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Separable Jordan Algebras over Commutative Rings, III

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We continue the study undertaken in [2] and [5] of separable Jordan algebras over commutative rings. In Section 1 we prove that a central separable Jordan algebra $J = J(p, q)$ has a generic minimum polynomial. This yields the generic trace and norm maps of J, and we show in Section 2 that these maps satisfy the standard properties. We use these maps in Section 3 together with the construction of Freudenthal–Springer–McCrimmon [13] to complete the determination begun in [5] of all separable Jordan algebras over commutative rings. If $J_i = J_i(p, q)$ is a central separable R-algebra, $i \in \{1, 2\}$, we prove in Section 4 that an R-module isomorphism ϕ of J_1 and J_2 is an isotopy if and only if $\phi \otimes 1$ is a norm similarity of $J_1 \otimes_R S$ and $J_2 \otimes_R S$ for every commutative associative R-algebra S. Special cases of Wedderburn decomposition and Malcev uniqueness theorems are established in Section 5.

Throughout this paper we assume that all commutative rings contain $\frac{1}{2}$ and that all algebras are unital. J is always a Jordan algebra over a commutative ring R. J is called *separable* if its unital universal multiplication envelope is a separable associative R-algebra. J is called *central* if the map $\alpha \to \alpha 1$ from R to the center of J is bijective.

References of the form $[Si]$, i an integer, refer to the list in [5] of basic properties of separable Jordan algebras over commutative rings. In particular, we recall that a separable R-algebra J is the direct sum of homogeneous components $J(p, q)$ $[S7, S8]$.

1. GENERIC MINIMUM POLYNOMIALS

Generic minimum polynomials of finite-dimensional Jordan algebras over fields are studied in Chapter VI of [9]. In this section we prove the existence of a generic minimum polynomial for a central separable R-algebra

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$J = J(p, q)$. We reduce to the case where (R, m) is complete local Noetherian and J/mJ is reduced, then we apply the classification of such algebras [2, p. 132].

Faithfully flat descent is used by Knus and Ojanguren in [11, pp. 108–110] to prove the existence of generic minimum polynomials for separable associative algebras over commutative rings. One could give a proof that avoids faithfully flat descent by applying the techniques of this section to associative algebras.

Let Q be the rational numbers, and let Z_2 be the subring of Q generated by $\frac{1}{2}$.

LEMMA 1.1. *Let J be central separable over a complete local Noetherian ring (R, m). Then there is a commutative associative R-algebra S and a Jordan Z_2-algebra J' such that*

(i) *S is a free R-module of finite rank and (S, mS) is complete local Noetherian,*

(ii) *J' is a central separable Z_2-algebra and a free Z_2-module of finite rank, and*

(iii) *$J \otimes_R S \cong J' \otimes_{Z_2} S$ as S-algebras.*

Proof. There is a commutative associative R-algebra S such that $(J \otimes_R S)/mS(J \otimes_R S)$ is reduced over S/mS and (i) is satisfied [2, p. 134]. $J \otimes_R S$ is central separable over S [S6], so $J \otimes_R S$ has one of the following forms [2, p. 132]:

(a) $J \otimes_R S \cong S$,

(b) $J \otimes_R S$ has a free basis $\{1, v_1, \ldots, v_n\}$, $n \geqslant 2$, over S such that $v_1^2 = 1$, v_i^2 is a unit in S for $i > 1$, and $v_i \cdot v_j = 0$ for $i \neq j$,

(c) $J \otimes_R S \cong H(M_n(D), j_\gamma)$, $n \geqslant 3$, where D is a composition algebra over S which is associative if $n \geqslant 4$ (notation as in [2, pp. 131–132]).

In case (a), take $J' = Z_2$. In case (b), we can enlarge S so that S/mS contains square roots of the images in S/mS of the v_i^2, while condition (i) is preserved [2, p. 130]. Then the proof of [2, p. 132] shows that $J \otimes_R S$ has a free basis $\{1, w_1, \ldots, w_n\}$ such that $w_i^2 = 1$ and $w_i \cdot w_j = 0$ for $i \neq j$. It suffices to take $J' = Z_2 1 + \sum Z_2 w_i \subset J \otimes_R S$, since [2, p. 119] implies that J' is central separable over Z_2. In case (c), let D be built by a doubling operation which adjoins elements $q_i \in D$ such that $q_i^2 = \mu_i$ is a unit of S [2, p. 131]. Enlarge S so that S/mS contains square roots of the images of all γ_i and μ_i, while condition (i) is preserved [2, p. 130]. The proof of Lemma 3.2 of [4] shows that the γ_i and μ_i have square roots in S. It follows that $J \otimes_R S \cong H(M_n(D), j_1)$ [9, p. 60] and that D is built by a doubling process which adjoins elements $\mu_i^{-1/2} q_i$ whose squares are 1. Let D' be the composition algebra over Z_2 whose Z_2-rank equals the S-rank of D and which is built by a doubling process that adjoins elements whose squares are 1. It suffices to let J' be the Z_2-algebra $H(M_n(D'), j_1)$, since [2, p. 118] implies that J' is central separable over Z_2. ▮

A finite-dimensional Jordan algebra over a field is said to have degree p if the generic minimum polynomials of its elements have degree p [9, p. 223].

LEMMA 1.2. *If $J = J(p, q)$ is central separable over field R, then J has degree p.*

Proof. Let F be the algebraic closure of R. $J \otimes_R F$ is a central separable F-algebra and $J \otimes_R F = (J \otimes_R F)(p, q)$ [S6, S8]. Since the center of $J \otimes_R F$ is a field, $J \otimes_R F$ is finite-dimensional central simple over F [2, p. 118]. It follows from [S7] and [9, p. 233] that $J \otimes_R F$ has degree p over F, so J has degree p over R. ▮

Let $R[\eta]$ be the polynomial ring over R in indeterminates η_i indexed by the positive integers. Let $R[\eta_1, ..., \eta_s]$ be the R-subalgebra of $R[\eta]$ generated by $\eta_1, ..., \eta_s$. If R is a field, let $R(\eta)$ be the quotient field of $R[\eta]$.

Let λ be an indeterminate independent of the η_i, and let $f(\lambda) \in R[\eta][\lambda]$ be a polynomial in λ with coefficients in $R[\eta]$. We call $f(\lambda)$ monic of degree p if λ^p is the highest power of λ appearing in $f(\lambda)$ and the coefficient of λ^p in $R[\eta]$ is 1. We call $f(\lambda)$ homogeneous of degree p if it is homogeneous of total degree p in λ and the η_i. If $x \in J \otimes_R R[\eta]$, let $f(x)$ be the element of $J \otimes_R R[\eta]$ obtained by substituting x for λ in $f(\lambda)$, considering $J \otimes_R R[\eta]$ as an $R[\eta]$-algebra. We call x a root of $f(\lambda)$ if $f(x) = 0$.

LEMMA 1.3. *Let $J = J(p, q)$ be central separable over R. If $x_1, ..., x_s \in J$, set $x = \sum \eta_i x_i \in J \otimes_R R[\eta]$. Then there is $m_x(\lambda) \in R[\eta][\lambda]$ such that $m_x(\lambda)$ is monic and homogeneous of degree p and has x as a root.*

Proof. Case 1: There is a central separable Z_2-algebra J' such that J' is a free Z_2-module of finite rank and $J' \otimes_{Z_2} R \cong J$. Since $J = J(p, q)$, [S8] implies that $J' = J'(p, q)$ and $J' \otimes_{Z_2} Q = (J' \otimes_{Z_2} Q)(p, q)$. Since $J' \otimes_{Z_2} Q$ is central separable over Q [S6], Lemma 1.2 shows that $J' \otimes_{Z_2} Q$ has degree p over Q.

Let $y_1, ..., y_t$ be a basis for J' as a free Z_2-module. Set $y = \sum \eta_j y_j \in J' \otimes_{Z_2} Q(\eta)$. Since $J' \otimes_{Z_2} Q(\eta)$ is finite dimensional over $Q(\eta)$, let $m_y(\lambda)$ be the minimum polynomial of y over $Q(\eta)$. Since the y_i form a basis for $J' \otimes_{Z_2} Q$ over Q, y is a generic element of $J' \otimes_{Z_2} Q$. Hence the preceding paragraph and [9, p. 222] imply that $m_y(\lambda) \in Q[\eta][\lambda]$ and that $m_y(\lambda)$ is monic and homogeneous of degree p.

We write $Z_2[\eta_1, ..., \eta_t]$ as S. Consider $y \in J' \otimes_{Z_2} S$, and let $S[y]$ be the S-subalgebra of $J' \otimes_{Z_2} S$ generated by y. S is Noetherian [1, pp. 80–81] and

$J' \otimes_{Z_2} S$ is finitely spanned over S, so $J' \otimes_{Z_2} S$ is a Noetherian S-module [1, p. 76]. Thus $S[y]$ is finitely spanned over S, so y is a root of a monic polynomial with coefficients in S [1, pp. 59 and 75]. Hence $m_y(\lambda) \in Z_2[\eta][\lambda]$, by Gauss's lemma [12, pp. 127–128].

Let $x_i = \sum_j a_{ij} y_j$, $a_{ij} \in R$, where we identify $J' \otimes_{Z_2} R \cong J$. Then $x = \sum \eta_i x_i = \sum \eta_i a_{ij} y_j$. Let $m_x(\lambda) \in R[\eta][\lambda]$ be obtained from $m_y(\lambda) \in Z_2[\eta][\lambda]$ by substituting $\sum_i \eta_i a_{ij}$ for each η_j. x is a root of $m_x(\lambda)$, since y is a root of $m_y(\lambda)$. Since $m_y(\lambda)$ is monic and homogeneous of degree p, so is $m_x(\lambda)$.

