

Connections between Two Theories of Concurrency: Metric Spaces and Synchronization Trees

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A connection is established between the semantic theories of concurrency and communication in the works of de Bakker and Zucker, who develop a denotational semantics of concurrency using metric spaces instead of complete partial orders, and Milner, who develops an algebraic semantics of communication based upon observational equivalence between processes. His *rigid synchronization trees* (RSTs) are endowed with a simple pseudometric distance induced by Milner's *weak equivalence* relation and the quotient space is shown to be complete. An isometry between this space and the solution to a domain equation of de Bakker and Zucker is established, presenting their solution in a conceptually simpler framework. Under an additional assumption, the equivalence between the weak equivalence relation over RSTs and the elementary equivalence relation induced by the sentences of a modal logic due to Hennessy and Milner is established.

0. INTRODUCTION

In this paper we establish a fundamental connection between the semantic theories of concurrency and communication in de Bakker and Zucker (1982) and Milner (1980). In de Bakker and Zucker (1982) the authors develop a denotational semantics of concurrency using metric spaces instead of complete partial orders as the underlying mathematical structures. They solve several reflexive domain equations, their method essentially entailing the abstract completion of a metric space recursively constructed from metric spaces which utilize a Hausdorff distance between closed sets. (See Arnold and Nivat (1980) or Nivat (1979) for examples of metric topology applied to various problems concerning infinite words, ω -CFL languages and the modelling to nondeterministic computations.) In Milner (1980) the author develops an algebraic framework for specifying and reasoning about processes which behave synchronously. Central to his ideas is the notion of behavioral or observational equivalence over processes; ideally processes which are in some sense externally indistinguishable short of taking them

apart are equivalent behaviorally and will be congruent with respect to some natural collection of algebraic operations. The conceptual process model underlying Milner's approach is the synchronization tree, an arc labeled tree over an event alphabet, representing the structure of the communication requirements of a process.

Our intentions in this paper are twofold. First we shall establish Milner's *rigid synchronization trees* (RSTs) as a complete pseudometric space with a simple pseudometric distance induced by his *weak observational equivalence* relation. Second, we proceed to establish isometries between the resulting quotient space, considered separately with a countable and a finite alphabet, and appropriate metric spaces using the de Bakker–Zucker construction. As a benefit of these isometries, note that one does not necessarily have to use the complicated notions of Hausdorff distance and the attendant machinery of metric space completions; as the processes of de Bakker and Zucker (1982) can be represented concretely as RSTs modulo weak equivalence, one can work directly with the trees as graphs and use a simple metric defined directly on the graph structure.

While the construction in this paper allows the alphabet to be countable, we can prove that our metric space is compact if and only if the alphabet is finite. In this case it turns out that the weak observational equivalence relation is exactly the elementary equivalence relation induced by the sentences of a simple modal logic due to Hennessy and Milner (1980). The statement that our space is compact is exactly the assertion of the compactness theorem for the Hennessy–Milner logic (HML). Since the HML compactness theorem follows from a direct translation into first-order logic, this gives us an elegant but nonconstructive proof of completeness for the finite alphabet case.

The rest of the paper is organized as follows: Section 1 is preliminary, defining the domain of trees and establishing some necessary properties. Section 2 presents the rigid synchronization trees of Milner and defines weak equivalence. The third section constructs the metric space and proves its completeness. The fourth section recalls the necessary definitions and results from de Bakker and Zucker (1982) and establishes the isometries between the metric spaces of this paper and those constructed using the de Bakker–Zucker method. Finally, Section 5 establishes the connections between HML and our metric space.

1. PRELIMINARIES

We regard a tree as a directed, unordered graph on a countable set of nodes with arcs labeled from an alphabet Σ . The graph must have the

obvious tree shape and arcs leaving the same node may have the same label. We define the set of trees \mathcal{E} as follows.

DEFINITION. S is a tree ($S \in \mathcal{E}$) iff S is a 4-tuple $S = (N, E, l, v_0)$, where

- | | |
|---------------------------|--|
| N | is a set of vertices or nodes; |
| $v_0 \in N$ | is the root; |
| $E \subseteq N \times N$ | is the edge relation, antisymmetric and irreflexive; |
| $l: E \rightarrow \Sigma$ | assigns a label to each edge. |

In addition the following properties are satisfied:

- (1) all nodes are reachable from the root: $\forall v \in N - \{v_0\} \langle v_0, v \rangle \in E^+$, where E^+ is the transitive closure of E ;
- (2) each node has only one ancestor: $\forall u, v, w \in N, \langle u, w \rangle \in E$ and $\langle v, w \rangle \in E$ implies $u = v$.

We say two trees are *isomorphic* if each can be transformed into the other preserving structure and labeling.

DEFINITION. $S = (N, E, l, v_0)$ and $S' = (N', E', l', v'_0)$ are isomorphic iff there is a bijection $f: N \rightarrow N'$ such that

- (1) $f(v_0) = v'_0$ (identification of roots);
- (2) $\langle v, w \rangle \in E \Leftrightarrow \langle f(v), f(w) \rangle \in E'$ (identification of edges);
- (3) $\forall \langle v, w \rangle \in E, l(\langle v, w \rangle) = l'(\langle f(v), f(w) \rangle)$ (identified edges have same label).

When S and S' are isomorphic, we shall write $S = S'$.

The notions of path, path length, and finite and infinite paths are the usual ones. We say a tree is *bounded* if there is a finite bound on all path lengths. For a tree S , the *height* $|S|$ is the length of its longest path if bounded, ω otherwise. A node is *finitely branching* if it has finite number of direct descendants. A tree is *finite branching* if all its nodes are. We allow *countable branching* at any node.

The k th cross section $S^{(k)}$ of a tree S is just S restricted so that no path has a length exceeding k .

DEFINITION. For $S \in \mathcal{E}$, let the k th *cross section* of $S = (N, E, l, v_0)$ be:

$$S^{(0)} = (\{v_0\}, \phi, \phi, v_0), \quad S^{(k)} = (N_k, E_k, l_k, v_0), \quad k \geq 1,$$

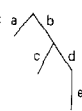
where


$$E_k = \{ \langle v, w \rangle \in E \mid \text{the path } \langle v_0, w \rangle \text{ has length at most } k \};$$

$$N_k = \{ v \in N \mid \text{the path } \langle v_0, v \rangle \text{ has length at most } k \};$$

$$l_k = l \upharpoonright E_k.$$

EXAMPLES. (1) $S^{(0)}$ is just the root, which we call *nil*.

(2) For $S =$  we have $S^{(0)} = \text{nil}$, $S^{(1)} = a \wedge b$,

$S^{(2)} =$  and $S^{(k)} = S$ for $k \geq 3$.

We have the following relationship between a tree and its cross sections.

LEMMA 1.1. For any $S = (N, E, l, v_0)$, let $\{S^{(k)}\}$ be the set of all its cross sections, $k \geq 0$. Then

- (a) $\forall k \geq 0, E_k \subseteq E_{k+1}$, and $E = \bigcup E_k$,
- (b) $\forall k \geq 0, N_k \subseteq N_{k+1}$, and $N = \bigcup N_k$,
- (c) $\forall k \geq 0, l_k \subseteq l_{k+1}$, and $l = \bigcup l_k$ (viewing l_k as a set of ordered pairs $\langle e_k, a \rangle$ from E_k and Σ).

