$, and $\mu(w',w)\neq 0$. Let Γ_W be the graph whose vertices are the elements of W and whose edges are the subsets of the form $\{w',w\}$ with $w'\prec w$, or, of course, $w\prec w'$. We label the graph Γ_W by assigning the set $L(w)=\{\alpha\in S\colon l(s_\alpha w)<l(w)\}$ to the vertex w.

Again, as in [7, §1], we define $w'\ll w$ if there exists a chain $w_1=w, w_2,\dots,w_n=w'$ such that $\{w_i,w_{i+1}\}$ is an edge of Γ_W and $L(w_i)\not\subset L(w_{i+1})$ for $i=1,2,\dots,n-1$. We write $w\approx w'$ if $w\ll w'$ and $w'\ll w$. The equivalence relation $\approx$ decomposes Γ_W into a disjoint union of subgraphs (called the KL-graphs) whose vertices form an equivalence class with respect to $\approx$, and the graph structure is induced by the structure of Γ_W.

Let $\Gamma(w)$ denote the KL-graph containing $w\in W$ as a vertex. If w_0 is the unique element of W of maximal length, then the graph $\Gamma(w_0w)$ depends only on $\Gamma(w)$ (see [7, 3.3]) and will be called the dual KL-graph to the graph $\Gamma(w)$.

Since $w_0^2=1$, the correspondence $\Gamma(w)\to\Gamma(w_0w)$ is an involution. It is known that $w_0(\alpha_i)=-\alpha_{\varepsilon(i)}$, where $\varepsilon:\{1,2,\dots,r\}\to\{1,2,\dots,r\}$ is a certain permutation (identical if G is not of type A_r, D_r (r odd), or E_6).

2.3. Proposition. *Let $\Gamma=\Gamma(w)$ be a KL-graph and $\hat\Gamma=\Gamma(w_0w)$ its dual graph. Then the graphs Γ and $\hat\Gamma$ are isomorphic as unlabelled graphs; for each vertex w' of Γ, the corresponding vertex of $\hat\Gamma$ is w_0w'. Also,*

$$L(w_0w')=S\setminus\varepsilon(L(w'))$$

(where, of course, we mean that $\varepsilon(\alpha_i)=\alpha_{\varepsilon(i)}$).

Proof. This easily follows from the definitions and [7, Corollary 3.2], which says that $x\prec y$ if and only if $w_0y\prec w_0x$, for any $x,y\in W$. □

2.4 L. Lemma. *Let $w_1 < w_2$ in W. Suppose that there exists $s \in S$ such that $sw_2 < w_2$, $sw_1 > w_1$ and $w_2 \neq sw_1$. Then $w_1 \nprec w_2$.*

2.4 R. Lemma. *The same as above, with s multiplying on the right.*

Proof. [7, 2.3e, f]. □

3. The Kazhdan–Lusztig graph of a subregular element

Here, we will prove that the graph, Γ_u, for a subregular unipotent $u \in U$ can be found among the KL-graphs of W, and only when $G \neq G_2$. In the next section, it will be shown that the corresponding dual KL-graph is isomorphic to the graph, $\Gamma_{u'}$, for u' belonging to the minimal conjugacy class in U, for the cases $G = A$, D, and E.

Analogously to L, let

$$R(w) = \{\alpha \in S: l(ws_\alpha) < l(w)\} \quad (= L(w^{-1})).$$

Let $\alpha \in S$.

3.1 L. Lemma. $\alpha \in L(w) \Leftrightarrow w^{-1}(\alpha) < 0$.

3.1 R. Lemma. $\alpha \in R(w) \Leftrightarrow w(\alpha) < 0$.

Proof. [3A, Lemma 2.2.1]. □

3.2 L. Lemma. $\alpha \notin L(w) \Leftrightarrow \alpha \in L(s_\alpha w)$.

3.2 R. Lemma. $\alpha \notin R(w) \Leftrightarrow \alpha \in R(ws_\alpha)$.

Proof. By Lemma 3.1, $\alpha \notin R(w)$ if and only if $w(\alpha) > 0$ i.e. $ws_\alpha(\alpha) < 0$ i.e. $\alpha \in R(ws_\alpha)$. The proof for L is entirely similar. □

3.3. Lemma. *Let γ and δ be elements of the root system R. Then*

$$s_\gamma(\delta) = \delta - \gamma^\vee(\delta)\gamma$$

where $\gamma^\vee$ is the co-root of γ. It has the form $\gamma^\vee(\delta) = 2(\gamma, \delta)\|\gamma\|^{-2}$ where $(\ ,\ )$ is W-invariant bilinear form on the vector space containing R. Assume that $(\gamma, \delta) \neq 0$. Then

$$\|\gamma\| \geq \|\delta\| \Rightarrow \gamma^\vee(\delta) = \pm 1,$$

and

$$\|\gamma\| < \|\delta\| \Rightarrow \gamma^\vee(\delta) = \begin{cases} \pm 2 & \text{if } G \neq G_2, \\ \pm 3 & \text{if } G = G_2. \end{cases}$$

If γ and δ are distinct simple roots, then $(\gamma,\delta)\le 0$; and $(\gamma,\delta)=0$ precisely when γ and δ are not adjacent in the Dynkin diagram of G.