Case 2: (R, m) complete local Noetherian. By Lemma 1.1, [S6], and [S8], there is a commutative associative R-algebra S which is a free R-module of finite rank such that $J \otimes_R S$ satisfies the hypotheses of Case 1 as an S-algebra. Consider $x_i \otimes 1 \in J \otimes_R S$ and $x \otimes 1 \in (J \otimes_R R[\eta]) \otimes_R S$. If we identify

$$(J \otimes_R R[\eta]) \otimes_R S \cong (J \otimes_R S) \otimes_S S[\eta],$$

then $x \otimes 1 = \sum \eta_i(x_i \otimes 1)$, so Case 1 implies that $x \otimes 1$ is a root of a polynomial $m_{x \otimes 1}(\lambda) \in S[\eta][\lambda]$ which is monic and homogeneous of degree p. Since S is a free R-module of finite rank, it contains R as a direct summand [6, p. 24]. Let $m_x(\lambda) \in R[\eta][\lambda]$ be obtained by applying the projection of S onto R to each coefficient of $m_{x \otimes 1}(\lambda)$. Since $m_{x \otimes 1}(\lambda)$ is monic and homogeneous of degree p, so is $m_x(\lambda)$. x is a root of $m_x(\lambda)$, since $x \otimes 1$ is a root of $m_{x \otimes 1}(\lambda)$.

Case 3: R Noetherian. Let S_k be the R-submodule of $R[\eta]$ spanned by all monomials of total degree k in the η_i. Let

$$N = \sum_{0 \leqslant k < p} S_{p-k} x^k \subset J \otimes_R R[\eta],$$

where $x^0 = 1 \in J \otimes_R R[\eta]$. Let $M = Rx^p + N \subset J \otimes_R R[\eta]$. Let m be a maximal ideal of R, and let R^* be the completion of R in the m-topology. $J \otimes_R R^*$ is central separable over R^*, and $J \otimes_R R^* = (J \otimes_R R^*)(p, q)$ [S6, S8]. Consider $x_i \otimes 1 \in J \otimes_R R^*$ and $x \otimes 1 \in (J \otimes_R R[\eta]) \otimes_R R^*$. If we identify

$$(J \otimes_R R[\eta]) \otimes_R R^* \cong (J \otimes_R R^*) \otimes_{R^*} R^*[\eta],$$

then $x \otimes 1 = \sum \eta_i(x_i \otimes 1)$, so Case 2 implies that $x \otimes 1$ is a root of a polynomial $m_{x \otimes 1}(\lambda) \in R^*[\eta][\lambda]$ monic and homogeneous of degree p. It follows that $M \otimes_R R^* = N \otimes_R R^*$, since we can identify $M \otimes_R R^*$ and $N \otimes_R R^*$ with their images in $(J \otimes_R R[\eta]) \otimes_R R^*$ because R^* is flat over R [1, p. 109]. Thus $(M/N) \otimes_R R^* = 0$. Since this holds for every maximal ideal m of R, $M/N = 0$ [1, pp. 40, 108, 110]. Hence x is a root of a polynomial $m_x(\lambda) \in R[\eta][\lambda]$ monic and homogeneous of degree p.

Case 4: General case. There is a Noetherian subring R' of R and a central separable R'-subalgebra J' of J such that $J \cong J' \otimes_{R'} R$ [2, p. 134]. [$S8$] implies that $J' = J'(p, q)$. R' remains Noetherian when finitely many elements of R are adjoined to it [1, p. 81], so we can assume that $x_1, ..., x_s \in J$. The lemma follows by applying Case 3 to J' as an R'-algebra. ∎

We recall that a central separable R-algebra is finitely spanned over R [$S3$].

THEOREM 1.4. *Let $J = J(p, q)$ be central separable over R. Let $x_1, ..., x_s$ span J over R, and set $x = \sum \eta_i x_i \in J \otimes_R R[\eta]$. Then there is a unique polynomial $m_x(\lambda) \in R[\eta][\lambda]$ such that x is a root of $m_x(\lambda)$ and $m_x(\lambda)$ is monic of degree p. $m_x(\lambda)$ is homogeneous of degree p.*

Proof. By Lemma 1.3, it suffices to prove that the powers x^j, $0 \leqslant j < p$, are linearly independent over $R[\eta]$. Let $g_j(\eta) \in R[\eta]$, $0 \leqslant j < p$, be such that $\sum g_j(\eta) x^j = 0$.

Case 1: R is a field. Assume that $g_k(\eta) \neq 0$ for some k. Renumber the x_i so that $x_1, ..., x_t$ form a basis of J over R, $t \leqslant s$. By field extension, we can assume that R is infinite. Then there are $\alpha_i \in R$ such that $\phi g_k(\eta) \neq 0$, where ϕ is the R-algebra endomorphism of $R[\eta]$ fixing $\eta_1, ..., \eta_t$ and taking η_i to α_i for $i > t$. $1 \otimes \phi$ is an R-algebra endomorphism of $J \otimes_R R[\eta]$ such that

$$(1 \otimes \phi) x = \eta_1 x_1 + \cdots + \eta_t x_t + a,$$

$a \in J$. Write $a = \beta_1 x_1 + \cdots + \beta_t x_t$, $\beta_i \in R$, so

$$(1 \otimes \phi) x = (\eta_1 + \beta_1) x_1 + \cdots + (\eta_t + \beta_t) x_t.$$

Let ψ be the R-algebra automorphism of $R[\eta]$ taking η_i to $\eta_i - \beta_i$ for $i \leqslant t$ and fixing η_i for $i > t$. $1 \otimes \psi$ is an R-algebra automorphism of $J \otimes_R R[\eta]$ such that

$$(1 \otimes \psi)(1 \otimes \phi) x = \eta_1 x_1 + \cdots + \eta_t x_t.$$

If we set $y = \eta_1 x_1 + \cdots + \eta_t x_t$ and $h_j(\eta) = \psi \phi g_j(\eta)$, applying $(1 \otimes \psi)(1 \otimes \phi)$ to $\sum g_j(\eta) x^j = 0$ yields $\sum h_j(\eta) y^j = 0$, $0 \leqslant j < p$. Each $h_j(\eta) \in R[\eta_1, ..., \eta_t]$, and $h_k(\eta) \neq 0$. Since y is a generic element of J, this contradicts the fact that J has degree p (Lemma 1.2).

Case 2: (R, m) local. For any positive integer v, let M_v be the set of all monic monomials in $R[\eta_1, ..., \eta_v]$ of total degree $\leqslant v$. Let N_v be the R-submodule of $J \otimes_R R[\eta]$ spanned by the elements of $M_v J$. There is an integer w such that N_w contains every $g_j(\eta) x^j$. Let

$$T = N_w \cap \{f(\eta) x^j \mid f(\eta) \in M_w, 0 \leqslant j < p\}.$$

Case 1 applies to J/mJ as an R/m-algebra [$S6$, $S8$], so the elements of T have images in $(J/mJ) \otimes_{R/m} (R/m)[\eta]$ linearly independent over R/m. The image of N_w in $(J/mJ) \otimes_{R/m} (R/m)[\eta]$ is isomorphic to N_w/mN_w, since N_w is a direct summand of $J \otimes_R R[\eta]$ over R. Thus the elements of T have linearly independent images in N_w/mN_w. J is finitely spanned and projective over R [$S3$], hence so is N_w. It follows that the elements of T are linearly independent over R [6, p. 24]. Thus the relation $\sum g_j(\eta) x^j = 0$ implies that each $g_j(\eta) = 0$.

Case 3: General case. Let R_m be the localization of R at a maximal ideal m. Case 2 applies to $J \otimes_R R_m$ as an R_m-algebra [$S6$, $S8$], so the images of the $g_j(\eta)$ in $R_m[\eta]$ are zero. Since this holds for every maximal ideal m of R, each $g_j(\eta) = 0$ [1, p. 40]. ∎

Let $m_x(\lambda)$ be as in Theorem 1.4, and let $m'_x(\lambda)$ be obtained from $m_x(\lambda)$ by substituting 0 for every η_i with $i > s$. x is a root of $m'_x(\lambda)$ and $m'_x(\lambda)$ is monic of degree p, so Theorem 1.4 implies that $m'_x(\lambda) = m_x(\lambda)$. Thus $m_x(\lambda)$ contains no η_i with $i > s$.

THEOREM 1.5. *Let $J = J(p, q)$ be central separable over R. Let $x_1, \ldots, x_s$ span J over R, and let $x = \sum \eta_i x_i \in J \otimes_R R[\eta]$. Let $m_x(\lambda) \in R[\eta][\lambda]$ be as in Theorem 1.4. Let $a \in J$, and write $a = \sum r_i x_i$, $r_i \in R$. Let $m_a(\lambda) \in R[\lambda]$ be obtained by substituting r_i for each η_i in $m_x(\lambda)$. Then $m_a(\lambda)$ depends only on a, J, and R, and not on the choice of the x_i or r_i.*

Proof. It suffices to prove the theorem when R is localized at any maximal ideal m [1, p. 40], so we can assume that (R, m) is local. J is a free R-module of finite rank, since it is finitely spanned and projective over the local ring (R, m) [$S3$], [6, p. 24]. Let $\{z_j\}$ be a finite free basis of J over R, and set $z = \sum \eta_j z_j \in J \otimes_R R[\eta]$. Let $a = \sum c_j z_j$ and $x_i = \sum_j d_{ij} z_j$, c_j, $d_{ij} \in R$. Then

$$\sum c_j z_j = a = \sum r_i x_i = \sum r_i d_{ij} z_j,$$

so

$$c_j = \sum_i r_i d_{ij}, \tag{1}$$

since the z_j are linearly independent over R.

Let $m_z(\lambda) \in R[\eta][\lambda]$ be as in Theorem 1.4, and let $m'(\lambda)$ be the result of substituting $\sum_i \eta_i d_{ij}$ for each η_j in $m_z(\lambda)$. $m'(\lambda)$ is monic of degree p. $z = \sum \eta_j z_j$ is a root of $m_z(\lambda)$, and

$$x = \sum \eta_i x_i = \sum \eta_i d_{ij} z_j,$$

so x is a root of $m'(\lambda)$. Hence $m_x(\lambda) = m'(\lambda)$ (Theorem 1.4). Thus substituting r_i for each η_i in $m_x(\lambda)$ gives the same result as substituting $\sum_i r_i d_{ij}$ for each η_j in $m_z(\lambda)$. Hence (1) shows that substituting c_j for each η_j in $m_z(\lambda)$ gives $m_a(\lambda)$. Holding the z_j and c_j fixed as the x_i and r_i vary shows that $m_a(\lambda)$ does not depend on the choice of the x_i and r_i. ∎

In the notation of Theorem 1.5, since $m_x(\lambda) \in R[\eta][\lambda]$ is monic of degree p and has x as a root, $m_a(\lambda) \in R[\lambda]$ is monic of degree p and has a as a root. We call $m_a(\lambda)$ the *generic minimum polynomial of* a. If R is a field, $m_a(\lambda)$ is the generic minimum polynomial of a in the sense of [9, p. 223].