Proof. (a) $E_k \subseteq E_{k+1}$ directly from the definition. Now clearly $E_k \subseteq E$ for all k so $\bigcup E_k \subseteq E$. Let $\langle v, w \rangle \in E$. Then there is a path $\langle v_0, w \rangle$ in S and therefore $\langle v, w \rangle \in E_k$ for any k not less than the path length of $\langle v_0, w \rangle$. Therefore $\langle v, w \rangle \in \bigcup E_k$, whereby $E \subseteq \bigcup E_k$.

The proofs of (b) and (c) are similar.

This lemma suggests that any tree can be represented as a union of its cross sections, leading to the

DEFINITION. Let $\{S_k\} \subseteq \mathcal{S}$. $\{S_k\}$ is a cross-sectional sequence (written $\langle S_k \rangle$ an XSS) iff

- (1) each S_k is bounded, say with maximum path length of $b(k)$;
- (2) $\forall m \geq k, S_m^{(b(k))} = S_k^{(b(k))}$.

The last condition ensures that the $b(k)$ th cross sections of S_k, S_{k+1}, \dots , are all isomorphic, that is, only the leaves of S_k with path length $b(k)$ can be

extended to form S_{k+1} . For convenience, in any sequence $\langle S_k \rangle$, we shall take S_0 to be the *nil* tree and $b(0) = 0$.

DEFINITION. Let $\langle S_k \rangle$ be an XSS. The *union tree* of $\langle S_k \rangle$ is

$$\bigcup S_k = \left(\bigcup N_k, \bigcup E_k, \bigcup l_k, v_0 \right).$$

We collect some facts about XSS which will be useful later.

LEMMA 1.2. Let $\langle S_k \rangle$ be an XSS.

- (a) $k \leq n$ implies $b(k) \leq b(n)$,
- (b) $S_k = S_k^{(b(k))}$,
- (c) $\forall m \geq k \forall j \leq b(k) S_m^{(j)} = S_k^{(j)}$,
- (d) $\bigcup S_k$ is a tree and $(\bigcup S_k)^{(b(k))} = S_k$.

Proof. Straightforward.

We wish to define two additional operators on trees, prefixing and joining, enabling us to create complex trees from simpler ones.

Notation. $S[v/w]$ means the tree S with the node w replaced by v .

DEFINITION. For $S = (N, E, l, v_0)$ and $a \in \Sigma$ let

$$aS = (N \cup \{v_a\}, E \cup \{\langle v_a, v_0 \rangle\}, l \cup \{\langle \langle v_a, v_0 \rangle, a \rangle\}, v_a),$$

where $v_a \notin N$. We call aS a *prefixed* (sub)tree.

DEFINITION. We say $\{S_k\}$ are *disjoint* if $\{N_k\}$ are pairwise disjoint.

DEFINITION. Let $\{S_k\} \subseteq \mathcal{E}$, $S_k = (N_k, E_k, l_k, v_{0,k})$, $\{S_k\}$ disjoint. The *join* of $\{S_k\}$ is

$$\sum S_k = \bigcup S_k[v_0/v_{0,k}].$$

For k finite, for example 2, we often write $\sum S_k$ as $S_1 + S_2$. So $S + T$ is the tree obtained by joining S and T at the root. We view the expression $S + S$ to be well defined, representing the join of two disjoint isomorphic copies of S . We represent by S^n the joining of n copies of S for $1 \leq n \leq \omega$. In a similar spirit, $S + T$ will always be taken to be well defined through an inessential relabeling of nodes if necessary.

LEMMA 1.3. aS and $\sum S_k$ are trees.

Proof. Clear.

Finally we establish another representation of an arbitrary tree.

LEMMA 1.4. *For $S \in \mathcal{E}$, there is a set $\{a_i S_i\} \subseteq \mathcal{E}$ such that $S = \sum a_i S_i$.*

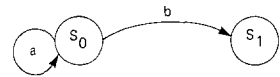
Proof. Clearly we can represent S as the join of its prefixed subtrees.

For convenience we shall often represent our tree algebraically. For example, $ab + ac$ represents the tree with two paths ab and ac joined at the root. The expression $a(b + c)$ represents the tree with a emanating from the root leading to a fork with labels b and c . Here juxtaposition binds tighter than joining, unless overruled by parenthesization.

As will be evident shortly the structures of interest to us are actually isomorphism classes of trees, structures with unlabeled nodes. For convenience we shall hereafter refer to such structures with labeled arcs and unlabeled nodes as *trees* and to a tuple (N, E, l, v_0) as a *representation* of the particular tree in mind.

2. RIGID SYNCHRONIZATION TREES AND WEAK EQUIVALENCE

In the spirit of Milner (1980) we regard a *rigid synchronization tree* (RST) as the unfolding of a state transition graph of a nondeterministic machine. For example, given the transition system



we associate the RST



. Note that state names are no longer

important; the tree nodes are nameless. The arc labels are chosen from an event alphabet Σ , reflecting the communication requirements of the process from its environment. We depart from Milner (1980) and allow the nodes to have countable branching.

Nondeterministic choice exists in the tree $ab + ac$. Given an a , the machine must choose between two paths, arriving at either a state where only b is acceptable or one in which only c is. Now consider the tree $a(b + c)$. If viewed as acceptors, both of these trees are equivalent, accepting the language $\{ab, ac\}$. But are they equivalent behaviorally? After one step the second tree is in a state where either a b or c is acceptable, and so it never deadlocks on input from $\{ab, ac\}$. However, the first tree can deadlock on either ab or ac ; after a has been consumed it will be in a state waiting for a specific event and will fail if the environment offers an incompatible input. Note that nondeterministic trees do not necessarily “choose correctly”; they

react only to the current event, not to future ones. Since the trees behave differently on inputs from $\{ab, ac\}$, it is reasonable to maintain that they are not equivalent behaviorally.

Several different equivalence relations have been proposed to describe behavioral or observational equivalence (Milner, 1980). The relation appropriate for this paper is the *weak equivalence* relation and is defined as follows:

Notation. When we write $S \rightarrow^a T$ we mean there is some a transition from the root of S leading to T , or that aT is a prefixed subtree of S .

DEFINITION. For $S, T \in \mathcal{C}$, S is weakly equivalent to T , $S \equiv_w T$, iff $\forall k S \equiv_k T$, where the equivalences \equiv_k are defined as

$$(1) \quad S \equiv_0 T \text{ for all } S, T;$$

$$(2) \quad S \equiv_{k+1} T$$

$$\Leftrightarrow \forall a \in \Sigma \forall S' \in \mathcal{C}, S \rightarrow^a S' \Rightarrow \exists T' \in \mathcal{C} \exists T \rightarrow^a T' \text{ and } S' \equiv_k T' \text{ and}$$

$$\forall a \in \Sigma \forall T' \in \mathcal{C}, T \rightarrow^a T' \Rightarrow \exists S' \in \mathcal{C} \exists S \rightarrow^a S' \text{ and } T' \equiv_k S'.$$

We write $S \equiv T$ for $S \equiv_w T$.