Proof. These facts are standard. See e.g. [9]. $\square$

Since G is simple, the root system R is irreducible, and at most 2 root lengths occur (long and short), each being an orbit of the action of W. When only one root length occurs (i.e. when G is of type A, D or E) it is called 'long'. These facts are discussed in [6, §10.4].

Let $\alpha,\beta\in S$ be distinct.

3.4 L. Lemma. $\{\alpha,\beta\}\cap L(w)=\emptyset\Rightarrow\beta\notin L(s_\alpha w)$.

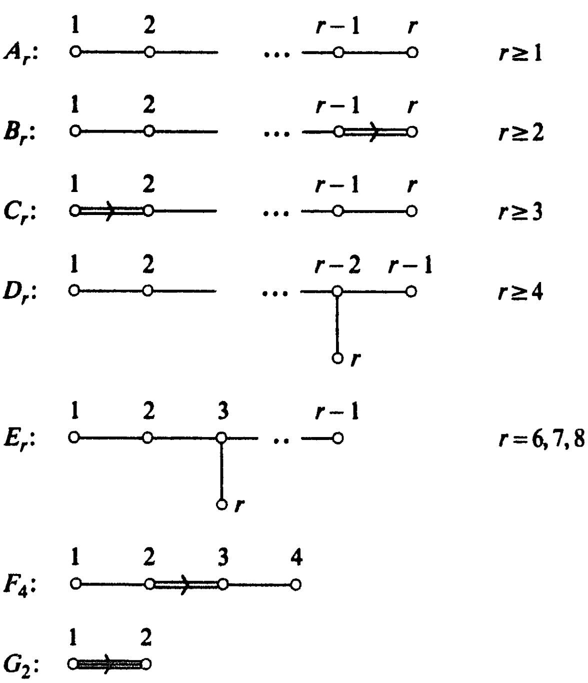
3.4 R. Lemma. $\{\alpha,\beta\}\cap R(w)=\emptyset\Rightarrow\beta\notin R(ws_\alpha)$.

Proof. We give the proof only for R. Recall that $s_\alpha(\beta)=\beta-\alpha^\vee(\beta)\alpha$ where $\alpha^\vee$ is the co-root of α, and $\alpha^\vee(\beta)\le 0$. Thus, $ws_\alpha(\beta)=w(\beta)-\alpha^\vee(\beta)w(\alpha)>0$. Hence $\beta\notin R(ws_\alpha)$. $\square$

We will be using an explicit numbering of the set of simple roots $S=\{\alpha_1,\dots,\alpha_r\}$. This is described in Table 1.

Table 1. Dynkin diagrams

The node numbered i represents the simple root α_i. As usual, the arrows point toward the short roots.



3.5. We proceed, now, to construct the KL-graph $\Gamma(s_1)$ containing the simple reflection s_1. The reader may verify that, for each group G not of type G_2, the list, below, contains all sequences $(i_1,\dots,i_k)$, $1\le i_j\le r$, constructed according to the following 3 rules. (If G is of type G_2, the rules do not apply, and we just list the sequences that concern us, for future reference.)

Rule 1. The sequence begins with 1, i.e. $i_1=1$.

Rule 2. The segment $(\dots i,j\dots)$ can occur only if $i\neq j$ and α_i is adjacent to α_j in the Dynkin diagram, D_G, of G.

Rule 3. The segment $(\dots i,j,i\dots)$ can occur only if $\alpha_i \Rightarrow \alpha_j$ occurs in D_G.

3.6. *Case* A_r $(1,2,\dots,k)$ $1\le k\le r$.

Case B_r $(1,2,\dots,k)$ $1\le k\le r$, $(1,2,\dots,r,r-1,\dots,j)$ $1\le j<r$.

Case C_r $(1,2,\dots,k)$ $1\le k\le r$, $(1,2,1)$.

Case D_r $(1,2,\dots,k)$ $1\le k<r$, $(1,2,\dots,r-2,r)$.

Case E_r $(1,2,\dots,k)$ $1\le k<r$, $(1,2,3,r)$.

Case F_4 $(1,2,\dots,k)$ $1\le k\le 4$, $(1,2,3,2)$, $(1,2,3,2,1)$.

Case G_2 (1), $(1,2)$, $(1,2,1)$, $(1,2,1,2)$, $(1,2,1,2,1)$.

The sequence $(i_1,\dots,i_k)$ determines the Weyl group element $w=s_{i_1}\circ\cdots\circ s_{i_k}$. We will see that the inverses of the elements determined from the above listed sequences form the vertices of $\Gamma(s_1)$.

3.7. Lemma. *The expression for each w determined from the List* 3.6 *is reduced. Also, if w ends in s_i, i.e. $i_k=i$, then $R(w)=\{\alpha_i\}$.*

Proof. The proof for G_2 is by inspection. For $G\neq G_2$, we do an induction on k, the length of the sequence determining w. The case $k=1$ is clear.

