Pattern Recognition Letters 2 (1984) 159–163
North-Holland March 1984

# *Digital metrics: A graph-theoretical approach*

Frank HARARY
*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA*

Robert A. MELTER
*Department of Mathematics, Southampton College of Long Island University, Southampton, NY 11968, USA*

Ioan TOMESCU
*Faculty of Mathematics, University of Bucharest, Bucharest, Romania*

Received 3 August 1983

*Abstract:* Consider the following two graphs $M$ and $N$, both with vertex set $Z \times Z$, where $Z$ is the set of all integers. In $M$, two vertices are adjacent when their euclidean distance is 1, while in $N$, adjacency is obtained when the distance is either 1 or $\sqrt{2}$. By definition, $H$ is a metric subgraph of the graph $G$ if the distance between any two points of $H$ is the same as their distance in $G$. We determine all the metric subgraphs of $M$ and $N$. The graph-theoretical distances in $M$ and $N$ are equal respectively to the city block and chessboard matrics used in pattern recognition.

*Key words:* Digital metrics, graph theory, city block distance, chessboard distance.

## 1. Introduction

We follow the notation and terminology of the book [3]. A subgraph $H$ of $G$ is a *metric subgraph* if the distance between any two points of $H$ is the same as their distance in $G$. Graphs in which every connected induced subgraph is metric are said to be *distance-hereditary*. A characterization of distance-hereditary graphs was derived by Howorka [6]. (Two diagonals $e_1$, $e_2$ or a cycle $\varphi$ are called a pair of skew diagonals of $\varphi$ if the graph $\varphi + e_1 + e_2$ is homeomorphic with $K_4$.) He showed, for example, that a graph $G$ is distance-hereditary if and only if each cycle of $G$ of length at least five has a pair of skew diagonals. (Figure 1 illustrates, as in [6], a distance-hereditary graph with 6 points.) Metric subgraphs have also been studied by Kundu [7] who showed that if $G$ has a unique metric spanning tree then $G$ is regular. He thus provided an answer to a question posed by Chartrand and Schuster [1]. Other results on isometric graphs are due to Chartrand and Steward [2].

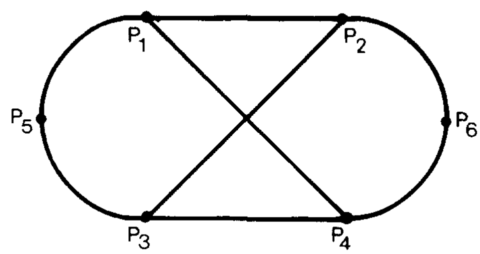


Fig. 1. A distance-hereditary graph.

In work on pattern recognition (see [10]) one considers a variety of distances defined on $Z \times Z$, the set of all integral points in the plane. For example, the city block distance $d_4$ and chessboard distance $d_8$ are defined by

$$d_4[(X_1, Y_1), (X_2, Y_2)] = |X_1 - X_2| + |Y_1 - Y_2|,$$
$$d_8[(X_1, Y_1), (X_2, Y_2)] = \max(|X_1 - X_2|, |Y_1 - Y_2|).$$

Other distances for $Z \times Z$ have recently been studied in [8].

If $u, v$ are points of $Z \times Z$, then $d_4(u,v)$ and $d_8(u,v)$ are equal respectively to the usual graph theoretic distance in the graphs $M$ and $N$, both of which have $Z \times Z$ as vertex set. In $M$ two vertices are adjacent when their euclidean distance is 1, while in $N$ adjacency is obtained when this distance is either 1 or $\sqrt{2}$. The graph $M$ is often called the *Manhattan graph*. One could refer to $N$ as a kind of diagonalized Manhattan graph. It can also be appropriately called the *King's graph* since adjacency is equivalent to two points being a King's move apart on an infinite chessboard. In Figure 2 we show some metric subgraphs of $M$ and $N$. Our object is to provide characterizations of the metric subgraphs of the Manhattan graph and the King's graph.