We consider examples of generic minimum polynomials. Let A be a central separable associative R-algebra having rank p^2 as an R-module [$M3$]. A^+ is a central separable Jordan R-algebra [5, Corollary 3.5]. Let $x \in A^+ \otimes_R R[\eta]$ and $m_x(\lambda) \in R[\eta][\lambda]$ be as in Theorem 1.4. If m is a maximal ideal of R and F is the algebraic closure of R/m, then $A/mA \otimes_{R/m} F$ is isomorphic to the algebra of p–by–p matrices over F [11, p. 93]. It follows that $A^+ = A^+(1,1)$ if $p=1$, $A^+ = A^+(p,3)$ if $p=2$, and $A^+ = A^+(p,2)$ if $p \geqslant 3$ [2, p. 137]. Hence $m_x(\lambda)$ has degree p. $A \otimes_R R[\eta]$ is a central separable associative $R[\eta]$-algebra of rank p^2 [6, pp. 27, 43]. Let $f(\lambda) \in R[\eta][\lambda]$ be the characteristic polynomial of x as an element of $A \otimes_R R[\eta]$ in the sense of Knus and Ojanguren [11, p. 108]. $f(\lambda)$ is a monic polynomial of degree p having x as a root, so $f(\lambda) = m_x(\lambda)$ (Theorem 1.4). Specializing the η_i in R shows that the characteristic polynomial of any $a \in A$ in the sense of Knus and Ojanguren equals the generic minimum polynomial of the image of a in A^+.

Let $Q(a,b)$ be a nondegenerate symmetric bilinear form on a finitely spanned, projective R-module M of rank $q \geqslant 2$ [$M3$, $M5$]. Let $J = R \oplus M$ be the Jordon algebra determined by Q. J has multiplication.

$$(\alpha, a) \cdot (\beta, b) = (\alpha\beta + Q(a,b), \alpha b + \beta a),$$

$\alpha, \beta \in R$, $a, b \in M$. J is central separable over R, and $J = J(2,q)$ [2, p. 140]. Let Q' be the bilinear form induced by Q on $M \otimes_R R[\eta]$ over $R[\eta]$. We identify $J \otimes_R R[\eta]$ with the Jordan algebra $R[\eta] \oplus (M \otimes_R R[\eta])$ determined by Q'. Let $x \in J \otimes_R R[\eta]$ be as in Theorem 1.4, and write $x = (\alpha', a')$, $\alpha' \in R[\eta]$, $a' \in M \otimes_R R[\eta]$. (α', a') is a root of $\lambda^2 - 2\alpha'\lambda + (\alpha'^2 - Q'(a',a'))$, so this polynomial equals $m_x(\lambda)$ (Theorem 1.4). Specializing the η_i in R shows that the generic minimum polynomial of any $(\alpha, a) \in R \oplus M$ is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - Q(a,a))$.

If S and T are commutative associative R-algebras and $\phi : S \to T$ is an R-algebra homomorphism, let $\phi_\lambda : S[\lambda] \to T[\lambda]$ be the homomorphism of $R[\lambda]$-algebras that agrees with ϕ on $S \subset S[\lambda]$. Define $\phi_{\eta\lambda} : S[\eta][\lambda] \to T[\eta][\lambda]$ analogously.

PROPOSITION 1.6. *Let $J = J(p, q)$ be central separable over R.*

(i) *Let S be a commutative associative R-algebra, and let $\phi : R \to S$ take $\alpha \in R$ to $\alpha 1$, $1 \in S$. If $a \in J$, then $m_{a \otimes 1}(\lambda) = \phi_\lambda m_a(\lambda)$ for $a \otimes 1 \in J \otimes_R S$.*

(ii) *Let m be a maximal ideal of R, and let $p : R \to R/m$ be the projection map. If $a \in J$ and a' is the image of a in J/mJ, then $m_{a'}(\lambda) = p_\lambda m_a(\lambda)$. $m_{a'}(\lambda)$ is the generic minimum polynomial of a' over R/m in the sense of* [9, p. 223].

(iii) *If f is homomorphism of J onto an R-algebra J', then $m_{f(a)}(\lambda) = m_a(\lambda)$ for $a \in J$.*

(iv) *Let S be a commutative associative R-algebra, and let $a \in J \otimes_R R[\eta]$. Let $a' \in J \otimes_R S$ be obtained from a by replacing each η_i with $s_i \in S$. Then $m_{a'}(\lambda) \in S[\lambda]$ is obtained from $m_a(\lambda) \in R[\eta][\lambda]$ by replacing each η_i with s_i.*

Proof. (i) $J \otimes_R S$ is central separable over S and equals $(J \otimes_R S)(p, q)$ [$S6$, $S8$]. Let x_i, x, and $m_x(\lambda)$ be as in Theorem 1.4. Consider $x_i \otimes 1 \in J \otimes_R S$ and $x \otimes 1 \in (J \otimes_R R[\eta]) \otimes_R S$. The $x_i \otimes 1$ span $J \otimes_R S$ over S, and $x \otimes 1 = \sum \eta_i(x_i \otimes 1)$ under the identification

$$(J \otimes_R R[\eta]) \otimes_R S \cong (J \otimes_R S) \otimes_S S[\eta].$$

$\phi_{\eta\lambda} m_x(\lambda) \in S[\eta][\lambda]$ is a monic polynomial of degree p having $x \otimes 1$ as a root, so

$$m_{x \otimes 1}(\lambda) = \phi_{\eta\lambda} m_x(\lambda) \tag{2}$$

(Theorem 1.4). If $a = \sum r_i x_i$ for $r_i \in R$, then $a \otimes 1 = \sum r_i(x_i \otimes 1)$ and specializing each η_i to r_i in (2) yields (i) (Theorem 1.5). (ii) follows by taking $S = R/m$ in (i).

(iii) J' is central separable over R and equals $J'(p, q)$ [$S6$, $S8$]. Let x_i, x, and $m_x(\lambda)$ be defined for J as in Theorem 1.4. Consider

$$f \otimes 1 : J \otimes_R R[\eta] \to J' \otimes_R R[\eta].$$

$(f \otimes 1)\, x = \sum \eta_i f(x_i)$, where the $f(x_i)$ span J' over R. Since $f \otimes 1$ is a homomorphism of $R[\eta]$-algebras and x is a root of $m_x(\lambda)$, $(f \otimes 1)\, x$ is also a root of $m_x(\lambda)$. Since $m_x(\lambda)$ is monic of degree p, Theorem 1.4 implies that

$$m_{(f \otimes 1)x}(\lambda) = m_x(\lambda). \tag{3}$$

If $a = \sum r_i x_i$ for $r_i \in R$, substituting r_i for each η_i in (3) yields (iii).

(iv) Since S is an R-algebra, we can make S an $R[\eta]$-algebra such that each η_i acts on S as multiplication by s_i. Appplying (i) with R replaced by $R[\eta]$ and J replaced by $J \otimes_R R[\eta]$ yields

$$m_{a\otimes 1}(\lambda) = \phi_\lambda m_a(\lambda), \tag{4}$$

where $a \otimes 1 \in (J \otimes_R R[\eta]) \otimes_{R[\eta]} S$ and $\phi : R[\eta] \to S$ takes $\alpha \in R[\eta]$ to $\alpha 1$, $1 \in S$. The isomorphism

$$(J \otimes_R R[\eta]) \otimes_{R[\eta]} S \cong J \otimes_R S$$

takes $a \otimes 1$ to a'. Then (iv) follows from (4), since $\phi_\lambda m_a(\lambda) \in S[\lambda]$ is obtained from $m_a(\lambda)$ by replacing each η_i with s_i. ∎

2. Generic Traces and Norms

Throughout this section we assume that $J = J(p, q)$ *is central separable over* R. If $a \in J$, let $T(a)$ be the negative of the coefficient of λ^{p-1} in $m_a(\lambda)$. Let $N(a)$ be $(-1)^p$ times the constant coefficient of $m_a(\lambda)$. T and N are functions from J to R called the *generic trace* and *generic norm* of J. We occasionally write them as T_J and N_J. We use the following lemma in Proposition 2.2 to prove that T and N have the same basic properties in general as they do when R is a field.

LEMMA 2.1. *Let $R[\xi]$ be a polynomial ring in any number of indeterminates ξ_k. Let $F = F(r_i, a_j, R, J, \xi_k, p, q)$ be an element of R, J, $R[\xi]$, or $J \otimes_R R[\xi]$ formed using ring and algebra operations on* (i) $r_i \in R$, (ii) $a_j \in J$ *and the images* $a_j \otimes 1 \in J \otimes_R R[\xi]$, (iii) ξ_k, (iv) *coefficients of generic minimum polynomials of elements of J formed from the r_i and a_j using operations of J, and* (v) *coefficients of generic minimum polynomials over $R[\xi]$ of elements of $J \otimes_R R[\xi]$ formed from the r_i, ξ_k, and $a_j \otimes 1$ using operations of $J \otimes_R R[\xi]$. Assume that the method of forming F is independent of the choice of R, J, the r_i, and the a_j. Hold p, q, and the ξ_k fixed as R, J, the r_i, and the a_j vary. If $F = 0$ whenever R is a field, then $F = 0$ when R is arbitrary.*

Proof. Any change of rings preserves the standing hypothesis that $J = J(p, q)$ is central separable over R. Coefficients of generic minimum polynomials are preserved under any change of rings (Proposition 1.6). Thus any change of rings preserves F.