Write $w=w's_i$. Here, w' is determined by a sequence from the list of length $k-1$, and $R(w')=\{\alpha_j\}$, $j\neq i$, by the inductive assumption. Hence by Lemma 3.1 R, the expression for w is reduced. By Lemmas 3.2 R and 3.4 R, we have $\alpha_i\in R(w)\subset\{\alpha_i,\alpha_j\}$. It remains to see that $\alpha_j\notin R(w)$.

Write $w'=w''s_j$, perhaps $w''=1$. Now,

$$w(\alpha_j)=w''s_js_i(\alpha_j)=w''((\alpha_i^\vee(\alpha_j)\alpha_j^\vee(\alpha_i)-1)\alpha_j-\alpha_i^\vee(\alpha_j)\alpha_i),$$

by Lemma 3.3. By Rule 2 of 3.5, we know that α_i and α_j are adjacent in D_G, and hence, that w'' is acting on a positive root, by Lemma 3.3. Hence, if $w''=1$, then $w(\alpha_j)>0$, as desired. So, assume $w''\neq 1$. Then $w''=w'''s_n$, perhaps $w'''=1$, and $R(w'')=\{\alpha_n\}$, $n\neq j$. If, also, $n\neq i$, then we clearly have $w(\alpha_j)>0$. So, let us assume that $n=i$. But, now, $(\dots i,j,i)$ occurs in the sequence for w, so by Rule 3 $\alpha_i \Rightarrow \alpha_j$ occurs in D_G. We have $\alpha_i^\vee(\alpha_j)=-1$ and $\alpha_j^\vee(\alpha_i)=-2$. Thus, $w(\alpha_j)=w'''s_i(\alpha_j+\alpha_i)=w'''(\alpha_j)$.

Again, if $w'''=1$, then $w(\alpha_j)>0$. So, assume that $w'''\neq 1$. Then, $R(w''')=\{\alpha_m\}$ by induction. By Rule 3, we know that $m\neq$ j. Hence, $w'''(\alpha_j)>0$. $\square$

3.8. Lemma. *Let w^{-1} be in the List 3.5, and $\sigma \in W$. Suppose that $\{w, \sigma\}$ is an edge of Γ_W (see 2.2) and $L(w) \not\subset L(\sigma)$. Then $\sigma = s_j w$ for some simple reflection s_j.*

Proof. By Lemma 3.7, we have $L(w) = \{\alpha_i\}$. By hypothesis, $\alpha_i \notin L(\sigma)$.

Case $w < \sigma$. Let $\alpha_j \in L(\sigma)$. Then $\alpha_j \neq \alpha_i$, $s_j w > w$ and $s_j \sigma < \sigma$. So, $\sigma = s_j w$, by Lemma 2.4 L.

Case $\sigma < w$. Then, $s_i \sigma > \sigma$ and $s_i w < w$. Again, by Lemma 2.4 L, $\sigma = s_i w$. □

3.9. Lemma. *Let $\sigma \in W$. If $\sigma \approx s_1$ (see 2.2), then σ^{-1} is determined by an element of the List 3.6.*

Proof. For $G = G_2$, the proof is by inspection. For $G \neq G_2$, let $w_1, \dots, w_n$ be a sequence in W with $w_1 = s_1$ and $w_n = \sigma$ such that $\{w_{i-1}, w_i\}$ is an edge of Γ_W and $L(w_{i-1}) \not\subset L(w_i)$ for $i = 2, \dots, n$.

Assume that σ^{-1} does not occur in the List. We will arrive at a contradiction. Since w_1^{-1} is in the List, choose an m so that for $i < m$, w_i^{-1} is in the List and w_m^{-1} is not. Change notation and put $w = w_{m-1}$ and $\sigma = w_m$. By Lemma 3.8, $\sigma = s_j w$ for some j. Also, $L(w) = \{\alpha_i\}$ and $j \neq i$ since σ^{-1} is not in the List.

We will show that $\alpha_i \in L(\sigma)$, contradicting the hypothesis that $L(w) \not\subset L(\sigma)$. If α_i and α_j are not adjacent in D_G, then $s_j(\alpha_i) = \alpha_i$ and $\sigma^{-1}(\alpha_i) = w^{-1}(\alpha_i) < 0$, as desired. So, assume that α_i and α_j are adjacent in D_G.

Since σ^{-1} is not in the List, while w^{-1} is, appending j to the sequence for w^{-1} must violate Rule 3. Thus, the sequence for w^{-1} ends with the segment $(\dots j, i)$ and either α_i is joined to α_j in D_G by a single bond, or $\alpha_i \Rightarrow \alpha_j$ occurs in D_G. In the first case, put $w = s_i w'$ with $L(w') = \{\alpha_j\}$. Then

$$\sigma^{-1}(\alpha_i) = w'^{-1} s_i s_j(\alpha_i) = w'^{-1}(\alpha_j) < 0,$$

as desired.