An alternate way of presenting $(k+1)$ -equivalence which we shall find convenient is

$$S \equiv_{k+1} T \Leftrightarrow \text{for every prefixed subtree } aS' \text{ of } S, \text{ there is a prefixed subtree } aT' \text{ of } T \text{ such that } S' \equiv_k T' \text{ (and vice versa).}$$

EXAMPLES. (1) $(ab + ac) \not\equiv_1 a(b + c)$ since they are $\not\equiv_2$. To see this, note that nodes are \equiv_1 if the set of events which can occur next are the same. The tree $b + c$ is $\not\equiv_1$ to either b or c .

(2) Let A_k be the tree consisting of a single path of length k , labeled entirely with a 's: $aa \cdots a$. Let $A_* = \sum A_k, k \geq 1$. So A_* has arbitrarily long finite paths of every length and a countably branching root. Let A_ω be the infinite tree $aa\dots$, and let $A_\infty = A_* + A_\omega$. Note that for all $k, A_\infty^{(k)} = A_*^{(k)}$ as each k th cross section contains one path each of lengths $1, \dots, k-1$ and a countable number of paths of length k . We claim that $A_\infty \equiv_k A_*$ for all k and thus $A_\infty \equiv A_*$, as can be seen from

LEMMA 2.1. *If $S^{(k)} = T^{(k)}$ then $S \equiv_k T$.*

Proof. Induction on k . For $k=0$ the result is immediate.

Assume the lemma holds for k .

Suppose now $S^{(k+1)} = T^{(k+1)}$. As the prefixed subtrees of S and T are in 1-1 correspondence, we can write $S^{(k+1)} = \sum a_i S_i^{(k)} = \sum a_i T_i^{(k)} = T^{(k+1)}$,

where $S_i^{(k)} = T_i^{(k)}$. Therefore, by the induction hypothesis we have $S_i \equiv_k T_i$. Clearly now we have $S \equiv_{k+1} T$.

We remark that the converse is false: $a + a \equiv a$, but not isomorphic. Finally, we collect some easy and useful facts.

- LEMMA 2.2. (1) $S \equiv_k T$ implies $\forall j \leq k, S \equiv_j T$,
 (2) $S \not\equiv_k T$ implies $\forall j \geq k, S \not\equiv_j T$,
 (3) $S \equiv_k S^{(k)} \equiv_k S^{(n)}, n \geq k$.

Proof. Straightforward.

3. THE METRIC SPACE OF RSTs

In this section the completeness of the metric space on \mathcal{E} induced by the weak equivalence relation is demonstrated. For topological definitions and related items, the reader is referred to Dugundji (1966).

We define the metric on \mathcal{E} .

DEFINITION. For $S, T \in \mathcal{E}$, let $d_w(S, T) = 2^{-k}$, where $k = \max_j S \equiv_j T$. If the maximum does not exist, we take k to be infinite.

As k -equivalence examines no nodes which are a distance greater than k from the root, we see that the larger value of k , the more alike the two trees are, the smaller the value of d_w .

- EXAMPLES. (1) $d_w(a + b, a) = 1$ since $S \not\equiv_1 T$,
 (2) $d_w(a + ab, a) = \frac{1}{2}$ since $S \equiv_1 T$ but $S \not\equiv_2 T$,
 (3) $d_w(a + a, a) = d_w(A_*, A_\infty) = 0$.

LEMMA 3.1. $\langle \mathcal{E}, d_w \rangle$ is an ultra pseudometric space.

Proof. (1) $d_w(S, T) = 0 \Leftrightarrow \forall k, S \equiv_k T \Leftrightarrow S \equiv T$ (pseudo),
 (2) $d_w(S, T) = d_w(T, S)$,
 (3) $d_w(S, T) \leq \max(d_w(S, U), d_w(U, T))$ (ultra). Let $d_w(S, T) = 2^{-k}$ and suppose (wlog) $d_w(S, U) < 2^{-k}$. Then $S \equiv_{k+1} U$. Since both $S \equiv_k U$ and $S \equiv_k T$, we have $U \equiv_k T$. However, $U \not\equiv_{k+1} T$ as $S \not\equiv_{k+1} T$. Therefore, $d_w(U, T) = 2^{-k}$.

We define the notions of Cauchy sequence and limit.

DEFINITION. $\langle S_n \rangle$ is a Cauchy sequence (CS) iff

$$\forall k \geq 0, \exists N \forall m, n \geq N, S_m \equiv_k S_n.$$

DEFINITION. S is a *limit* of a CS $\langle S_n \rangle$ (written $S \in \perp \lim S_{n\downarrow}$) iff

$$\forall k \geq 0, \exists N \forall n \geq N, S \equiv_k S_n.$$

Remarks. (1) The above definitions are equivalent to the more usual presentations, for example, $\forall \varepsilon > 0, \exists N \forall m, n \geq N, d_w(S_m, S_n) < \varepsilon$.

(2) We must deal with equivalence classes of CS limits. Recall that $\langle \mathcal{E}, d_w \rangle$ is a pseudometric space and if $S_n = \sum_{j=1}^n A_j$ (for example $S_3 = a + aa + aaa$), we have that $\langle S_n \rangle$ is a CS, and for all n $A_* \equiv_n S_n \equiv_n A_\infty$, and, therefore, $\{A_*, A_\infty\} \subseteq \perp \lim S_{n\downarrow}$.

Proceeding to the completeness proof, we will establish that any XSS $\langle S_n \rangle$ in $\langle \mathcal{E}, d_w \rangle$ is a CS with a well-defined constructible limit, the union tree: $\bigcup S_n \in \perp \lim S_{n\downarrow}$. An operator on trees, \mathcal{E} , yielding a fully expanded countably branching tree in a sense made precise below, will be defined and shown to possess the following special properties:

(1) weak equivalence is the same as isomorphism, that is,

$$\mathcal{E}(S) = \mathcal{E}(T) \Leftrightarrow \mathcal{E}(S) \equiv \mathcal{E}(T) \quad \text{for bounded } S, T,$$

(2) for any bounded $S, S \equiv \mathcal{E}(S)$.

Now given a CS $\langle S_n \rangle$, $\mathcal{E}(S_n^{(n)})$ will be shown to be an XSS (due to (1)) and, therefore, possesses a limit which by (2) is the same as the limit of $\langle S_n \rangle$; the completeness of $\langle \mathcal{E}, d_w \rangle$ follows directly.

LEMMA 3.2. *If $\langle S_n \rangle$ is an XSS, then it is also a CS in $\langle \mathcal{E}, d_w \rangle$.*

Proof. Recall $\langle S_n \rangle$ is a CS $\Leftrightarrow \forall k \geq 0, \exists N \forall m, n \geq N, S_m \equiv_k S_n$. We have two cases:

(a) $\langle S_n \rangle$ is bounded (i.e., $\{b(n)\}$ is bounded). Then after some $N_0, \forall m, n \geq N_0, S_m = S_n$. Then for any $k, S_m^{(k)} = S_n^{(k)}$ and so $S_m \equiv_k S_n$ (Lemma 2.1).