Let R be arbitrary, and fix $r_i \in R$ and $a_j \in J$. There is a Noetherian subring R' of R and a central separable R'-subalgebra J' of J such that $J' \otimes_{R'} R$ is isomorphic to J [2, p. 134]. R' remains Noetherian when finitely many elements of R are adjoined to it [1, p. 81], so we can ensure that each

$r_i \in R'$ and $a_j \in J'$. Thus we can assume that R is Noetherian. It is enough to prove that $F=0$ under localization at any maximal ideal m of R [1, p. 40], so we can assume that (R, m) is Noetherian local. It suffices to show that $F=0$ after tensoring with the completion of R at m [1, pp. 108–110], so we can assume that (R, m) is complete local Noetherian. By Lemma 1.1, there are a central separable Z_2-algebra J' and a commutative associative R-algebra S such that J' is a free Z_2-module of finite rank, S is a free R-module of finite rank, and $J' \otimes_{Z_2} S \cong J \otimes_R S$.

Let $\{d_k\}$ be a basis of J' as a free Z_2-module, and let η_i and ζ_{ju} be indeterminates. By hypothesis, $F=0$ holds for $J' \otimes_{Z_2} Q(\eta, \zeta)$ over $Q(\eta, \zeta)$. Thus Proposition 1.6(i) implies that $F=0$ holds for $J' \otimes_{Z_2} Z_2[\eta, \zeta]$ over $Z_2[\eta, \zeta]$, since J' is a free Z_2-module. In particular, $F=0$ when we take each r_i to be $\eta_i \in Z_2[\eta, \zeta]$ and each a_j to be $\sum_u \zeta_{ju} d_u \in J \otimes_{Z_2} Z_2[\eta, \zeta]$. Specializing the η_i and ζ_{ju} to arbitrary elements of S shows that $F=0$ for $J' \otimes_{Z_2} S$ over S (Proposition 1.6(iv)). Since $J' \otimes_{Z_2} S \cong J \otimes_R S$ and S is a free R-module, it follows that $F=0$ holds for J over R. ∎

PROPOSITION 2.2. *If $a, b, c \in J$ and $r, r_i, r_j \in R$, then:*

(i) $T \in \operatorname{Hom}_R(J, R)$ *and* $N(ra) = r^p N(a)$.

(ii) $m_1(\lambda) = (\lambda - 1)^p$, *so* $T(1) = p$ *and* $N(1) = 1$.

(iii) $T((a \cdot b) \cdot c) = T(a \cdot (b \cdot c))$.

(iv) $N(U_a b) = N(a)^2 N(b)$.

(v) $N(\lambda 1 - a) = m_a(\lambda)$, *where N is the generic norm of $J \otimes_R R[\lambda]$ over $R[\lambda]$.*

(vi) $N[(\sum r_i a^i) \cdot (\sum r_j a^j)] = N(\sum r_i a^i) N(\sum r_j a^j)$. *Thus* $N(b \cdot c) = N(b) N(c)$ *if b and c belong to the subalgebra of J generated by a single element* (*and* 1).

(vii) *If $a \in J$ is a root of $f(\lambda) \in R[\lambda]$, then $f(\lambda)^p$ is divisible by $m_a(\lambda)$ in $R[\lambda]$.*

Proof. (i)–(vi) follow from Lemma 2.1 and [9, pp. 224, 227, and 235]. (vii) follows from (i), (ii), (v) and (vi), by the proof in [9, pp. 225–226]. ∎

A symmetric bilinear form $Q(a, b)$ on a finitely spanned projective R-module M is called *nondegenerate* if it induces a nondegenerate bilinear form on M/mM for every maximal ideal m of R. If Q is nondegenerate, $M \otimes_R M$ is isomorphic to $\operatorname{Hom}_R(M, M)$ via $a \otimes b \to Q_{ab}$, where $Q_{ab}(x) = Q(a, x)\, b$ for $x \in M$ [2, p. 114]. Since $\operatorname{Hom}_R(M, M)$ contains the identity map on M, there are $a_i, b_i \in M$ such that $x = \sum Q(a_i, x)\, b_i$ for all $x \in M$. Thus, if $x \in M$ is such that $Q(x, y) = 0$ for all $y \in M$, then $x = 0$.

Set $T(a, b) = T(a \cdot b)$ for $a, b \in J$. We show in Lemma 4.2 that $T(a, b)$ equals the logarithmic derivative of the generic norm of J at 1.

PROPOSITION 2.3. *$T(a, b)$ is a nondegenerate symmetric bilinear form on J such that $T(a \cdot b, c) = T(a, b \cdot c)$ for all a, b, $c \in J$.*

Proof. Proposition 2.2 implies that $T(a, b)$ is a symmetric bilinear form such that $T(a \cdot b, c) = T(a, b \cdot c)$. For every maximal ideal m of R, $T(a, b)$ induces the classical generic trace form on J/mJ over R/m (Proposition 1.6(ii)). Since the latter is nondegenerate [9, p. 240], so is $T(a, b)$. ∎

If $a \in J$, we write $m_a(\lambda) = \lambda f(\lambda) + (-1)^p N(a) 1$, $f(\lambda) \in R[\lambda]$, and set $a^\# = (-1)^{p-1} f(a) \in J$. Since a is a root of $m_a(\lambda)$, $a \cdot a^\# = N(a) 1$.

PROPOSITION 2.4. *$a \in J$ is invertible if and only if $N(a)$ is a unit in R. If so, $a^{-1} = N(a)^{-1} a^\#$ and $N(a^{-1}) = N(a)^{-1}$.*

Proof. If a is invertible, $1 = N(1) = N(U_a a^{-2}) = N(a)^2 N(a^{-2})$ (Proposition 2.2), so $N(a)$ is a unit. Conversely, assume that $N(a)$ is a unit. The equation $a \cdot a^\# = N(a) 1$ implies that $a \cdot [N(a)^{-1} a^\#] = 1$ and $a^2 \cdot [N(a)^{-1} a^\#] = a$, since J is power-associative. Thus $a^{-1} = N(a)^{-1} a^\#$ [9, p. 52], and a^{-1} is in the subalgebra of J generated by a. Together with the relation $a \cdot a^{-1} = 1$, this implies that $N(a) \cdot N(a^{-1}) = 1$ (Proposition 2.2(ii) and (vii)), so $N(a^{-1}) = N(a)^{-1}$. ∎

Combining Proposition 2.4 with the proof of Proposition 1.4 of [3] yields the following result.

LEMMA 2.5. *Let $F(x_i) = 0$ be an identity which holds for all invertible $x_i \in J \otimes_R S$ for every commutative associative R-algebra S. Then $F(x_i) = 0$ holds for all $x_i \in J$.* ∎

PROPOSITION 2.6. *$N(a^\#) = N(a)^{p-1}$ and $a^{\#\#} = N(a)^{p-2} a$ for all $a \in J$.*

Proof. The standing hypothesis that $J = J(p, q)$ is central separable is preserved under a change of rings [S6, S8]. Thus, by Lemma 2.5, we can assume that a is invertible. Then

$$N(a^\#) = N(N(a) a^{-1}) = N(a)^p N(a^{-1}) = N(a)^{p-1}$$

(Propositions 2.2 and 2.4), and

$$\begin{aligned} a^{\#\#} &= N(a^\#)(a^\#)^{-1} = N(a)^{p-1} (N(a) a^{-1})^{-1} \\ &= N(a)^{p-2} a. \quad ∎ \end{aligned}$$

3. Determination of Separable Algebras

All special separable Jordan algebras over commutative rings were determined in Theorem 3.9 of [5]. In this section we combine the preceding results on generic traces and norms with the construction of Freudenthal–Springer–McCrimmon [13] to study central separable algebras $J = J(3, 8)$. This enables us to determine all separable Jordan algebras over commutative rings.

Let S be a commutative ring, let M be a finitely spanned projective S-module, and let c be a distinguished element of M. Let $(N, \partial N)$ be a cubic form on M, that is, $N : M \to S$ and $\partial N : M \times M \to S$ are such that, if $\alpha \in S$ and $x, y \in M$, $\partial_x N|_y \equiv \partial N(x, y)$ is linear in x and quadratic in y, $N(\alpha x) = \alpha^3 N(x)$, and $N(x+y) = N(x) + \partial_x N|_y + \partial_y N|_x + N(y)$. ∂N is uniquely determined by N, since one verifies that

$$\partial_x N|_y = \tfrac{1}{2}N(x+2y) - N(x+y) - 3N(y) + \tfrac{1}{2}N(x). \tag{5}$$

If $x, y \in M$, set

$$T(x, y) = (\partial_x N|_c)(\partial_y N|_c) - \partial_{x,y}(\partial_c N|), \tag{6}$$

where $\partial_{x,y}(\partial_c N|) \equiv \partial_c N|_{x+y} - \partial_c N|_x - \partial_c N|_y$. Set $T(x) = T(x, c)$, $x \in M$. Assume that $T(x, y)$ is nondegenerate, so $M \cong \mathrm{Hom}_S(M, S)$ via $z \to T(z, (\ ))$ [2, p. 114]. Thus, if $x \in M$, there is a unique element $x^{\#} \in M$ such that

$$T(x^{\#}, y) = \partial_y N|_x \tag{7}$$

for all $y \in M$. $x \to x^{\#}$ is a quadratic map from M to itself, and we set $x \times y = (x+y)^{\#} - x^{\#} - y^{\#}$, $x, y \in M$. Let $J(N, M, S, c)$ be the nonassociative S-algebra composed of the S-module M with multiplication

$$x \cdot y = \tfrac{1}{2}[x \times y + T(x)\,y + T(y)\,x - T(x \times y)\,c], \tag{8}$$

$x, y \in M$. We call N *admissible* if $T(x, y)$ is nondegenerate, $N(c) = 1$, and $x^{\#\#} = N(x)\,x$ for all $x \in J \otimes_S U$, where U is any commutative associative S-algebra and we take the unique extension of N to a cubic form on $J \otimes_S U$ over U. c is called the *basepoint* of N.

If Q is a symmetric bilinear form on a module M over a commutative ring S, let $J(Q, M, S)$ be the Jordan S-algebra composed of the S-module $S \oplus M$ with multiplication

$$(\alpha, a) \cdot (\beta, b) = (\alpha\beta + Q(a, b), \alpha b + \beta a),$$

$\alpha, \beta \in S$, $a, b \in M$.