In the second case, the segment $(\dots j, i)$ of w^{-1} is moving to the left along D_G. Since D_G has only one double bond, w^{-1} must end with $(\dots i, j, i)$. Put $w = s_i s_j w''$ with $L(w'') = \{\alpha_i\}$. Then

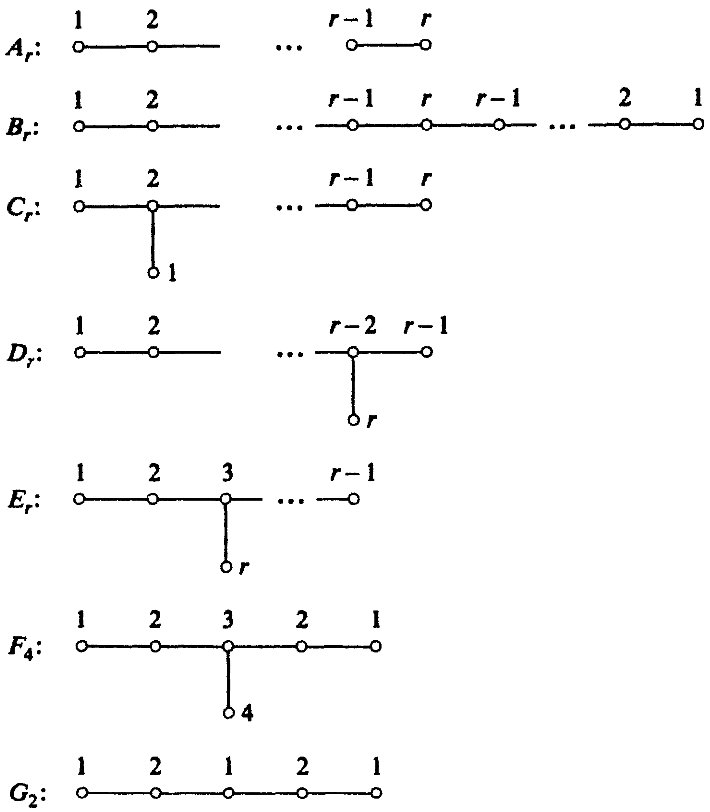
$$\begin{aligned}\sigma^{-1}(\alpha_i) &= w''^{-1} s_j s_i s_j(\alpha_i) = w''^{-1} s_j s_i(\alpha_i + 2\alpha_j)\\ &= w''^{-1} s_j(\alpha_i + 2\alpha_j) = w''^{-1}(\alpha_i) < 0. \qquad \square\end{aligned}$$

3.10. Theorem. *The KL-graph, $\Gamma(s_1)$, containing the simple reflection s_1 is as described in Table 2. For G not of type G_2, $\Gamma(s_1)$ is equal to the graph Γ_u of a subregular unipotent element.*

Proof. The last statement is verified by comparing with Steinberg [14, p. 148]. By Lemma 3.9, the vertices for $\Gamma(s_1)$ are amongst the inverses of the elements determined by the sequences in List 3.6. To determine the $\approx$ equivalence class of s_1, we need only determine the edges of Γ_W occurring amongst the inverses of the

Table 2. The KL-graph containing s_1

The number i beside a node represents the index set $L=\{\alpha_i\}$.



elements determined by the List. We do this by direct computation in each of the possible types of G.

Case A_r. Let $\sigma_k=s_k\cdots s_2s_1$, $1\le k\le r$. By Lemma 3.7, $l(\sigma_k)=k$ and $L(\sigma_k)=\{\alpha_k\}$. Thus, for $i<j$ we have $\sigma_i<\sigma_j$, $s_j\sigma_i>\sigma_i$ and $s_j\sigma_j<\sigma_j$. Hence, by Lemma 2.4 L all the $\prec$ relations are

$$\sigma_1\prec\sigma_2\prec\cdots\prec\sigma_r.$$

Moreover, $L(\sigma_{i-1})\not\subset L(\sigma_i)$ and $L(\sigma_i)\not\subset L(\sigma_{i-1})$, so we have

$$\sigma_1\approx\sigma_2\approx\cdots\approx\sigma_r.$$

Case B_r. Let $\sigma_k=s_k\cdots s_2s_1$, $1\le k\le r$, and $\tau_j=s_j\cdots s_{r-1}\sigma_r$, $1\le j<r$. Then $l(\sigma_k)=k$, $L(\sigma_k)=\{\alpha_k\}$, $l(\tau_j)=2r-j$ and $L(\tau_j)=\{\alpha_j\}$. As in the case A_r, we have

$$\sigma_1\approx\sigma_2\approx\cdots\approx\sigma_r\approx\tau_{r-1}\approx\cdots\approx\tau_1.$$

The only possible omitted edges must be between σ_k and τ_j, and Lemma 2.4 L implies this can happen only if $k=j$. But $l(\tau_k)-l(\sigma_k)$ is even, so that $\sigma_k\not\prec\tau_k$.

Case C_r. Let $\sigma_k=s_k\cdots s_2s_1$, $1\le k\le r$ and $\tau=s_1s_2s_1$. Then $l(\sigma_k)=k$, $L(\sigma_k)=\{\alpha_k\}$,

$l(\tau)=3$ and $L(\tau)=\{\alpha_1\}$. We have

$$\begin{array}{l} \sigma_1 \approx \sigma_2 \approx \cdots \approx \sigma_r \\ \quad \wr\wr \\ \quad \tau \end{array}$$

with no other edges amongst the σ's. Moreover, $\{\sigma_1, \tau\}$ is not an edge since $l(\sigma_1)-l(\tau)$ is even, and $\{\sigma_i, \tau\}$ is not an edge for $i \geq 3$ since $\tau \not< \sigma_i$; cf. 2.2.