(b) $\langle S_n \rangle$ is not bounded. Choose N such that $b(N) \geq k$. Then as $S_N = S_N^{(b(N))}$ (Lemma 1.2b) we have $\forall m, n \geq N, S_m^{(k)} = S_n^{(k)}$ (Lemma 1.2c) and, therefore, $S_m \equiv_k S_n$ (Lemma 2.1).

THEOREM 3.3. *Let $\langle S_n \rangle$ be an XSS. Then $\perp \lim S_{n\downarrow}$ exists and $\bigcup S_n \in \perp \lim S_{n\downarrow}$.*

Proof. $\bigcup S_n$ exists by Lemma 1.2d. Now as $(\bigcup S_n)^{(b(m))} = S_m$ (Lemma 1.2d) we have $\bigcup S_n \equiv_{b(m)} S_m$ (Lemma 2.1). As in Lemma 3.2 we have two cases:

(a) $\langle S_n \rangle$ is bounded. Clearly $\bigcup S_n = S_m$ for some m and, therefore, $\bigcup S_n \in \text{lim } S_n$.

(b) $\langle S_n \rangle$ is unbounded. We establish

$$\forall k \geq 0, \exists N \forall m \geq N, \bigcup S_n \equiv_k S_m.$$

Given $k \geq 0$ find $N \geq 0$ such that $b(N) \geq k$. Now

$$\forall m \geq N, \bigcup S_n \equiv_{b(N)} S_m \quad \text{and so} \quad \bigcup S_n \equiv_k S_m.$$

Our \mathcal{E} operator is defined as:

DEFINITION. For any bounded tree S , let $\mathcal{E}(S)$ be

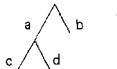
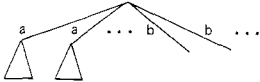
$$\mathcal{E}(nil) = nil, \quad \mathcal{E}\left(\sum a_i S_i\right) = \left[\sum a_i \mathcal{E}(S_i)\right]^\omega,$$

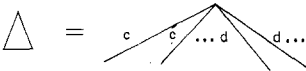
where $[S]^\omega$ is understood to be ω copies of S joined at the root.

To aid the intuition, $\mathcal{E}(S)$ can be constructed for any bounded tree S as follows:

- (1) mark all leaf nodes as *ready*;
- (2) repeat until the root is marked *ready*:

if all of the descendants of a node are *ready*,
 then replace each prefixed subtree of the node with ω copies
 of the subtree and
 mark the node *ready*;

For example, if $S =$
, then $\mathcal{E}(S) =$


where $\triangle =$


LEMMA 3.4. For S bounded, $\mathcal{E}(S)$ is a tree.

Proof. Straightforward.

The utility of \mathcal{E} -trees becomes evident in the following theorem and corollary in which weak equivalence is seen to be the same as isomorphism.

THEOREM 3.5. Let $C = \mathcal{E}(S)$ and $D = \mathcal{E}(T)$ for some bounded S, T . Then $C \equiv_k D \Leftrightarrow C^{(k)} = D^{(k)}$.

Proof. (\Leftarrow) Lemma 2.1. (\Rightarrow) Induction on k .

Case $k = 0$. Immediate.

Assume for k .

Case $k + 1$. Suppose $C \equiv_{k+1} D$. Partition the prefixed subtrees of both C and D into $(k + 1)$ -equivalence classes. As $C \equiv_{k+1} D$, these equivalence classes of C and D are in 1-1 correspondence. Now consider a representative of any $(k + 1)$ -equivalence class of C and a corresponding representative of D ; they will be of the form aC' and aD' , where $C' \equiv_k D'$. By the induction hypothesis $C'^{(k)} = D'^{(k)}$ and so $(aC')^{(k+1)} = (aD')^{(k+1)}$, or the representatives of corresponding $(k + 1)$ -equivalence classes of prefixed subtrees have $(k + 1)$ -isomorphic cross sections. Therefore, the two trees obtained from C and D by the joining of their $(k + 1)$ -representatives are $(k + 1)$ -isomorphic. As C and D are bounded \mathcal{C} -trees, every representative prefixed subtree of C or D contributes ω copies of itself to C or D . So the number of subtrees represented by any class is ω . Therefore, we have $C^{(k+1)} = D^{(k+1)}$.

COROLLARY. Let C and D be as in Theorem 3.5. Then $C \equiv D \Leftrightarrow C = D$.

Proof. As C, D bounded, just take k as the section height of C or D .

The last result we need prior to proving completeness is

LEMMA 3.6. For S bounded, $S \equiv \mathcal{C}(S)$.

Proof. We show $\forall k S \equiv_k \mathcal{C}(S)$ by induction on k .

Case $k = 0$. Immediate.

Assume for k .

Case $k + 1$. Let $S = \sum a_i S_i$, $\mathcal{C}(S) = [\sum a_i \mathcal{C}(S_i)]^\omega$.

Now $S \equiv_{k+1} \mathcal{C}(S) \Leftrightarrow \forall a \forall S', S \rightarrow^a S'$ implies $\exists C', \mathcal{C}(S) \rightarrow^a C'$ and $S' \equiv_k C'$, and vice versa. If $a_i S_i$ is a prefixed subtree of S , then $a_i \mathcal{C}(S_i)$ is a prefixed subtree of $\mathcal{C}(S)$. We have $S_i \equiv_k \mathcal{C}(S_i)$ by the induction hypothesis and so the required C' exists. A similar argument for the reverse direction establishes the lemma.

We are now ready to prove

THEOREM 3.7. $\langle \mathcal{C}, d_w \rangle$ is complete.

Proof. Let $\langle S_n \rangle$ be any arbitrary CS in $\langle \mathcal{C}, d_w \rangle$, that is,

$$\forall k \geq 0, \quad \exists N \forall m, n \geq N \quad S_m \equiv_k S_n.$$

By passing to a subsequence if necessary, we can assume $\forall k, \forall n \geq k, S_k \equiv_k S_n$. Consider now the sequence $\langle S_k^{(k)} \rangle$. Clearly $\langle S_k^{(k)} \rangle$ is a CS as

$S_k^{(k)} \equiv_k S_n^{(n)}, \forall n \geq k$. Since $S_k^{(k)}$ is bounded, $S_k^{(k)} \equiv \mathcal{E}(S_k^{(k)})$ by Lemma 3.6. Therefore, $\langle S_k^{(k)} \rangle$ has a limit iff $\langle \mathcal{E}(S_k^{(k)}) \rangle$ does. Since $S_k^{(k)} \equiv_k S_{k+1}^{(k+1)}$ we have $\mathcal{E}(S_k^{(k)}) \equiv_k \mathcal{E}(S_{k+1}^{(k+1)})$ so that $\mathcal{E}(S_k^{(k)})^{(k)} = \mathcal{E}(S_{k+1}^{(k+1)})^{(k)}$ by Theorem 3.5. Since $\mathcal{E}(S_k^{(k)})^{(k)} = \mathcal{E}(S_k^{(k)})$, we have that $\langle \mathcal{E}(S_k^{(k)}) \rangle$ is an XSS and has a limit (Theorem 3.3).