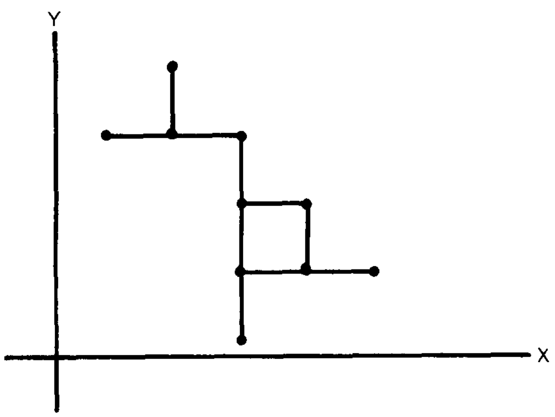


Fig. 2a. A metric subgraph of $M$ (this graph is axially convex but not diagonally convex).

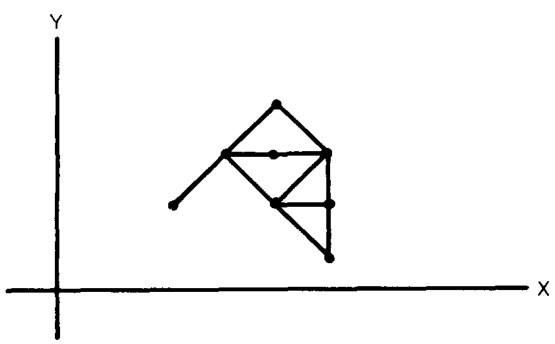


Fig. 2b. A metric subgraph of $N$ (this graph is diagonally convex but not axially convex).

## 2. Metric subgraphs of the Manhattan graph

A general notion of convexity in graphs has been defined by Harary and Nieminen [5]. A set $S \subset V(G)$ is *convex* if for all $u, v \in S$, every vertex on all $u-v$ geodesics is also in $S$. If $G$ were not mentioned in the preceding sentence, this definition would be the same as that of a convex set in any other metric space. It will be useful, however, to define the following related but different concept. A subgraph $G$ of $M$ is *axially convex* if for any two points of $G$ lying on a line parallel to the coordinate axes, all points on the line segment connecting them belong to $V(G)$.

Rosenfeld [9] characterized geodesics for $M$ in the following way: A path

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

of $M$ is a geodesic if and only if

$$X_1 \le X_2 \le \cdots \le X_n \quad \text{and} \quad Y_1 \le Y_2 \le \cdots \le Y_n.$$

We have assumed without loss of generality that $X_1 \le X_n$ and $Y_1 \le Y_n$.

We now proceed to the main theorem of this section.

**Theorem 1.** *A subgraph $G$ of the Manhattan graph $M$ is a metric subgraph if and only if $G$ is both connected, and axially convex.*

**Proof.** If $G$ is a metric subgraph of $M$, then $G$ obviously is connected. Suppose that $G$ is not axially convex. It follows that there are two points $a, b \in V(G)$ such that the line through $a$ and $b$ is parallel to one of the coordinate axes, but at least one point of the segment connecting $a$ and $b$ does not belong to $V(G)$. This implies that $d_G(a,b) \ge d_M(a,b) + 2$, which contradicts the hypothesis.