Theorem 3.1. *J is a separable Jordan R-algebra if and only if*

$$J \cong H(A, j) \oplus J(Q, M_1, S_1) \oplus J(N, M_2, S_2, c), \tag{9}$$

where

(i) *A is a separable associative R-algebra with involution j,*

(ii) *S_1 is a commutative separable associative R-algebra, M_1 is a finitely spanned projective S_1-module, and Q is a nondegenerate symmetric bilinear form on M_1 over S_1, and*

(iii) *S_2 is a commutative separable associative R-algebra, M_2 is a finitely spanned projective S_2-module of rank 27, and N is an admissible cubic form on M_2 with basepoint c.*

Proof. First assume that J is a separable Jordan R-algebra. Let $B = \bigoplus J(p,q)$ for $(p,q) \neq (3,8)$, so $J = B \oplus J(3,8)$ [S7]. B and $J(3,8)$ are separable over R [S6]. B is special over R [5, Corollary 3.4], so $B \cong H(A,j) \oplus J(Q, M_1, S_1)$, where conditions (i) and (ii) are satisfied [5, Theorem 3.9]. Set $S_2 = Z[J(3,8)]$, $M_2 = J(3,8)$, and $c = 1 \in J(3,8)$. S_2 is a commutative separable associative R-algebra [5, Proposition 1.1], and M_2 is finitely spanned and projective of rank 27 over S_2 [S3, S7]. Let T and N be the generic trace and norm of $J(3,8)$ over S_2. Let $T(x,y) = T(x \cdot y)$, and define $x^{\#}$ as in Section 2. $N(c) = 1$, $T(x,y)$ is nondegenerate, and $x^{\#\#} = N(x)\, x$ for $x \in J$ (Propositions 2.2, 2.3, 2.6). Moreover, the relation $x^{\#\#} = N(x)\, x$ is preserved under a change of rings (Proposition 1.6(i)). Equations (6) and (7) hold when S_2 is a field [9, pp. 217, 243, 244], so (5) and Lemma 2.1 imply that they hold when S_2 is arbitrary. $J(3,8)$ satisfies (8) when S_2 is an algebraically closed field [S7], [13, p. 503]. By field extension, (8) holds whenever S_2 is a field. Thus Lemma 2.1 implies that (8) holds when S_2 is arbitrary. Hence $J(3,8) \cong J(N, M_2, S_2, c)$, and (iii) is satisfied.

Conversely, assume that J satisfies (9) and conditions (i)–(iii). $H[A,j] \oplus J(Q, M_1, S_1)$ is separable over R [5, Theorem 3.9]. If we prove that $J(N, M_2, S, c)$ is separable over R, it follows that J is separable over R [5, Theorem 3.2], as required. We write $J(N, M_2, S_2, c)$ as K. K is a Jordan algebra, by the proof of Theorem 5 of [13]. Since S_2 is a separable associative R-algebra, it suffices to prove that K is separable over S_2 [5, Theorem 3.1]. Since K is finitely spanned over S_2, it suffices to prove that K/mK is separable over S_2/m for every maximal ideal m of S_2 [S2]. Thus we only need to show that K is separable over S_2 when S_2 is a field. Reference [13, pp. 496–497] shows that $T(x,y) = T(x)\, T(y) - T(x \times y)$ and $T(c) = 3$. Then applying T to (8) yields $T(x \cdot y) = T(x,y)$. Since M_2 is 27-dimensional over S_2, it follows that $x \to T(x)$ is the generic trace map of K over S_2 [13, pp. 506–507]. Thus $T(x,y) = T(x \cdot y)$ is the generic trace form of K over S_2. Since $T(x,y)$ is nondegenerate, it follows that K is separable over S_2 [9, p. 240], as required. ∎

4. Isotopies and Strict Norm Similarities

Let $J = J(p, q)$ and $J' = J'(p, q)$ be central separable R-algebras. A *norm similarity* $\phi : J \to J'$ is an R-module isomorphism such that there is a unit $r \in R$ satisfying $N_{J'}(\phi x) = rN_J(x)$ for all $x \in J$. A norm similarity is called *norm-preserving* if $r = 1$. A *strict norm similarity* $\phi : J \to J'$ is an R-module isomorphism such that $\phi \otimes 1 : J \otimes_R S \to J' \otimes_R S$ is a norm similarity for every commutative associative R-algebra S. In this section we prove that an R-module isomorphism $\Phi : J \to J'$ is an isotopy if and only if it is a strict norm similarity. Our proof parallels the proof in [9, pp. 241–245] of the case where R is a field, except that we use generic elements to replace Zariski topology arguments which apply only when R is a field.

If a is an invertible element of J, let (J, a) be the a-isotope of J. If J is separable over R, so is (J, a) [2, p. 119]. Since isotopy is an equivalence relation [9, p. 58], it follows that J is central separable over R if and only if (J, a) is. If R is an algebraically closed field, J is isomorphic to (J, a) [9, p. 242]. It follows that if J is separable over any commutative ring R, then $J = J(p, q)$ if and only if $(J, a) = (J, a)(p, q)$.

Lemma 4.1. *If $J = J(p, q)$ and $J' = J'(p, q)$ are central separable R-algebras, an isotopy $\phi : J \to J'$ is a strict norm similarity.*

Proof. $\phi \otimes 1$ is an isotopy of $J \otimes_R S$ and $J' \otimes_R S$ for every commutative associative R-algebra S, so it suffices to prove that ϕ is a norm similarity. There is an invertible element $a \in J$ such that ϕ is an R-algebra isomorphism of (J, a) and J'. Since algebra isomorphisms are norm-preserving (Proposition 1.6(iii)), it suffices to prove that

$$N_{(J,a)}(x) = N_J(a)\, N_J(x), \qquad x \in J. \tag{10}$$

As in the proof of Lemma 2.1, we can assume that (R, m) is complete local Noetherian. Let $\psi : J \to J/mJ$ be the canonical map. There is a finite-dimensional extension field F of R/m such that $m_{\psi a}(\lambda)$ is a product of linear factors in $F[\lambda]$ and every root of $m_{\psi a}(\lambda)$ in F has a square root in F. There is a commutative associative R-algebra S such that S is a free R-module of finite rank, (S, mS) is complete local Noetherian, and $S/mS \cong F$ [2, p. 130]. It suffices to prove (10) with R replaced by S and J replaced by $J \otimes_R S$, so we can assume that $m_{\psi a}(\lambda)$ is a product of linear factors in $(R/m)[\lambda]$ and that every root of $m_{\psi a}(\lambda)$ has a square root in R/m. It follows that there is $c \in J$ such that $(\psi c)^2 = \psi a$, ψc is in the subalgebra of J/mJ generated by ψa, and ψc is invertible [9, p. 242]. We can assume that c is in the subalgebra of J generated by a. Since ψc is invertible, $N_{J/mJ}(\psi c)$ is nonzero, so $N(c)$ is a unit. Then c is invertible, and c^{-1} is in the subalgebra of J generated by a (Proposition 1.4). Since J is finitely spanned over R [S3], J is complete in the m-topology [1, p. 108]. Since $c^{-2} \cdot a \equiv 1 \pmod{mJ}$, there is $d \in J$ such that $d^2 = c^{-2} \cdot a$ and d is a formal power series in $1 - c^{-2} \cdot a$ [14, p. 944]. Since J is power-associative, $(c \cdot d)^2 = a$. Then $U_{c \cdot d}$ is a R-algebra isomorphism of (J, a) onto J [9, p. 242], so

$$N_{(J,a)}(x) = N_J(U_{c \cdot d}(x)) = N_J(c \cdot d)^2 N_J(x)$$
$$= N_J(a) N_J(x),$$

$x \in J$ (Proposition 2.2). ∎

Let $R[\xi]$ be the polynomial ring over R in indeterminates ξ_{ij}, where i varies over the positive integers and $j \in \{1, 2, 3\}$. For $s \in \{1, 2\}$, define $\Delta^s : R[\xi] \to R[\xi]$ by

$$\Delta^s f = \sum_i \xi_{is}(\partial f / \partial \xi_{i3}), \tag{11}$$

$f \in R[\xi]$, where $\partial f / \partial \xi_{i3}$ is the formal partial derivative of f with respect to ξ_{i3}. If R is a field, let $R(\xi)$ be the quotient field of $R[\xi]$, and define $\Delta^s : R(\xi) \to R(\xi)$ by (11).

LEMMA 4.2. *Let $J = J(p, q)$ be central separable over R. Let $\{x_i\}$ be a finite set of elements of J, and set $a = \sum \xi_{i1} x_i$, $b = \sum \xi_{i2} x_i$, and $c = \sum \xi_{i3} x_i$ in $J \otimes_R R[\xi]$. Then*

$$T(a, U_{c^{\#}}(b)) = -N(c) \Delta^2 \Delta^1 N(c) + (\Delta^1 N(c))(\Delta^2 N(c)), \tag{12}$$

where T and N are defined for $J \otimes R[\xi]$ over $R[\xi]$.

Proof. First assume that R is a field and that $\{x_i\}$ is a basis of J over R. If we specialize the ξ_{ij} in R so that c becomes $1 \in J$, $N_{J \otimes R[\xi]}(c)$ specializes to $N_J(1) = 1$ (Proposition 1.6(iv), Proposition 2.2). Then $N_{J \otimes R[\xi]}(c) \neq 0$, so $N_{J \otimes R(\xi)}(c) \neq 0$ (Proposition 1.6(i)), and c is invertible in $J \otimes R(\xi)$ (Proposition 2.4). Since $\{x_i\}$ is a basis of $J \otimes R(\xi)$ over $R(\xi)$, [9, pp. 217, 244] shows that

$$T(a, U_{c^{-1}}(b)) = N(c)^{-2} [-N(c) \Delta^2 \Delta^1 N(c) + (\Delta^1 N(c))(\Delta^2 N(c))]$$

in $J \otimes R(\xi)$ over $R(\xi)$. Multiplying this equation by $N(c)^2$ establishes (12) for $J \otimes R(\xi)$ over $R(\xi)$ (Proposition 2.4). Then Proposition 1.6(i) implies that (12) holds for $J \otimes R[\xi]$ over $R[\xi]$.