Finally, we observe that by construction $\langle S_k \rangle$ has the same limit as $\langle S_k^{(k)} \rangle$, completing the proof of the theorem.

At this point we would like to remark that our construction not only incorporates countably branching trees, but requires them for our space to be complete. That arbitrary finite branching is not enough can be seen from the following. Recall that A_j is the tree $aa \cdots a$ (j times). Now suppose that $S \equiv_k A_j$ for $j < k$. Then all paths in S must necessarily have length exactly j . For the case when $j = k$, all paths in S must have length at least j .

Now suppose $S \equiv_{k+1} A_*$, where we now write $A_* = \sum aA_j$ for j a natural number. Then for all $j \leq k$ there is a prefixed subtree aS_j of S such that $A_j \equiv_k S_j$. By the above, each S_j is different, establishing

LEMMA 3.8. *If $S \equiv_k A_*$, then S has at least a k -way branching root.*

THEOREM 3.9. *Without countable branching, $\langle \mathcal{E}, d_w \rangle$ is incomplete.*

Proof. $\langle A_1, A_1 + A_2, \dots \rangle$ is a finitely branching CS with limit A_* , which by the lemma is not equivalent to any finitely branching tree.

4. AN ISOMETRY WITH A METRIC SPACE OF DE BAKKER AND ZUCKER

En route to their denotational semantics of concurrency, de Bakker and Zucker wish to find a metric space $\langle P, d_B \rangle$ which solves

$$P \cong \{p_0\} \cup \mathcal{P}_c(\Sigma \times P), \tag{4.1}$$

where \mathcal{P}_c refers to the set of all subsets closed with respect to d_B . A space which works is the one obtained by completing the space $\langle \bigcup P_n, \bigcup d_n \rangle$, where:

DEFINITION. We let $\langle P_n, d_n \rangle$ be a series of metric spaces defined by

$$\begin{aligned} P_0 &= \{p_0\}, & p_0 & \text{is the nil process,} \\ P_{n+1} &= \{p_0\} \cup \mathcal{P}(\Sigma \times P_n), & \mathcal{P} & \text{is the power set operator,} \\ d_0(p, q) &= 0 & & \text{for all } p, q \in P_0, \end{aligned}$$

and

$$\begin{aligned}
 d_{n+1}(p, q) &= 0 && \text{for } p = q = p_0, \\
 &= 1 && \text{for } p = p_0 \text{ or } q = p_0, \text{ but not both,} \\
 &= \max(\sup_{p' \in p} \inf_{q' \in q} d'_{n+1}(p', q'), \sup_{q' \in q} \inf_{p' \in p} d'_{n+1}(p', q')) \\
 &&& \text{for both } p, q \subseteq \Sigma \times P_n,
 \end{aligned}$$

where

$$\begin{aligned}
 d'_{n+1}(p', q') &= 0 && \text{for } p' = q' = p_0, \\
 &= 1 && \text{for } p' = p_0 \text{ or } q' = p_0, \text{ but not both,} \\
 &= d_n(p'', q'')/2 && \text{for } p' = \langle a, p'' \rangle, q' = \langle b, q'' \rangle, \text{ and } a = b, \\
 &= 1 && \text{for } p' = \langle a, p'' \rangle, q' = \langle b, q'' \rangle, \text{ and } a \neq b.
 \end{aligned}$$

Note that d_{n+1} is the Hausdorff metric distance between the subsets of P_{n+1} induced by the metric d'_{n+1} on the points of P_{n+1} .

DEFINITION. Let $\langle P, d_B \rangle$ be the completion of $\langle \bigcup P_n, \bigcup d_n \rangle$.

THEOREM 4.1 (de Bakker and Zucker, 1982). $\langle P, d_B \rangle$ satisfies (4.1).

We wish to establish isometries between spaces of RSTs and spaces constructed under the de Bakker–Zucker (BZ) method above. If Σ is countable their solution space is quite large, for example, $\text{card}(P_1) = 2^\omega$ and $\text{card}(P_{n+1}) = 2^{\text{card}(P_n)}$. So in P_2 , for example, processes exist which exhibit uncountable nondeterminism, such as the process $\{\langle a, Q \rangle \mid Q \in P_1\}$. This necessarily precludes any isometry with an RST space; our construction, because it does not admit uncountable nondeterminism, leads to spaces with smaller cardinality. However, isometries do exist between two important RST spaces and appropriate BZ constructions: the space $\langle \mathcal{E}/\equiv, d_w \rangle$ with Σ finite and the same space with Σ countable (the space of Section 3). For the finite case the appropriate BZ construction is the obvious one over a finite alphabet. It turns out that this construction also satisfies the domain equation (4.1) over finite Σ . For the countable case we must restrict the power domain operator used in constructing the spaces P_n to the collection of all countable subsets. This is sufficient to induce an isometry as now, no P_n contains processes with uncountable nondeterminism.

DEFINITION. (1) Let $\langle P'_n, d'_n \rangle$ be like $\langle P_n, d_n \rangle$ except Σ is understood to be finite. Let $\langle P', d_B \rangle$ be the completion of $\langle \bigcup P'_n, \bigcup d'_n \rangle$.

(2) Let $\langle P''_n, d_n \rangle$ be like $\langle P_n, d_n \rangle$, where $\mathcal{P}(\cdot)$ is understood to be the collection of all countable subsets. Let $\langle P'', d_B \rangle$ be the completion of $\langle \bigcup P''_n, \bigcup d_n \rangle$.

The rest of this section will establish the isometries and investigate the domain equation in each context. We will establish the isometries through the quotient space of *reduced* trees. After the definitions we first show that bounded reduced trees exist and are unique, justifying their use as a quotient space.

DEFINITION. For any trees S and T , S *arbitrarily extends* T , $S \sqsubseteq T$, just when S can be embedded into T at the root, preserving structure and labeling:

$$S \sqsubseteq T \Leftrightarrow v_S \text{ for any representations of } S \text{ and } T, (N_S, E_S, l_S, v_S) \text{ and } (N_T, E_T, l_T, v_T),$$

there is an injection $\phi: N_S \rightarrow N_T$ such that $E'_S \subseteq E_T$ and $l_T|_{E'_S} = l_S$,

where $\langle u, v \rangle \in E'_S \Leftrightarrow \langle \phi^{-1}(u), \phi^{-1}(v) \rangle \in E_S$ and similarly for l'_S .

Assume henceforth for any tree S we have a fixed representation in mind. Whenever $S \sqsubseteq T$ via ϕ , we refer to ϕ as an *induced injection* between (the nodes of) S and T . If $\phi(u) = w$ then we say the nodes u and w are *associated*; we also refer to the subtrees rooted at u and w as associated. We say that ϕ *respects* weak equivalence, or that ϕ is *respectful* if it has the additional property that for any associated subtrees S' and T' we have $S' \equiv T'^{(k)}$, where $k = |S'|$. So, for example, if $S \equiv T$ and $S \sqsubseteq T$ via ϕ , ϕ is respectful if associated subtrees are equivalent.

DEFINITION. For bounded S and T , S is *reduced* with respect to \equiv and \sqsubseteq iff whenever $S \equiv T$ we have $S \sqsubseteq T$ via some induced injection respecting weak equivalence.