Suppose now that the subgraph $G$ is connected and axially convex. It remains to show that $d_G(a,b) = d_M(a,b)$ for any $a, b \in V(G)$. Since $G$ is connected there is a path in $G$ between any two points $a, b$ of $G$. Let a geodesic $P_{ab}$ be determined by the sequence of points

$$a = (X_1, Y_1), (X_2, Y_2), \ldots, (X_r, Y_r) = b$$

and suppose that $X_1 \le X_r$, $Y_1 \le Y_r$ and $d_G(a,b) > d_M(a,b)$. Since $P_{ab}$ is not a geodesic for $M$ it follows that there is an index $s \ge 1$ such that

$$X_1 \le X_2 \le \cdots \le X_s, \qquad Y_1 \le Y_2 \le \cdots \le Y_s$$

and

$$X_s > X_{s+1} \quad \text{or} \quad Y_s > Y_{s+1}.$$

We shall give details of the proof for the instance in which $X_s > S_{s+1}$. Since $P_{ab}$ is a geodesic in a subgraph of $M$, it follows that

$$X_{s-1} = X_s = X_{s+1} + 1 \quad \text{and} \quad Y_{s+1} = Y_s = Y_{s-1} + 1.$$

We will examine separately the two cases I: $X_1 \ge X_s$ and II: $X_1 < X_s$.

*Case I.* If $X_1 = X_s$ then, since $X_1 \le X_r$, there is a point $d = (X_p, Y_p)$ on $P_{ab}$ such that $p \le r$, $X_p = X_s$, and $Y_p > Y_s$. If $c = (X_s, Y_s)$ and $G$ is axially convex it follows that all points on the segment connecting $c$ and $d$ are in $V(G)$ (see Figure 3). If the subpath

$$(X_s, Y_s), (X_{s+1}, Y_{s+1}),, \ldots, (X_p, Y_p)$$

of $P_{ab}$ is replaced by the vertical path $P_{cd}$ between $c$ and $d$, then a path between $a$ and $b$ in $G$ is obtained which is shorter than $P_{ab}$; this contradicts the hypothesis that $P_{ab}$ is a geodesic in $G$.

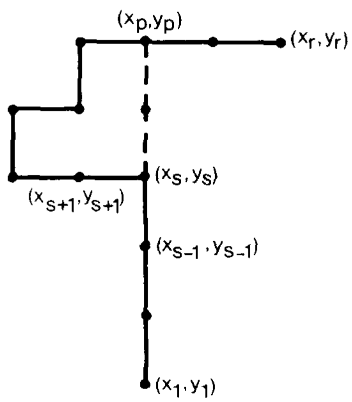


Fig. 3. Illustration for the proof of Theorem 1.

*Case II.* If $X_1 < X_s$ then there is an index $1 \le p \le s-2$ such that $X_p = X_{s+1}$ and $Y_p < Y_{s+1}$, i.e., the points $c = (X_p, Y_p)$ and $d = (X_{s+1}, Y_{s+1})$ lie on a line parallel to the $y$-axis. We now replace the subpath

$$(X_p, Y_p), (X_{p+1}, Y_{p+1}), \ldots, (X_{s+1}, Y_{s+1})$$

of $P_{ab}$ by the vertical path between $c$ and $d$; again, a path between $a$ and $b$ shorter than $P_{ab}$ has been constructed in $G$.

## 3. Metric subgraphs of the King's graph

The following variation of convexity is pertinent to the characterizations at hand. A subgraph $G$ of the King's graph $N$ is *diagonally* convex if for any two points of $G$ lying on a line with slope +1, all points of the line segment connecting them belong to $V(G)$.

Rosenfeld [9] characterized geodesics in $N$ as follows: A path

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

of $N$ is a geodesic if and only if

$$X_1 < X_2 < \cdots < X_n \quad \text{or} \quad Y_1 < Y_2 < \cdots < Y_n,$$

assuming without loss of generality that $X_1 \le X_n$ and $Y_1 \le Y_n$.

The principal result of this section can now be stated.

**Theorem 2.** *A subgraph G of the King's graph N is a metric subgraph of N if and only if G is*

(i) *connected,*

(ii) *diagonally convex, and*

(iii) *G does not contain as a subgraph any of the eight subgraphs illustrated in Figure* 4.