Next assume that R is a field and that $\{x_i\}$ is any finite set of elements of J. J is finite dimensional over R [S3]. Let $\{y_k\}$ be a basis of J over R, and set $a' = \sum \xi_{k1} y_k$, $b' = \sum \xi_{k2} y_k$, and $c' = \sum \xi_{k3} y_k$ in $J \otimes R[\xi]$. Let (12′) be the equation obtained from (12) by replacing a, b, and c with a', b', and c'.

The preceding paragraph shows that (12′) holds. Let $x_i = \sum_k r_{ik} y_k$, $r_{ik} \in R$. For $j \in \{1, 2, 3\}$,

$$\sum_i \xi_{ij} x_i = \sum_{i,k} \xi_{ij} r_{ik} y_k,$$

so a, b, and c are obtained from a', b', and c' by substituting $\sum_i \xi_{ij} r_{ik}$ for each ξ_{kj}. Let ϕ be the R-algebra endomorphism of $R[\xi]$ taking each ξ_{kj} to $\sum_i \xi_{ij} r_{ik}$ (where, if J is t-dimensional, we set $r_{ik} = 0$ for $k > t$). Both $\Delta^1 \phi$ and $\phi \Delta^1$ take each ξ_{k1} and ξ_{k2} to 0 and each ξ_{k3} ro $\sum_i \xi_{i1} r_{ik}$. Since Δ^1 is a derivation and ϕ is an R-algebra endomorphism of $R[\xi]$, it follows that $\Delta^1 \phi = \phi \Delta^1$. Similarly $\Delta^2 \phi = \phi \Delta^2$. Thus applying ϕ to (12′) establishes (12) (Proposition 1.6(iv)).

The general case follows from the case where R is a field, by Lemma 2.1. ∎

Equation (12) remains valid when the ξ_{ij} are specialized in R so that a and b become arbitrary elements of J, c becomes $1 \in J$, and T and N are defined for J over R (Proposition 1.6(iv)). Since $1^{\#} = 1$ (Propositions 2.2, 2.4), this shows that $T(a, b)$ equals the logarithmic derivative of the generic norm of J at 1, as in [9, pp. 217, 244].

LEMMA 4.3. *Let $J = J(p, q)$ and $J' = J'(p, q)$ be central separable R-algebras. If $\phi : J \to J'$ is a strict norm similarity such that $\phi 1 = 1'$, then ϕ is an isomorphism of R-algebras.*

Proof. Let $\{x_i\}$ be a finite set spanning J over R [S3]. Define a, b, $c \in J \otimes R[\xi]$ as in Lemma 4.2. Extend ϕ to a norm similarity of $J \otimes R[\xi]$ and $J' \otimes R[\xi]$ over $R[\xi]$. Since $N(\phi 1) = N(1') = N(1)$, ϕ is norm-preserving. Thus Lemma 4.2 implies that

$$T(\phi a, U_{(\phi c)^{\#}}(\phi b)) = T(a, U_{c^{\#}}(b)). \tag{13}$$

Specializing the ξ_{i3} so that c becomes 1 yields

$$T(\phi a, \phi b) = T(a, b), \tag{14}$$

since $1^{\#} = 1$ (Proposition 2.4). Specializing the ξ_{i2} in (14) so that b becomes $U_{c^{\#}}(b)$ and comparing the result with (13) yields

$$T(\phi a, U_{(\phi c)^{\#}}(\phi b)) = T(\phi a, \phi(U_{c^{\#}} b)).$$

The ξ_{i1} can be specialized so that ϕa becomes any element of $J' \otimes R[\xi]$. Since $T(x, y)$ is nondegenerate (Proposition 2.3), it follows that

$$U_{(\phi c)^{\#}}(\phi b) = \phi(U_{c^{\#}} b).$$

Specializing the ξ_{i2} so that b becomes c^2 yields

$$U_{(\phi c)^\#}\phi(c^2) = N(c)^2\, 1'. \tag{15}$$

Since $(\phi c)^\#$ is in the subalgebra of $J \otimes R[\xi]$ generated by ϕc,

$$U_{\phi c} U_{(\phi c)^\#} = U_{\phi c \cdot (\phi c)^\#} = N(\phi c)^2\, 1' = N(c)^2\, 1'$$

[9, p. 38]. Hence applying $U_{\phi c}$ to (15) yields

$$N(c)^2\, \phi(c^2) = N(c)^2\, (\phi c)^2. \tag{16}$$

The ξ_{i3} can be specialized so that c becomes 1, specializing $N(c)$ to 1 (Proposition 1.6(iv), Proposition 2.2(ii)). It follows that $N(c)$ annihilates no nonzero element of $J' \otimes R[\xi]$ [3, p. 419]. Thus (16) implies that $\phi(c^2) = (\phi c)^2$. Specializing the ξ_{i3} in R yields $\phi(x^2) = (\phi x)^2$ for all $x \in J$. Linearizing $x \to x, y$ and multiplying by $\frac{1}{2}$ yields $\phi(x \cdot y) = \phi x \cdot \phi y$ for all $x, y \in J$. ∎

THEOREM 4.4. *Let $J = J(p, q)$ and $J' = J'(p, q)$ be central separable R-algebras, and let ϕ be an R-module isomorphism of J onto J'. Then ϕ is a strict norm similarity if and only if ϕ is an isotopy.*

Proof. If ϕ is an isotopy, Lemma 4.1 shows that it is a strict norm similarity. Conversely, assume that ϕ is a strict norm similarity. If $a = \phi 1$, then $N(a)$ is a unit, so a is invertible (Proposition 2.4). By Lemma 4.1, the identity map on the R-module J' induces a strict norm similarity ψ of J' onto (J', a^{-1}). $\psi\phi$ is a strict norm similarity of J onto (J', a^{-1}) taking $1 \in J$ to the identity element a of (J', a^{-1}). Then $\psi\phi$ is an algebra isomorphism (Lemma 4.3), so ϕ is an isotopy. ∎

5. WEDDERBURN DECOMPOSITION

Let J be finitely spanned over R, and let N be a Penico-solvable ideal of J such that J/N is separable over R. The Wedderburn decomposition theorem states that there is a separable subalgebra S of J such that $J = S + N$. This was proved by Albert, Penico, and Taft when R is a field [9, p. 292]. In this section we prove the Wedderburn theorem when R is a commutative ring and one of the following additional conditions is satisfied: J/N is projective over R, R is complete local Noetherian, or R is a Dedekind domain. If $1/3 \in R$ and J has separable subalgebras S_i such that $J = S_i \oplus N$, $i \in \{1, 2\}$, there is an R-algebra automorphism T of J such that $T(S_1) = S_2$ and $T(N) = N$. This was proved by McCrimmon when R is a field [14, p. 937]. The general result follows from his proof and the fact that all derivations of a separable

Jordan algebra over a commutative ring with 1/3 into its bimodules are generalized inner derivations [5, Theorem 5.6].

We use factor sets to study Wedderburn decompositions, as in [9, pp. 288–289]. If M is a J-bimodule, let $Z(J, M)$ be the R-module composed of all R-bilinear maps $h : J \times J \to M$ such that $h(a, b) = h(b, a)$ and

$$a(bh(a, a)) + ah(a^2, b) + h(a^2 \cdot b, a) = a^2h(a, b) + (a \cdot b)\, h(a, a) + h(a^2, a \cdot b).$$

If $\mu : J \to M$ is an R-module homomorphism, define $\beta(\mu) : J \times J \to M$ by

$$\beta(\mu)(a, b) = \mu(a \cdot b) - a\mu(b) - b\mu(a),$$

$a, b \in J$. Let $B(J, M)$ be the R-module composed of all maps $\beta(\mu)$, $\mu \in \operatorname{Hom}_R(J, M)$. Reference [9, p. 94] shows that $B(J, M) \subset Z(J, M)$.

LEMMA 5.1. *Let J be separable, finitely spanned, and projective over R. If M is a J-bimodule, then*

$$B(J, M) = Z(J, M). \tag{17}$$

Proof. Let $h \in Z(J, M)$, and set $N = U_R(J)\, h(J, J)$. Since J is finitely spanned over R, so is $U_R(J)$ [9, p. 97], and thus so is N. h induces $h' \in Z(J, N)$, and it suffices to prove that $h' \in B(J, N)$, so we can assume that M is finitely spanned over R.

First assume that (R, m) is complete local Noetherian. Let $h \in Z(J, M)$. h induces $h' \in Z(J/mJ, M/mM)$. Since (17) holds when R is a field [9, p. 292], there is $\mu' \in \operatorname{Hom}_{R/m}(J/mJ, M/mM)$ such that $h' = \beta(\mu')$. μ' has a preimage $\mu_1 \in \operatorname{Hom}_R(J, M)$, since J is projective over R. Thus $h - \beta(\mu_1) \in Z(J, mM)$. Hence it follows by induction that for every positive integer n there is $\mu_n \in \operatorname{Hom}_R(J, m^{n-1}M)$ such that

$$h - \beta(\mu_1 + \cdots + \mu_n) \in Z(J, m^nM).$$

Since M is finitely spanned over R, it is complete in the m-topology [1, p. 108]. Thus $h = \beta(\mu)$ for $\mu = \sum \mu_i \in \operatorname{Hom}_R(J, M)$.