EXAMPLE. Consider the weakly equivalent trees $S = \sum aS_k$ for $k \geq 1$, where $S_k = 1 + \dots + k$ and $T = S + aS_1$. It is easy to see that S and T are equivalent under \sqsubseteq . However, S is reduced while T is not. (No injection from T to S establishing the \sqsubseteq -relation respects equivalence.)

We turn now to establishing the existence and uniqueness of reduced trees.

THEOREM 4.2. (Existence of reduced trees). *For bounded trees, within each weak equivalence class there exists a reduced tree with respect to \sqsubseteq .*

Proof. Clearly within any weak equivalence class all trees have the same height. So we proceed by induction over the height of the class.

Case $h = 0$. Immediate, as the only tree in this class is *nil*.

Assume for h .

Case $h + 1$. Let $|S| = h + 1$, where $S = \sum a_i S_i$. Let S' be the tree with exactly one prefixed reduced representative (as guaranteed by the induction hypothesis) for each equivalence class represented by the prefixed subtrees of S . Clearly $S \equiv S'$. Now for any $T \equiv S'$, the equivalence classes represented by the prefixed subtrees are the same. By construction S' has precisely one representative from every class. To show S' is reduced, we construct an injection ϕ respecting equivalence by first associating an equivalent prefixed subtree in T for every prefixed subtree in S' , say aT'' for aS'' , and then by letting ϕ assume the values of the equivalence respecting injection ϕ'' between S'' and T'' guaranteed by the induction hypothesis, that is, $\phi \upharpoonright N_{S''} = \phi''$.

To demonstrate uniqueness we need some preliminary results.

LEMMA 4.3. *If S is bounded and reduced then all of its prefixed subtrees are pairwise nonequivalent.*

Proof. Suppose not. Let T be S with only one prefixed subtree of S for each equivalence class represented by the prefixed subtrees of S . Clearly $T \equiv S$, but no \equiv -respecting injection between S and T can exist, contradicting the hypothesis that S is reduced.

LEMMA 4.4. *If S is bounded and reduced then so are all of its subtrees.*

Proof. It suffices to consider just prefixed subtrees. Let aS' be any prefixed subtree of S and let T' be a reduced tree equivalent to S' , as guaranteed by Theorem 4.2. Let T be the tree S with the prefixed subtree aS' replaced by aT' . Since S is reduced, any \equiv -respecting injection ϕ between S and T must associate S' and T' by Lemma 4.3. Furthermore as T' is reduced, for any $Q' \equiv T'$ there is an induced injection ϕ' between T' and Q' respecting equivalence. So now $\phi \upharpoonright N_{S'} \circ \phi'$ is an \equiv -respecting injection between S' and Q' . As Q' is arbitrary, S' is reduced.

THEOREM 4.5 (Uniqueness of reduced trees). *If S and T are bounded and reduced then $S \equiv T$ implies $S = T$.*

Proof. By induction on the heights of S and T .

Case $h = 0$. Then $S = \text{nil} = T$.

Assume for h .

Case $h + 1$. Let $S \equiv T$ and both be reduced, where $S = \sum a_i S_i$ and $T = \sum b_i T_i$. By Lemma 4.4, S_i and T_i are reduced for every i . Since $\{a_i S_i\}$

and $\{b_i T_i\}$ are pairwise nonequivalent by Lemma 4.3, the sets are necessarily in a 1-1 correspondence under \equiv . Now by the induction hypothesis the elements under the correspondence are isomorphic. Therefore $S = T$.

DEFINITION. For any bounded S , let $\mathcal{R}(S)$ designate the reduced tree equivalent to S . Let R_n denote the set of all reduced trees of height at most n . Let $\langle \mathcal{R}, d_w \rangle$ be the completion of $\langle \bigcup R_n, d_w \rangle$.

- EXAMPLES. (1) $S = a + a, \mathcal{R}(S) = a,$
 (2) $S = a + ab + ac + ab, \mathcal{R}(S) = a + ab + ac,$
 (3) $S = ab + a(b + b), \mathcal{R}(S) = ab.$

We note the follows consequences of the existence and uniqueness theorems.

- COROLLARY. (1) $S \equiv \mathcal{R}(S)$ for S bounded;
 (2) $\langle \mathcal{R}/\equiv, d_w \rangle = \langle \mathcal{E}/\equiv, d_w \rangle.$

Proof. We prove (2). Since \mathcal{R} is a closed subspace of \mathcal{E} , we have immediately $\langle \mathcal{R}/\equiv, d_w \rangle \subseteq \langle \mathcal{E}/\equiv, d_w \rangle$. Now let $S \in \mathcal{E}$. Recall that $S = \bigcup S^{(n)}$; we have $S^{(n)} \equiv \mathcal{R}(S^{(n)})$ by (1). Therefore, $\langle \mathcal{R}(S^{(n)}) \rangle$ is a CS in $\langle \mathcal{R}/\equiv, d_w \rangle$ and has some limit $T \in \mathcal{R}$. Clearly, $T \equiv S$.

With the reduced trees in hand we can proceed to establish the isometries. We first must verify

THEOREM 4.6. $\langle \mathcal{E}/\equiv, d_w \rangle$ is complete when Σ is finite.

Proof. The results of the last section can be quoted in toto as they only require $\text{card}(\Sigma) \leq \omega$.

As the proofs for establishing the isometries for either cardinality of Σ are identical we let $\mathcal{E}, \mathcal{R}, P$, etc., stand for both spaces in what follows. Since $\langle \mathcal{R}/\equiv, d_w \rangle = \langle \mathcal{E}/\equiv, d_w \rangle$, to demonstrate the isometry between $\langle \mathcal{E}/\equiv, d_w \rangle$ and $\langle P, d_B \rangle$ it will suffice to establish an isometry between $\langle \bigcup R_n, d_w \rangle$ and $\langle \bigcup P_n, d_n \rangle$ as then their completions, $\langle \mathcal{R}/\equiv, d_w \rangle$ and $\langle P, d_B \rangle$, will necessarily be isometric and thereby our desired result follows.

DEFINITION. Let $\phi: \bigcup R_n \rightarrow \bigcup P_n$ by $\phi(\text{nil}) = p_0; \phi(\sum a_i S_i) = \langle \langle a_i, \phi(S_i) \rangle \rangle$.

THEOREM 4.7. $\phi | R_n$ is a bijection between R_n and P_n .

Proof. Induction on n .

Case $n = 0$. Immediate.

Assume for n .

Case $n + 1$. ϕ is 1-1: Let $S, T \in R_{n+1}$, and suppose $\phi(S) = \phi(T)$.