**Proof.** It is clear that if $G$ does not satisfy any of (i), (ii), or (iii), then it is not a metric subgraph of $N$. In particular, in each of the graphs of Figure 4, we have

$$d_N(C,D) = d_G(A,B) + 1 = d_G(C,D) - 1.$$

Conversely assume that a subgraph $G$ of $N$ satisfies (i), (ii) and (iii) and is not a metric subgraph of $N$. It follows that there exist two points $u, v \in V(G)$ such that $d_G(u,v) > d_N(u,v)$. Since $G$ is connected, there is a shortest path $P_{uv}$ between $u$ and $v$. Let $P_{uv}$ be determined by the sequence of points

$$u = (X_1, Y_1), (X_2, Y_2), \ldots, (X_r, Y_r) = v$$

and suppose that $X_1 \le X_r$ and $Y_1 \le Y_r$. Since $P_{uv}$ is not a geodesic in $N$ it follows that there are indices $i, j$, $1 \le i, j \le r-1$ such that

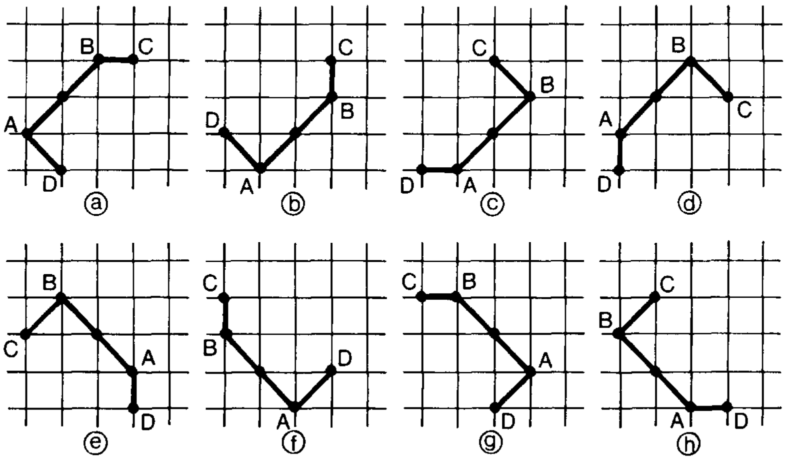


Fig. 4. Forbidden subgraphs in metric subgraphs of $N$.

$$X_1<X_2<\cdots<X_i, \quad Y_1<Y_2<\cdots<Y_j,$$
$$X_i\geq X_{i+1} \quad \text{and} \quad Y_j\geq Y_{j+1}.$$

We need to consider separately the cases I: $i=j$ and II: $i\neq j$.

*Case I.* If $i=j$ then

$$X_{i+1}=X_i-1 \quad \text{and} \quad Y_{i+1}=Y_i$$

or

$$X_{i+1}=X_i \quad \text{and} \quad Y_{i+1}=Y_i-1.$$

Since $X_{i-1}=X_i-1$ and $Y_{i-1}=Y_i-1$ it follows that in either case the path

$$(X_{i-1},Y_{i-1}),(X_{i+1},Y_{i+1})$$

of length one in $G$ is shorter than the subpath

$$(X_{i-1},Y_{i-1}),(X_i,Y_i),(X_{i+1},Y_{i+1})$$

of $P_{uv}$. This contradicts the assumption that $P_{uv}$ is a geodesic in $G$.

*Case II.* Assume without loss of generality that $i<j$. Since $X_{i+1}\leq X_i$ and $Y_{i+1}>Y_i$ it follows that

$$X_{i+1}=X_i \quad \text{and} \quad Y_{i+1}=Y_i+1$$

or

$$X_{i+1}=X_i-1 \quad \text{and} \quad Y_{i+1}=Y_i+1.$$

There are in fact three subcases which must now be examined:

II.1. $X_j=X_{j-1}+1$ and $Y_j=Y_{j-1}+1$,
II.2. $X_j=X_{j-1}$ and $Y_j=Y_{j-1}+1$,
II.3. $X_j=X_{j-1}-1$ and $Y_j=Y_{j-1}+1$.