Next assume that R is Noetherian. Let m be a maximal ideal of R, and let R^* be the completion of R in the m-topology. R^* is complete local Noetherian [1, pp. 109, 113]. For any R-module N, let N^* denote $N \otimes_R R^*$. Since R^* is a flat R-module [1, p. 109], we identify $Z(J, M)^*$ and $B(J, M)^*$ as submodules of $\operatorname{Hom}_R(J \otimes_R J, M)^*$. Since J is finitely spanned and projective over R, it follows that

$$\operatorname{Hom}_R(J \otimes_R J, M)^* \cong \operatorname{Hom}_{R^*}(J^* \otimes_{R^*} J^*, M^*)$$

[6, p. 15]. This isomorphism takes $Z(J, M)^*$ to a submodule of $Z(J^*, M^*)$. It takes $B(J, M)^*$ onto $B(J^*, M^*)$, since

$$\mathrm{Hom}_R(J, M)^* \cong \mathrm{Hom}_{R^*}(J^*, M^*)$$

because J is finitely spanned R-projective [6, p. 15]. The preceding paragraph shows that $B(J^*, M^*) = Z(J^*, M^*)$. Since $B(J, M) \subset Z(J, M)$, it follows that $B(J, M)^* = Z(J, M)^*$. Since this holds for every maximal ideal m of R, $B(J, M) = Z(J, M)$ [1, pp. 40, 108, 110].

Finally, let R be arbitrary. Let $h \in Z(J, M)$. There is a Noetherian subring R' of R and an R'-subalgebra J' of J such that J' is separable, finitely spanned, and projective over R' and $J \cong J' \otimes_{R'} R$ [2, p. 134]. Set $M' = U_{R'}(J')\, h(J', J')$. As in the first paragraph of the proof, M' is finitely spanned over R' and h induces $h' \in Z(J', M')$. By the preceding paragraph, there is $\mu' \in \mathrm{Hom}_{R'}(J', M')$ such that $h' = \beta(\mu')$. Since $J \cong J' \otimes_{R'} R$, μ' extends to $\mu \in \mathrm{Hom}_R(J, M)$ such that $h = \beta(\mu)$, establishing (17). ∎

If N is an ideal of J, set $N^{(0)} = N$ and $N^{(i+1)} = (N^{(i)} \cdot N^{(i)}) \cdot J$, $i \geqslant 0$. $(N^{(i)})^2 \subset N^{(i+1)}$ and each $N^{(i)}$ is an ideal of J [9, p. 190]. We call N Penico-solvable if $N^{(t)} = 0$ for some positive integer t.

THEOREM 5.2. *Let N be a Penico-solvable ideal of J such that J/N is separable, finitely spanned, and projective over R. Then there is a separable subalgebra S of J such that $J = S \oplus N$ as R-modules.*

Proof. First assume that $N^2 = 0$. Let $\phi : J \to J/N$ be the canonical map. Since J/N is projective over R, there is an R-module homomorphism $\delta : J/N \to J$ such that $\phi\delta$ is the identity map on J/N. Define a bilinear map $h : J/N \times J/N \to N$ by $h(a, b) = a^\delta \cdot b^\delta - (a \cdot b)^\delta$, $a, b \in J/N$. Reference [9, pp. 92–93] shows that $h \in Z(J/N, N)$. By Lemma 5.1, there is $\mu \in \mathrm{Hom}_R(J/N, N)$ such that $h = \beta(\mu)$. Set $S = \{a^\delta + a^\mu \mid a \in J/N\}$. S is a nonunital subalgebra of J such that $J = S \oplus N$ over R [9, pp. 94, 289]. Write $1 \in J$ as $1 = e + x$, $e \in S$, $x \in N$. Since $x^2 = 0$, $x = (e + x) \cdot x = e \cdot x$ and $e + x = (e + x)^2 = e^2 + 2e \cdot x = e^2 + 2x$. Then $x = e - e^2 = 0$, since $S \cap N = 0$. Thus $1 = e \in S$, and S is a unital subalgebra of J.

In the general case, we induct on the least nonnegative integer t such that $N^{(t)} = 0$, the case $t = 0$ being trivial. If we write $N^{(1)}$ as M, N/M is an ideal of J/M such that $(N/M)^2 = 0$ and $(J/M)/(N/M) \cong J/N$ is separable, finitely spanned, and projective over R. By the preceding paragraph, there is a unital subalgebra S' of J/M such that $J/M = S' \oplus N/M$. If we let S_1 be the preimage of S' in J, $J = S_1 + N$ and $S_1 \cap N = M$. S_1 is a unital subalgebra of J such that $S_1/M \cong S' \cong J/N$ is separable, finitely spanned, and projective over R. Since $M^{(t-1)} = 0$, induction implies that there is a unital subalgebra S of S_1 such that $S_1 = S \oplus M$. Then S is a unital subalgebra of J such that $S+N=(S+M)+N=S_1+N=J$ and $S\cap N=S\cap(S_1\cap N)=S\cap M=0$. Thus $J=S\oplus N$, so $S\cong J/N$ is separable over R. ∎

We remark that the hypotheses on J/N in Theorem 5.2 are satisfied if J/N is central separable over R [S3].

In Theorem 5.3, we establish Wedderburn's theorem for J finitely spanned over (R, m) complete local Noetherian. This result and its proof parallel Ingraham's work on associative algebras [8]. In the statement and proof of Theorem 5.3 we waive our assumption that all Jordan algebras and subalgebras are unital; in particular, J need not be unital. Of course, separable Jordan algebras must be unital, and we continue to assume that $\frac{1}{2}\in R$.

THEOREM 5.3. *Let J be finitely spanned over (R, m) complete local Noetherian. If N is an ideal of J such that J/N is separable over R, there is a subalgebra S of J such that S is separable over R and $J=S+N$.*

Proof. Set $J'=J/mJ$ and $N'=(N+mJ)/mJ$. N' is an ideal of J' and

$$J'/N'\cong J/(N+mJ)\cong (J/N)/m(J/N). \tag{18}$$

Since J/N is separable over R, the last algebra in (18) is separable over R/m [S6], hence so is J'/N'. By the Wedderburn theorem for algebras over fields, there is a subalgebra S' of J' such that $J'=S'\oplus N'$ [9, p. 292]. Let S_1 be the preimage of S' in J, so S_1 is a subalgebra of J, $mJ\subset S_1$, $J=S_1+N$, and $S_1\cap N\subset mJ$. S_1 is finitely spanned over R, since R is Noetherian. $S_1\cap N$ is an ideal of S_1 such that $S_1/(S_1\cap N)\cong(S_1+N)/N=J/N$ is separable over R. Thus we can repeat the above argument with J and N replaced by S_1 and $S_1\cap N$, respectively. Hence induction shows that, for every positive integer n, there is a subalgebra S_n of S_{n-1} (taking $S_0=J$) such that

$$mS_{n-1}\subset S_n, \tag{19}$$

$$S_{n-1}=S_n+(S_{n-1}\cap N), \tag{20}$$

$$S_n\cap N\subset mS_{n-1}. \tag{21}$$

Equation (19) implies that

$$m^nJ\subset S_n. \tag{22}$$

We claim that

$$S_n\subset S_{n+1}+m^nJ \tag{23}$$

for $n\geqslant 0$, taking $m^0=R$. We induct on n, the case $n=0$ being trivial.

Assuming that $S_{n-1} \subset S_n + m^{n-1}J$, then

$$\begin{aligned} S_n &= S_{n+1} + (N \cap S_n) && \text{(by (20))} \\ &\subset S_{n+1} + mS_{n-1} && \text{(by (21))} \\ &\subset S_{n+1} + m(S_n + m^{n-1}J) && \text{(by induction)} \\ &= S_{n+1} + mS_n + m^nJ \\ &= S_{n+1} + m^nJ && \text{(by (19))}, \end{aligned}$$

completing the induction.

Let $S = \bigcap S_i$. We claim that

$$S_n = S + m^nJ \tag{24}$$

for $n \geqslant 0$. Equation (22) implies that $S_n \supset S + m^nJ$. Conversely, let $x_n \in S_n$. By (23) and induction, for every integer $i > n$ there is $x_i \in S_i$ such that $x_i - x_{i-1} \in m^{i-1}J$. Let $x = \lim x_i$, since J is complete in the m-topology [1, p. 108]. If $j > k \geqslant n$, then $x_j - x_k \in m^kJ$, so $x - x_k \in m^kJ \subset S_k$, by (22). Thus $x \in S_k$ for all $k \geqslant n$, so $x \in S$. $x - x_n \in m^nJ$, so $x_n \in S + m^nJ$, proving (24).

S is a subalgebra of J. By (20) and (24), $J = S_1 + N = S + N + mJ$. Thus

$$J = S + N, \tag{25}$$

by Nakayama's Lemma. Equations (21) and (24) imply that

$$S \cap N \subset S_n \cap N \subset mS_{n-1} = m(S + m^{n-1}J) = mS + m^nJ$$

for $n \geqslant 1$, so $S \cap N \subset mS + (S \cap m^nJ)$. Since J is finitely spanned and R is Noetherian, $S \cap m^nJ \subset mS$ for sufficiently large n [1, p. 107]. Hence $S \cap N \subset mS$. Then the canonical homomorphism of S onto $(S + N)/(mS + N)$ has kernel $mS + (S \cap N) = mS$, so

$$\begin{aligned} S/mS &\cong (S + N)/(mS + N) \\ &= J/(mJ + N) \qquad \text{(by (25))} \\ &\cong (J/N)/m(J/N). \end{aligned}$$

The last algebra above is separable over R/m [$S6$], hence so is S/mS. Since J is finitely spanned over the Noetherian ring R, so is S. Together with the fact that S/mS is separable over R/m, this implies that S is separable over R [$S2$]. ∎

We reinstate our assumption that all algebras and subalgebras are unital. To see that Theorem 5.3 remains valid in this context, let J be a unital algebra satisfying the hypotheses of Theorem 5.3, and let S be a nonunital subalgebra of J satisfying the conclusion of Theorem 5.3. Set $S' = S + R1$, $1 \in J$. Let T be the R/m-algebra created by adjoining a new unit element to S/mS [9, p. 30]. Since S/mS is separable over R/m, so is T [9, p. 287]. There is a homomorphism of T onto S'/mS', so the latter algebra is separable over R/m [S6]. It follows as in the proof of Theorem 5.3 that S' is separable over R.

We combine Theorems 5.2 and 5.3 to prove the Wedderburn theorem for finitely spanned Jordan algebras over Dedekind domains. The corresponding result for associative algebras was proved by Ingraham in [7].