Now if $\phi(S) = \phi(T) = p_0$, then by the induction hypothesis, $S = T = \text{nil}$. Suppose now that $\phi(S) = \phi(T) \neq p_0$. Then $S = \sum a_i S_i$ and $T = \sum b_j T_j$ for $S_i, T_j \in R_n$.

$$\therefore \{\langle a_i, \phi(S_i) \rangle\} = \{\langle b_j, \phi(T_j) \rangle\},$$

$$\therefore \forall i, \exists j \langle a_i, \phi(S_i) \rangle = \langle b_j, \phi(T_j) \rangle \text{ and vice versa,}$$

$$\therefore \text{by the induction hypothesis } \forall i, \exists j a_i S_i = b_j T_j \text{ and vice versa,}$$

$$\therefore \sum a_i S_i \equiv \sum b_j T_j \text{ and, as } S \text{ and } T \text{ are reduced, we have } S = T \text{ by Theorem 4.5,}$$

$$\therefore \phi \text{ is 1-1.}$$

ϕ is onto: Let $p \in P_{n+1}$. If $p = p_0$, choose $\phi^{-1}(p) = \text{nil}$. Else $p = \{\langle a_i, p_i \rangle\}$, where $p_i \in P_n$ and $\text{card}(p) \leq \omega$. By the induction hypothesis ϕ is onto P_n . Denote by $\phi^{-1}(p_i)$ the unique (ϕ is 1-1) element of R_n such that $\phi(\phi^{-1}(p_i)) = p_i$. Let $\phi^{-1}(p) = \sum a_i \phi^{-1}(p_i)$. Because $\phi^{-1}(p_i) \in R_n$ we have $a_i \phi^{-1}(p_i) \in R_{n+1}$ and as $\text{card}(\{a_i \phi^{-1}(p_i)\}) \leq \omega$ we have $\phi^{-1}(p) \in \mathcal{E}$. It remains to show $\phi^{-1}(p)$ is reduced. All prefixed subtrees of $\phi^{-1}(p)$ are mutually nonisomorphic, else then we would have $a_i \phi^{-1}(p_i) = a_j \phi^{-1}(p_j)$ for $i \neq j$, which means $\langle a_i, p_i \rangle = \langle a_j, p_j \rangle$, a contradiction. As each prefixed subtree is reduced, each represents a different equivalence class (Theorem 4.5). For any $S \equiv \phi^{-1}(p)$ a respectful injection can easily be constructed. Therefore, $\phi^{-1}(p) \in R_{n+1}$. Furthermore, $\phi(\phi^{-1}(p)) = \{\langle a_i, p_i \rangle\} = p$, $\therefore \phi$ is onto.

COROLLARY. ϕ is a bijection between $\cup R_n$ and $\cup P_n$.

THEOREM 4.8. ϕ is an isometry between $\cup R_n$ and $\cup P_n$.

Proof. We shall establish that $\forall S, T \in R_n, d_w(S, T) = d_n(\phi(S), \phi(T))$ from which the conclusion follows. We proceed by induction on n .

Case $n = 0$: $d_w(S, T) = 0$ since $S = T = \text{nil}$ and $d_0(\phi(S), \phi(T)) = d_0(p_0, p_0) = 0$.

Assume for n .

Case $n + 1$: we shall establish

$$\forall S, T \in R_{n+1}, d_w(S, T) = 2^{-k} \Leftrightarrow d_{n+1}(\phi(S), \phi(T)) = 2^{-k}.$$

We let $S = \sum a_i S_i$ and $T = \sum a_j T_j$.

Induction on k .

Case $k = 0$. $d_w(S, T) = 0 \Leftrightarrow S = T$ (Theorem 4.5) $\Leftrightarrow \phi(S) = \phi(T) \Leftrightarrow d_{n+1}(\phi(S), \phi(T)) = 0$.

Assume for k .

Case $k + 1$. We know $d_w(S, T) = 2^{-(k+1)} \Leftrightarrow S \equiv_{k+1} T$ and $S \not\equiv_{k+2} T$. We claim that $S \equiv_{k+1} T \Leftrightarrow d_{n+1}(\phi(S), \phi(T)) \leq 2^{-(k+1)}$. If the claim is established, then the induction and theorem follow as

$$\begin{aligned} d_w(S, T) = 2^{-(k+1)} &\Leftrightarrow 2^{-(k+2)} < d_{n+1}(\phi(S), \phi(T)) \leq 2^{-(k+1)} \\ &\Leftrightarrow d_{n+1}(\phi(S), \phi(T)) = 2^{-(k+1)}. \end{aligned}$$

The first inequality above arises from the claim and the fact that $S \not\equiv_{k+2} T$. It remains to establish the claim.

Claim. $S \equiv_{k+1} T \Leftrightarrow d_{n+1}(\phi(S), \phi(T)) \leq 2^{-(k+1)}$.

Proof. (\Rightarrow) $S \equiv_{k+1} T \Leftrightarrow \forall a \forall S' S \rightarrow^a S' \Rightarrow \exists T' T \rightarrow^a T'$ and $S' \equiv_k T'$ and vice versa;

$$\begin{aligned} \therefore d_n(\phi(S'), \phi(T')) &\leq 2^{-k} \text{ by the induction hypotheses for } n \text{ and } k, \\ \therefore d'_{n+1}(\phi(aS'), \phi(aT')) &\leq 2^{-(k+1)}, \\ \therefore \inf_j d'_{n+1}(\phi(aS'), \phi(a_j T_j)) &\leq 2^{-(k+1)}. \end{aligned}$$

Since $S \equiv_{k+1} T$, $\forall i \inf_j d'_{n+1}(\phi(a_i S_i), \phi(a_j T_j)) \leq 2^{-(k+1)}$,

$$\text{we have } \sup_i \inf_j d'_{n+1}(\phi(a_i S_i), \phi(a_j T_j)) \leq 2^{-(k+1)}.$$

A similar argument establishes $\sup_j \inf_i d'_{n+1}(\cdot, \cdot) \leq 2^{-(k+1)}$,

$$\therefore d_{n+1}(\phi(S), \phi(T)) \leq 2^{-(k+1)}.$$

(\Leftarrow) Suppose now $d_{n+1}(\phi(S), \phi(T)) \leq 2^{-(k+1)}$. Then

$$\begin{aligned} \sup_i \inf_j d'_{n+1}(\phi(a_i S_i), \phi(b_j T_j)) &\leq 2^{-(k+1)}, \\ \therefore \forall i, \exists j d'_{n+1}(\phi(a_i S_i), \phi(b_j T_j)) &\leq 2^{-(k+1)}, \\ \therefore \forall i, \exists j a_i = b_j \text{ and } d_n(\phi(S_i), \phi(T_j)) &\leq 2^{-k}. \end{aligned}$$

By applying the induction hypotheses for n and k we have $d_w(S_i, T_j) \leq 2^{-k}$ and so it follows $S_i \equiv_k T_j$. Therefore, one half of the definition of $(k + 1)$ -equivalence is satisfied. We obtain the other half from

$\sup_j \inf_i d'_{n+1}(\cdot, \cdot) \leq 2^{-(k+1)}$. This completes the proof of the claim and the theorem.