Case D_r. Let $\sigma_k = s_k \cdots s_2 s_1$, $1 \leq k < r$ and $\tau = s_r s_{r-2} \cdots s_2 s_1$. Then $l(\sigma_k)=k$, $L(\sigma_k)=\{\alpha_k\}$, $l(\tau)=r-1$ and $L(\tau)=\{\alpha_r\}$. We have

$$\begin{array}{l} \sigma_1 \approx \cdots \approx \sigma_{r-2} \approx \sigma_{r-1} \\ \qquad\qquad \wr\wr \\ \qquad\qquad \tau \end{array}$$

with no other edges. The reasoning is the same as in the previous cases.

Case E_r. Let $\sigma_k = s_k \cdots s_2 s_1$, $1 \leq k < r$ and $\tau = s_r s_3 s_2 s_1$. Then $l(\sigma_k)=k$, $L(\sigma_k)=\{\alpha_k\}$, $l(\tau)=4$ and $L(\tau)=r$. We have

$$\begin{array}{l} \sigma_1 \approx \sigma_2 \approx \sigma_3 \approx \cdots \approx \sigma_{r-1} \\ \qquad\quad \wr\wr \\ \qquad\quad \tau \end{array}$$

with no other edges, as in the previous cases.

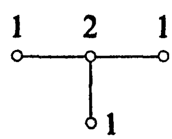
Case F_4. Let $\sigma_k = s_k \cdots s_2 s_1$, $1 \leq k \leq 4$, $\tau_1 = s_1 s_2 s_3 s_2 s_1$ and $\tau_2 = s_2 s_3 s_2 s_1$. Then $l(\sigma_k)=k$, $L(\sigma_k)=\{\alpha_k\}$, $l(\tau_i)=6-i$ and $L(\tau_i)=\{\alpha_i\}$. We have

$$\begin{array}{l} \sigma_1 \approx \sigma_2 \approx \sigma_3 \approx \sigma_4 \\ \qquad\quad \wr\wr \\ \qquad\quad \tau_2 \\ \qquad\quad \wr\wr \\ \qquad\quad \tau_1 \end{array}$$

with no other edges, as in the previous cases.

Case G_2. Let $\sigma_1 = s_1$, $\sigma_2 = s_2 s_1$, $\sigma_3 = s_1 s_2 s_1$, $\sigma_4 = s_2 s_1 s_2 s_1$ and $\sigma_5 = s_1 s_2 s_1 s_2 s_1$. Then $l(\sigma_i)=i$ and $L(\sigma_i)=\{\alpha_1\}(i \text{ odd}), \{\alpha_2\}(i \text{ even})$. We have $\sigma_1 \approx \sigma_2 \cdots \approx \sigma_5$ with no other edges, as in the previous cases. □

Remark. For G_2 there are 3 other KL-graphs, other than the one in Table 2, viz. the graph dual to the one in Table 2, the graph consisting of the identity element, and the graph consisting of the element of longest length in W. Evidently, the subregular resolution graph



does not appear.

4. The resolution graph of the minimal unipotent class

In this section, we show that the resolution graph, Γ_u, as explained in Section 1, of an element u in the minimal conjugacy class, has one vertex v_i for each long simple root α_i. Two vertices v_i and v_j are joined precisely when α_i and α_j are adjacent in the Dynkin diagram, D_G, of G. The weight associated to v_i is $I_i = S\setminus\{\alpha_i\}$. These graphs are displayed in Table 3. In particular, for G of type A, D or E, Γ_u occurs as dual to the subregular graph, and also as a KL-graph. This last statement follows from Theorem 3.10 and Proposition 2.3.

4.1. For each w, an element of the Weyl group W, let $B[w]$ denote the Bruhat cell BwB in G/B; see Borel [2, p. 347]. It is isomorphic to $\mathbb{C}^{l(w)}$, and the Euclidean closure in G/B is an irreducible algebraic variety of dimension $l(w)$. The Bruhat ordering on W can be defined by $w_1 < w_2$ when $B[w_1] \subset \overline{B[w_2]}$, and coincides with the ordering defined in 2.2.

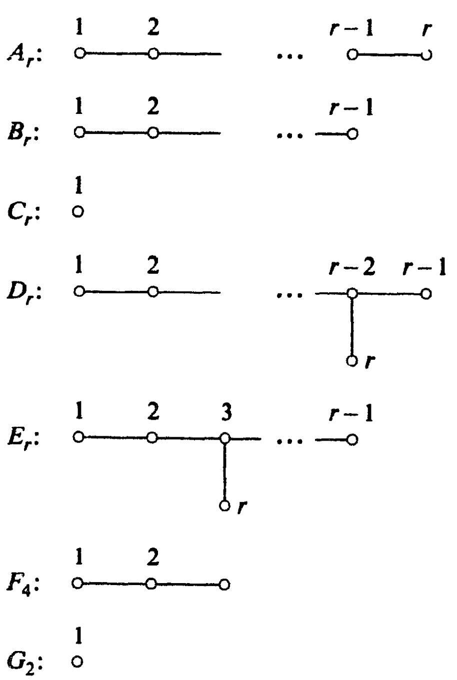
Recall that $w_0 \in W$ is the element of maximal length. The following is a formal consequence of the definition of Bruhat order in 2.1.