We shall present the details for subcase II.1. The other subcases can be dealt with in a similar manner.

In subcase II.1 since $Y_{j+1}\leq Y_j$ it follows that exactly one of the following statements is true:

($\alpha$) $X_{j+1}=X_j-1$ and $Y_{j+1}=Y_j$,
($\beta$) $X_{j+1}=X_j$ and $Y_{j+1}=Y_j-1$,
($\gamma$) $X_{j+1}=X_j+1$ and $Y_{j+1}=Y_j$,
($\delta$) $X_{j+1}=X_j+1$ and $Y_{j+1}=Y_j-1$.

If ($\alpha$) or ($\beta$) holds it is easy to see that, as in the proof of case I, we can replace a subpath of length two of $P_{uv}$ by a path of length one and hence $P_{uv}$ cannot be a geodesic for $G$.

If ($\gamma$) is true then there exists an index $k$, $i+1\leq k\leq j-1$ such that $X_k<X_{k+1}<\cdots<X_j$, $Y_k<Y_{k+1}<\cdots<Y_j$, and either

($\delta$1) $X_{k-1}=X_k$ and $Y_{k-1}=Y_k-1$ or
($\delta$2) $X_{k-1}=X_k+1$ and $Y_{k-1}=Y_k-1$.

In the case of ($\delta$1), the slope of the line passing through the points

$$C=(X_{j+1},Y_{j+1}) \quad \text{and} \quad D=(X_{k-1},Y_{k-1})$$

is equal to 1. Since $G$ is diagonally convex it follows that

$$d_G(C,D)=d_{P_{uv}}(C,D)-1$$

and hence $P_{uv}$ cannot be a geodesic for $G$.

If ($\delta$2) holds then the subpath of $P_{uv}$ between $C$ and $D$ is similar to the graph depicted in Figure 4a and hence (iii) implies that

$$d_G(C,D)=d_G(A,B)+1$$
$$<d_G(A,B)+2=d_{P_{uv}}(C,D)$$

which contradicts the assumption that $P_{uv}$ is a geodesic.

When $\delta$ holds we can show the existence of an index $k$ having the same properties, but if

$$X_{k-1} = X_k \quad \text{and} \quad Y_{k-1} = Y_k - 1$$

the graph of Figure 4d is obtained. Finally if

$$X_{k-1} = X_k + 1 \quad \text{and} \quad Y_{k-1} = Y_k - 1$$

the slope of $CD$ equals 1 and it follows that $P_{uv}$ is not a geodesic for $G$ since $G$ is diagonally convex.

## References

[1] Chartrand, G. and S. Schuster (1974). Which graphs have unique distance trees? *Amer. Math. Monthly* 81, 53–56.

[2] Chartrand, G. and M.J. Steward (1971). Geometric graphs. *Springer Lecture Notes Math.* No. 186, pp. 63–67.

[3] Harary, F. (1969). *Graph Theory.* Addison-Wesley, Reading, MA.

[4] Harary, F., *Achievement and avoidance games.* To appear.

[5] Harary, F. and J. Nieminen (1981). Convexity in graphs. *J. Differential Geometry* 16, 185–190.

[6] Howorka, E. (1977). A characterization of distance hereditary graphs. *Quart. J. Math. Oxford* 26, 417–420.

[7] Kundu, S. (1977). The Chartrand-Schuster conjecture: Graphs with unique distance trees are regular. *J. Combin. Theory B* 22, 233–245.

[8] Melter, R. and I. Tomescu (1983). Path-generated digital metrics. *Pattern Recognition Letters* 1, 151–154.

[9] Rosenfeld, A. (1978). Geodesics in digital pictures. *Information and Control* 36, 74–84.

[10] Rosenfeld, A. and A.C. Kak (1976). *Digital Picture Processing.* Academic Press, New York.