Let $J = \bigoplus J_i(e)$ be the Peirce decomposition of J with respect to an idempotent e, where $J_i(e)$ is the i-eigenspace of J under multiplication by e. If R is an integral domain and M is an R-module, let $T(M)$ be the submodule composed of all elements of M annihilated by a nonzero element of R. M is called torsion if $M = T(M)$.

THEOREM 5.4. *Let J be finitely spanned over a Dedekind domain R, and let N be a Penico-solvable ideal of J such that J/N is separable over R. Then there is a separable subalgebra S of J such that $J = S + N$.*

Proof. We write $Z(J/N)$ as C. Since R is a Dedekind domain, $C/T(C)$ is projective over R [10, p. 625]. Since J/N is separable over R, C is a separable associative R-algebra [5, Proposition 1.1]. Together with the facts that $T(C)$ is an ideal of C and $C/T(C)$ is projective over R, this implies that $C/T(C)$ is projective over C [6, p. 48]. Thus $T(C)$ is an ideal direct summand of C, so $T(C)$ has an identity element e'. Since $e' \in C$,

$$J/N = (J/N)_1\,(e') \oplus (J/N)_0\,(e') \tag{26}$$

is a direct sum of ideals, where $Z[(J/N)_1\,(e')] = T(C)$ is torsion and $Z[(J/N)_0\,(e')] \cong C/T(C)$ is projective over R. Since N is solvable, e' lifts to an idempotent $e \in J$, by the proof of [2, p. 130]. $J_1(e)$ and $J_0(e)$ have e and $1-e$ as identity elements. For $i \in \{0, 1\}$, $N \cap J_i(e)$ is a solvable ideal of $J_i(e)$ such that

$$J_i(e)/(N \cap J_i(e)) \cong (J/N)_i\,(e')$$

is separable over R, by (26) and [5, Theorem 3.2]. If $J_i(e)$ has a separable subalgebra S_i such that $J_i(e) = S_i + (N \cap J_i(e))$, $i \in \{0, 1\}$, then $S_0 + S_1 \cong S_0 \oplus S_1$ is separable over R [5, Theorem 3.2], and (26) implies that $J = S_0 + S_1 + N$. Thus it suffices to prove Theorem 5.4 with J replaced by $J_i(e)$ and N replaced by $N \cap J_i(e)$, $i \in \{0, 1\}$. Hence we can assume that $Z(J/N)$ is either projective or torsion over R.

First assume that $Z(J/N)$ is projective over R. J/N is projective over

$Z(J/N)$, since J/N is separable over R [S3]. Then J/N is projective over R, by the transitivity of projectivity [6, p. 5]. We are done by Theorem 5.2.

Next assume that $Z(J/N)$ is torsion. $1 \in Z(J/N)$ is annihilated by a nonzero element r of R. Then $r1 \in N$ for $1 \in J$. Since N is solvable, there is a positive integer n such that $0=(r1)^n = r^n 1$, so $r^n J = 0$. There are finitely many maximal ideals m_i such that $\prod m_i^d \subset r^n R$ for some positive integer d [10, p. 604]. Let $I_j = \prod_{i \neq j} m_i^d$, and let $J_j = I_j J$. J_j is an ideal of J such that $m_j^d J_j = 0$. $J = \sum J_j$, since $R = \sum I_j$ [10, p. 605]. $J_j J_k \subset r^n J = 0$ if $j \neq k$, so $J = \bigoplus J_j$ is a direct sum of ideals.

Each J_j is naturally an algebra over R_j^*, the completion of R in the m_j-topology. R_j^* is complete local Noetherian with maximal ideal $m_j R_j^*$ [1, pp. 109, 113]. By Theorem 5.3, there is a separable R_j^*-subalgebra S_j of J_j such that $J_j = S_j + (N \cap J_j)$. $S_j/m_j S_j$ is separable over $R_j^*/m_j R_j^* \cong R/m_j$ [S6], [1, p. 109]. If m is any maximal ideal of R besides m_j, the relations $m + m_j^d = R$ and $m_j^d J_j = 0$ imply that $S_j/mS_j = 0$. Thus S_j/mS_j is either zero or separable over R/m for every maximal ideal m of R. S_j is finitely spanned over R, since $S_j \subset J$ and R is Noetherian. Together the last two sentences imply that S_j is separable over R [S2]. Thus $\bigoplus S_j \cong \sum S_j \subset J$ is separable over R [5, Theorem 3.2], and $J = \sum S_j + N$. ∎

Finally, we prove Malcev's theorem on the uniqueness up to automorphism of split Wedderburn decompositions for Jordan algebras over commutative rings containing 1/3. We follow McCrimmon's proof of the corresponding result for algebras over fields [14].

Let $S_R(J)$ be the unital special universal envelope of J, and let $\sigma : J \to S_R(J)$ be the canonical map. We recall that latently and weakly 4-interconnected Jordan algebras are defined in [2, p. 139]. It is proved there that a weakly 4-interconnected algebra is special and reflexive. Thus, if J is weakly 4-interconnected and $x_1, \dots, x_n \in J$, let $\langle x_1, \dots, x_n \rangle$ be the unique preimage in J of $x_1^\sigma \cdots x_n^\sigma + x_n^\sigma \cdots x_1^\sigma \in S_R(J)$.

Let J_i be the i-eigenspace of J under multiplication by an idempotent of J. If J_1 is weakly 4-interconnected over R and $f, g, h \in J_1$, define $V_{f,g,h} \in \mathrm{Hom}_R(J, J)$ by

$$V_{f,g,h}(J_0) = 0,$$
$$V_{f,g,h}(x) = V_f V_g V_h(x), \qquad x \in J_{1/2},$$
$$V_{f,g,h}(x) = \langle f, g, h, x \rangle, \qquad x \in J_1.$$

Define $W_{f,g,h} \in \mathrm{Hom}_R(J, J)$ by

$$W_{f,g,h}(x) = x + V_{f,g,h}(x) + U_f U_g U_h(x),$$

$x \in J$.

THEOREM 5.5. *Let $1/3 \in R$. Let J contain a subalgebra B and a Penico-*

solvable ideal N such that $B \oplus N \cong B + N \subset J$. Let C be a separable subalgebra of J such that $C \subset B + N$ and C is finitely spanned over R. Then there is an automorphism T of J such that $T(C) \subset B$ and $T(N) \subset N$. If $J = C \oplus N$, it follows that $T(C) = B$ and $T(N) = N$.

Proof. We say that an algebra automorphism ϕ of J covers a derivation $D : C \to N/N^{(2)}$ if

(i) $\phi(x) \equiv x - D(x) \pmod{N^{(2)}}$, $x \in C$,

(ii) $\phi(x) \equiv x \pmod{N^{(2)}}$, $x \in N$, and

(iii) $\phi(I) \subset I$ for every ideal I of J.

The theorem follows from McCrimmon's arguments in [14, pp. 960–961] once we show that every derivation of C into $N/N^{(2)}$ is a sum of derivations which are each covered by an automorphism of J. The latter result follows from McCrimmon's work in [14] and the fact that every derivation of C into a bimodule is a generalized inner derivation [5, Theorem 5.6].

Specifically, let $a, b \in C(p, q)$, $p \geqslant 4$. Let N' be a C-subbimodule of $N/N^{(2)}$ such that N' is a unital bimodule for $C(p, q)$ over $Z[C(p, q)]$. Let $c \in N'$. Since $C(p, q)$ is finitely spanned over R, $C(p, q)$ is weakly 4-interconnected over both R and $Z[C(p, q)]$ [2, p. 139]. Hence so is the split null extension $C(p, q) \oplus N'$. Thus, if $x \in C(p, q)$, we can define $\langle a, b, c, x\rangle - \langle c, b, a, x\rangle \in N'$ by considering $C(p, q) \oplus N'$ over either R or $Z[C(p, q)]$. We get the same element of N' in either case, since there is a natural homomorphism

$$S_R(C(p, q) \oplus N') \to S_{Z[C(p,q)]}(C(p, q) \oplus N').$$

Define $D_{a,b,c} \in \mathrm{Hom}_R(C, N/N^{(2)})$ by

$$D_{a,b,c}(x) = \langle a, b, c, x\rangle - \langle c, b, a, x\rangle, \qquad x \in C(p, q),$$

and

$$D_{a,b,c}(C(p', q')) = 0, \qquad (p', q') \neq (p, q).$$

The argument before Theorem 5.6 of [5] shows that $D_{a,b,c}$ is a derivation.

Let J_i be the i-eigenspace of J with respect to the identity element of $C(p, q)$. c has a preimage $d \in N \cap J_1$. Since $C(p, q)$ is weakly 4-interconnected over R, so is J_1. Set $F = W_{a,b,-d} W_{d,b,a} \in \mathrm{Hom}_R(J, J)$. J_1 is latently 4-interconnected when localized at any maximal ideal m of R, so $F(1) \equiv 1 \pmod{N^{(2)}}$ under such localization [14, p. 950]. Then $F(1) \equiv 1 \pmod{N^{(2)}}$ holds without localization [1, p. 40]. It follows that there is an invertible element $w \in J$ such that $w^2 = F(1)$ and $w \equiv 1 \pmod{N^{(2)}}$ [14,

p. 944]. Set $T = U_w^{-1}F \in \mathrm{Hom}_R(J, J)$. If R_m is the localization of R at a maximal ideal m, $T \otimes 1$ is an automorphism of $J \otimes R_m$ covering

$$D_{a,b,c} \otimes 1 : C \otimes R_m \to N/N^{(2)} \otimes R_m$$

[14, p. 949]. Hence T is an automorphism of J covering $D_{a,b,c}$ [1, p. 40].

We have shown that every derivation of the form $D_{a,b,c}$ can be covered by an automorphism of J. There is an automorphism of J covering every inner derivation $D_{u,v}(x) = [u, x, v]$, $u, x \in C$, $v \in N/N^{(2)}$ [14, p. 945]. Since C is separable over R and $1/3 \in R$, every derivation of C into $N/N^{(2)}$ can be written as a sum of derivations of the forms $D_{a,b,c}$ and $D_{u,v}$ [5, proof of Theorem 5.6]. Since derivations of these two forms are covered by automorphisms of J, the theorem follows by McCrimmon's arguments in [14, pp. 960–961]. ∎

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