4.2. Lemma. *Let $w_1 < w_2$ in W. Then $w_1^{-1} < w_2^{-1}$ and $w_0 w_2 < w_0 w_1$.*

We view, now, the root systems, R, as a subset of $\mathfrak{h}^*$, the vector space dual of

Table 3. The resolution graph of the minimal unipotent conjugacy class

The label, i, represents the index set $I = S\setminus\{\alpha_i\}$.



the Lie algebra of a maximal torus of G. Fix a W-invariant inner product (,) on $\mathfrak{h}^*$, and an element $a \in \mathfrak{h}^*$. Let $E = \{w \in W: wa = a\}$. Assume that $(a, \alpha_i) \geq 0$ for each simple root α_i, i.e. that a is in the closure of the fundamental Weyl chambre. Then, E is generated by those simple reflections s_i for which $(a, \alpha_i) = 0$. For the proof of this, and other related results, see Carter [3A, §2.5].

4.3. Lemma. *Let σ be any element of W. Then, with the above notation and hypothesis, the coset σE contains elements w_1 and w_2, necessarily unique, such that*

$$w_1 < w < w_2 \quad \forall w \in \sigma E.$$

Proof. Let $w_1 \in \sigma E$ be the element of minimal length. For any $w \in \sigma E$, we may write $w = w_1 w'$ for some $w' \in E$. Moreover, $l(w) = l(w_1) + l(w')$. In particular, $w_1 < w$. Now, let $w_2' \in w_0 \sigma E$ be the element of minimal length. Put $w_2 = w_0 w_2' \in \sigma E$. By the first part of the proof, $w_2' < w_0 w$, whence, by Lemma 4.2, $w < w_0 w_2'$. □

Let $\mathscr{H} = \{\eta \in W: \eta^{-1}(h) > 0\}$, where h is the highest root of R. Let $d = \max\{l(\eta): \eta \in \mathscr{H}\}$, and let $\alpha_1, \ldots, \alpha_k$ be the long roots in the Dynkin diagram, D_G, of G; see Table 1. In [5, §§3,4], the following facts are proved.

4.4. Lemma. *The Springer fibre for an element u in the minimal conjugacy class is*

$$\mathscr{B}_u = \bigcup \{B[\eta]: \eta \in \mathscr{H}\}.$$

There are precisely k elements $\eta_1, \ldots, \eta_k \in \mathscr{H}$ of the maximal length d; each corresponds to a long simple root. They satisfy

(i) $\eta_i^{-1}(h) = \alpha_i$.

(ii) $\eta_i(\alpha) < 0$ *for every simple root α other than α_i.*

(iii) $\eta_i s_j = \eta_j s_i$ *when α_i is adjacent to α_j in D_G; i.e. when $(\alpha_i, \alpha_j) \neq 0$.*

(iv) η_i *is the unique Weyl group element of maximal length taking α_i to h.*

Note. The case $G = A_r$ is described in Vargas [16, p. 2]. In [13], Springer shows that $\mathscr{B}_u$, above, is a union of Bruhat cells. Also, condition iv follows from i and Lemma 4.3, since $(h, \alpha) \geq 0$ for each simple root α.

Let $C_i = \overline{B[\eta_i]}$. Then $\dim(C_i) = d$ and $\mathscr{B}_u \supset \bigcup C_i$, $i = 1, \ldots, k$. The next lemma is proved in Spaltenstein [10, 11] for *every* Springer fibre $\mathscr{B}_u$. It shows that the above containment is actually an equality.

4.5. Lemma. *$\mathscr{B}_u$ is pure d-dimensional.*

Proof. Let $\eta \in \mathscr{H}$. We wish to show that $\eta < \eta_i$ for some $i = 1, \ldots, k$. If $\eta^{-1}(h) = \alpha_i$, then we are done by Lemmas 4.3 and 4.4(iv).

If $\eta^{-1}(h)$ is not a simple root, choose any simple root α_j such that $\eta(\alpha_j) > 0$. This is possible since $\eta \neq w_0$. Then, by Lemma 3.1 R, $\eta < \eta s_j$. Also, $(\eta s_j)^{-1}(h) =$

$s_j\eta^{-1}(h)>0$, since s_j permutes the elements of $R^+\setminus\{\alpha_j\}$. We take ηs_j to be our new η, and must eventually obtain that $\eta^{-1}(h)$ is a simple root. This completes the proof. □

4.6 Theorem. *The weighted resolution graph, Γ_u, for u in the minimal unipotent conjugacy class, is as displayed in Table* 3.

Proof. By Lemmas 4.4 and 4.5, there is one vertex v_i for each long simple root α_i.

Step 1. v_i is joined to v_j if α_i is adjacent to α_j in D_G. (The converse statement is Step 2.)

With the notation of Lemma 4.4, let $\eta=\eta_i s_j=\eta_j s_i$. Since $\eta_i(\alpha_j)<0$, we have that $l(\eta)=l(\eta_i)-1=d-1$, and $\eta<\eta_i$. Similarly, $\eta<\eta_j$. Hence, $C_i\cap C_j\supset B[\eta]$, a subvariety of dimension $d-1$. This proves Step 1.

Step 2. If v_i is joined to v_j, then α_i and α_j are adjacent in D_G.

Suppose v_i is joined to v_j, but that α_i and α_j are not adjacent. We will arrive at a contradiction.

Let $\eta\in\mathscr{H}$, $l(\eta)=d-1$ and $B[\eta]\subset C_i\cap C_j$. As explained in 2.1, there are positive roots α and β so that $\eta_i s_\alpha=\eta=\eta_j s_\beta$. Renumber the long simple roots and choose a chain

$$\text{old } \alpha_i=\alpha_0,\alpha_1,\dots,\alpha_{m-1},\alpha_m=\text{old } \alpha_j,$$

without repetitions; α_{n-1} is adjacent to α_n in D_G for $n=1,\dots,m$, i.e. this is the subdiagram of D_G containing old α_i and old α_j. Also, we are assuming that $m\geq 2$.

We have that $\eta_0 s_\alpha=\eta_m s_\beta$, i.e. $s_\beta s_\alpha=\eta_m^{-1}\eta_0=\pi_m$, where

$$\pi_m=s_{m-1}s_m s_{m-2}s_{m-1}\cdots s_1 s_2 s_0 s_1,$$

as is easily checked by induction, using Lemma 4.4(iii). The following equations may also be verified:

$$\pi_m(\alpha_0)=\alpha_m, \tag{1}$$

$$\pi_m(\alpha_m)=\alpha_{m-2}, \tag{2}$$

$$\pi_m(\alpha_1)=-\alpha_0-\alpha_1-\cdots-\alpha_m. \tag{3}$$

Lemma 3.3 will be used in the remainder of the proof. Replace π_m by $s_\beta s_\alpha$, and rewrite (1) and (2) to obtain

$$\alpha_0-\alpha^\vee(\alpha_0)\alpha=\alpha_m-\beta^\vee(\alpha_m)\beta, \tag{4}$$

$$\alpha_m-\alpha^\vee(\alpha_m)\alpha=\alpha_{m-2}-\beta^\vee(\alpha_{m-2})\beta. \tag{5}$$

Since the α_i's form part of a basis, it is easy to see from equation (4) that neither $\alpha^\vee(\alpha_0)$ nor $\beta^\vee(\alpha_m)$ is zero. Moreover, they have the same sign. A similar statement holds for (5). Thus, $\alpha^\vee(\alpha_0)=\varepsilon\alpha^\vee(\alpha_m)$ and $\beta^\vee(\alpha_m)=\varepsilon\beta^\vee(\alpha_{m-2})$, where $\varepsilon=\pm 1$.

If $\varepsilon=1$, subtract (4) and (5).

$$\alpha_0-\alpha_m=\alpha_m-\alpha_{m-2}, \quad \text{whence} \quad \alpha_0+\alpha_{m-2}=2\alpha_m$$

This is impossible. Hence, $\varepsilon=-1$. Now, add (4) and (5).

$$\alpha_0+\alpha_m=\alpha_m+\alpha_{m-2}, \quad \text{whence} \quad \alpha_0=\alpha_{m-2}.$$

Thus, $m=2$ and $s_\beta s_\alpha=s_1s_2s_0s_1$, so that $(s_\beta s_\alpha)^2=1$, i.e. $\beta^\vee(\alpha)=0$.

Equation (3) simplifies to

$$(6) \qquad \alpha_0+2\alpha_1+\alpha_2=\alpha^\vee(\alpha_1)\alpha+\beta^\vee(\alpha_1)\beta.$$

Let $\alpha^\vee(\alpha_0)=\lambda$ and $\beta^\vee(\alpha_2)=\mu$, which as we have seen, are both non-zero and of the same sign. Equation (4) now reads

$$(7) \qquad \alpha_0-\alpha_2=\lambda\alpha-\mu\beta.$$

Evaluate $\alpha_1^\vee$ on both sides and obtain

$$(8) \qquad 0=\lambda\alpha_1^\vee(\alpha)-\mu\alpha_1^\vee(\beta),$$

since $\alpha_1^\vee(\alpha_0)=\alpha_1^\vee(\alpha_2)=-1$. Equation (6) implies that not both $\alpha_1^\vee(\alpha)$ and $\alpha_1^\vee(\beta)$ are zero. Thus, since α_1 is a long root, and λ and μ have the same sign, equation (8) implies that $\alpha_1^\vee(\alpha)=\alpha_1^\vee(\beta)=\pm1$ and $\lambda=\mu$. In particular, α and β have the same length and

$$\alpha^\vee(\alpha_1)=\beta^\vee(\alpha_1)=\varepsilon'\lambda, \quad \varepsilon'=\pm1.$$

We now use equations (6) and (7) to solve for α and β. There are two cases to consider.

Case $\varepsilon'=1$. Then $\alpha=\alpha_0+\alpha_1$. Recall that $\eta=\eta_0s_\alpha$ has length $d-1$. But

$$\begin{aligned}\eta_0(\alpha)&=\eta_0(\alpha_0)+\eta_0(\alpha_1)\\ &=h+\eta_0(\alpha_1), \qquad \text{Lemma 4.4(i)}\\ &>0,\end{aligned}$$

i.e. $\eta_0s_\alpha>\eta_0$, as explained in 2.1. This is a contradiction, since η_0 has length d.

Case $\varepsilon'=-1$. Then $\alpha=\alpha_1+\alpha_2$. We have $\alpha=s_1(\alpha_2)$, so $s_\alpha=s_1s_2s_1$. It is easily verified that $\eta_0(\alpha_1)<0$, $\eta_0s_1(\alpha_2)<0$ and $\eta_0s_1s_2(\alpha_1)<0$. This implies that $\eta=\eta_0s_\alpha$ has length $d-3$, contradicting the hypothesis that η has length $d-1$.

This completes the proof of Step 2.

Step 3. The weight associated to v_i is $I_i=S\setminus\{\alpha_i\}$.

For any root α, let $X_\alpha\subset G$ be the associated root subgroup. Let $U_i=\prod\{X_\alpha\colon \alpha>0,\ \eta_i^{-1}(\alpha)<0\}$. Now, C_i is the closure of

$$B[\eta_i]=U_i\eta_i\mathrm{o},$$

where o is the image under $G\to G/B$ of $1\in G$, and η_i is identified with some representation in the normalizer of the maximal torus in G.

Let α_j be any simple root, and P_j the associated minimal parabolic. The fibre of $G/B\to G/P_j$ through a point xB is $\{xpB\colon p\in P_j\}$. Thus, C_i is $\mathbb{P}^1$-saturated precisely when for each $z\in U_i$ and for each p in some neighborhood P_j^0 of 1 in P_j there is a z' in U_i with $z\eta_i pB = z'\eta_i B$. This condition may be rewritten as

$$(*) \qquad P_j^0 \subset \eta_i^{-1} U_i \eta_i B.$$

Since P_j is generated by B and $X_{-\alpha_j}$, $(*)$ is equivalent to having

$$X_{-\alpha_j} \subset \prod \{X_\beta : \beta = \eta_i^{-1}(\alpha) < 0, \alpha > 0\}.$$

Here, we have used the fact that $wX_\alpha w^{-1} = X_{w(\alpha)}$ for any $w \in W$ and any root α. Thus, [2, p. 347], $-\alpha_j = \eta_i^{-1}(\alpha)$ for some $\alpha > 0$, i.e. $\eta_i(\alpha_j) < 0$. The argument can be reversed, so that

$$\begin{aligned} I_i &= \{\alpha \in S : \eta_i(\alpha) < 0\} \\ &= S \setminus \{\alpha_i\} \qquad \text{by Lemma 4.4(ii).} \quad \square \end{aligned}$$

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References

[1] I.N. Bernstein, I.M. Gel'fand and S.I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28 (1973) 1-26.
[2] A. Borel, Linear Algebraic Groups (Benjamin, New York, 1969).
[3] N. Bourbaki, Groupes et Algèbres de Lie, Ch. IV-VI (Hermann, Paris, 1968).
[3A] R. Carter, Simple groups of Lie type, Pure and Applied Mathematics 28 (Wiley, New York, 1972).
[4] J. Dixmier, Algèbres Enveloppantes (Gauthiers-Villars, Paris, 1974).
[5] N. Goldstein, Ampleness and connectedness in complex G/P, Trans. Amer. Math. Soc. 274 (1982) 361-373.
[6] J. Humphreys, Introduction to Lie algebras and Representation Theory, 3rd printing, revised (Springer, Berlin, 1972).
[7] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Inventiones Math. 53 (1979) 165-184.
[8] G. Lusztig, Green polynomials and singularities of unipotent classes, Advances Math. 42 (1981) 169-179.
[9] J.-P. Serre, Algèbres de Lie Semi-simples Complexes (Benjamin, New York, 1966).
[10] N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology 16 (1977) 203-204.
[11] N. Spaltenstein, Classes Unipotentes et Sous-groupes de Borel, Lecture Notes in Mathematics 946 (Springer, Berlin, 1982).
[12] T. Springer, The unipotent variety of a semi-simple group, Bombay Colloq. Algebraic Geometry (Oxford Univ. Press, London, 1969) 373-391.
[13] T. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Inventiones Math. 36 (1976) 173-207.

[14] R. Steinberg, R. Conjugacy Classes in Algebraic Groups, Lecture Notes in Math. 366 (Springer, Berlin, 1974).
[15] R. Steinberg, Lectures on Chevalley groups (Yale University Press, New Haven, CO, 1967).
[16] J. Vargas, Fixed points under the action of unipotent elements of SL_n in the flag variety, Bol. Soc. Mat. Mexicana 24 (1979) 1-14.

Note added in proof

Since completion of the paper, it has been brought to our attention [17] that, for the classical groups, the structure of $\mathcal{B}_u$ (u minimal) was known to G. Lusztig. Also, Lusztig [18] had, independently, constructed the left cells containing a simple reflection, in any Coxeter group.

[17] G. Lusztig, Letter, 8 December 1982.
[18] G. Lusztig, Some examples of square integrable representations of semi-simple p-adic groups, Preprint, March 1982.