Let us consider now the relationship of $\langle P', d_B \rangle$ and $\langle P'', d_B \rangle$, the spaces constructed over finite and countable alphabets, respectively, to the domain equation (4.1). In de Bakker and Zucker (1982), Σ was arbitrary and, therefore, we can say $\langle P', d_B \rangle$ solves (4.1) for finite Σ . As for $\langle P'', d_B \rangle$ the construction took only countable subsets at each stage. Consider the following CS in $\langle P'', d_B \rangle$:

$$X_0 = \{p_0\}, \quad X_{n+1} = \{p_0\} \cup \{\langle 0, X' \rangle, \langle 1, X' \rangle \mid X' \in X_n\}.$$

Informally we may think of X_n as the set of all sequences over $\{0, 1\}$ with length at most n . Via the isometry we associate with each X_n the tree S_n consisting of each sequence joined at the root, for example, $S_2 = 0 + 1 + 00 + 01 + 10 + 11$. As representatives of $\lfloor \lim S_n \rfloor$ in $\langle \mathcal{R}'' / \equiv, d_w \rangle$ we may choose $\sum S_n$ or (more judiciously) the tree consisting of just the join of all the finite sequences. Both are countably branching trees. However, the BZ construction has a specific limit representative in mind, the one closed with respect to d_B , which gives for $\lim X_n$ the uncountable set of all finite and infinite sequences. (Note the countably branching tree suggested by the isometry is weakly equivalent to the limit representatives above.) We feel the space of reduced trees will satisfy a domain equation like (4.1) given a suitable extension of the notion of reduced trees to the unbounded case.

5. A CONNECTION WITH PROGRAMMING LOGIC

In this section we treat the case when our RSTs are labeled from a finite set Σ . We introduce the small modal logic HML (Hennessy and Milner, 1980). It turns out that for any trees S, T , $S \equiv T$ iff for every $\phi \in \text{HML}$, $S \models \phi \Leftrightarrow T \models \phi$. We exploit this fact to show that completeness of the space $\langle \mathcal{E}, d_w \rangle$ is a consequence of the compactness theorem for HML. This theorem in turn follows from the compactness theorem for first-order logic, so we have an alternative proof of completeness in this case. Finally, we observe that if our metric space is compact, then the HML compactness theorem follows as a consequence.

These results are in a sense already known in model theory. The relation \equiv can be defined on arbitrary first-order structures, and the equivalence $\mathcal{A} \equiv \mathcal{B}$ iff for all sentences ϕ , $\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi$ is part of the Ehrenfeucht–Fraïssé characterization of elementary equivalence (Monk, 1976); HML can be considered as a fragment of first-order logic and the general theory applied. However, the proofs in the HML case are simple and revealing, so we think it worthwhile to present them here.

DEFINITION. The set of formulas HML is given by the inductive clauses,
 $tt, ff \in \text{HML}$ (two Boolean constants),
 $\phi, \psi \in \text{HML}$ imply $\phi \wedge \psi \in \text{HML}$ and $\neg\phi \in \text{HML}$ (Boolean operations),
 $\phi \in \text{HML}$ and $a \in \Sigma$ imply $a\langle\phi\rangle \in \text{HML}$ (“possible” modality).

The formula $a\langle\phi\rangle$ is to be read: “From the initial state (root) it is possible to execute the atomic action a and arrive in a state satisfying ϕ .” Note: Σ is henceforth finite.

DEFINITION (Semantics of HML). Let S be an RST over Σ , and let $\phi \in \text{HML}$. We say S satisfies ϕ ($S \models \phi$) in case we can apply the inductive clauses

$$\begin{aligned} S \models tt & \quad \text{always;} \\ S \models ff & \quad \text{never;} \\ S \models \phi \wedge \psi & \quad \text{iff } S \models \phi \text{ and } S \models \psi; \\ S \models \neg\phi & \quad \text{iff not } (S \models \phi); \\ S \models a\langle\phi\rangle & \quad \text{iff } (\exists S') (S \xrightarrow{a} S' \text{ and } S' \models \phi). \end{aligned}$$

We proceed to develop some facts about HML and the relation \equiv .

DEFINITION. The depth $|\phi|$ of an HML formula is given by

$$\begin{aligned} |tt| = |ff| &= 0; & |\phi \wedge \psi| &= \max(|\phi|, |\psi|); \\ |\neg\phi| &= |\phi|; & |a\langle\phi\rangle| &= 1 + |\phi|. \end{aligned}$$

Let $\text{HML}_n = \{\phi \mid |\phi| \leq n\}$.

LEMMA 5.1. For all T, U , and n , if $T \equiv_n U$ then for all $\phi \in \text{HML}_n$ ($T \models \phi \Leftrightarrow U \models \phi$).

Proof. Easy induction on n .

The converse of Lemma 5.1 requires a little work, and is false unless Σ is finite.

DEFINITION. Two HML formulas ϕ, ψ are logically equivalent iff for all T , $T \models \phi$ iff $T \models \psi$.

LEMMA 5.2. *For each n , the relation of logical equivalence restricted to HML_n has only finitely many equivalence classes.*

Proof. Use induction on n ; the proof amounts to finding a DNF for the formulas in HML_n . Here the finiteness of Σ must be used.

THEOREM 5.3. *For any n , and any T, U , if for all $\phi \in \text{HML}_n$, $T \models \phi \Leftrightarrow U \models \phi$, then $T \equiv_n U$.*

Proof. Again, by induction on n . The result is clear when $n = 0$. Assume it for k , and all T', U' , and $\phi \in \text{HML}_k$. Suppose $T \rightarrow^a T'$. Let

$$F_k = \{\Theta_1, \dots, \Theta_p\}$$

be the set of representatives of the equivalence classes of logical equivalence restricted to HML_k , and suppose $\Theta_1, \dots, \Theta_i$ are the formulas in F_k satisfied by T' . Then $T \models a(\Theta_1 \wedge \dots \wedge \Theta_i \wedge \neg \Theta_{i+1} \wedge \dots \wedge \neg \Theta_p)$. This is a formula in HML_{k+1} , so by hypothesis, U satisfies it too. This gives a tree U' with $U \rightarrow^a U'$ and T' and U' satisfying exactly the same formulas in F_k . Since F_k is a complete set of representatives for logical equivalence, T' and U' satisfy exactly the same HML_k formulas. By inductive hypothesis, $T' \equiv_k U'$. The case $U \rightarrow^a U'$ is, of course, exactly similar, so the proof of Lemma 5.3 is complete.

COROLLARY 5.4. $S \equiv T$ iff $\forall \phi \in \text{HML}$, $S \models \phi \Leftrightarrow T \models \phi$.

COROLLARY 5.5 ("Master formula" theorem for HML). *For each $n \geq 0$ and each T , there is a formula $\phi(n, T)$ such that*

- (i) $T \models \phi(n, T)$.
- (ii) *For any U , if $U \models \phi(n, T)$ then $U \equiv_n T$.*

Proof. As in Theorem 5.3 let F_n be a representative system for logical equivalence in HML_n . Given T , let

$$\begin{aligned} \phi(n, T) = & \bigwedge \{ \phi \in F_n \mid T \models \phi \}, \\ & \wedge \bigwedge \{ \neg \phi \mid \phi \in F_n \text{ and not } T \models \phi \}. \end{aligned}$$

Clearly, $T \models \phi(n, T)$. Further, if $U \models \phi(n, T)$ then U and T agree on all formulas in F_n and thus on HML_n . The result follows from Theorem 5.3.

THEOREM 5.6 (Compactness theorem for HML). *Let $\Gamma \subseteq \text{HML}$. If for any finite $\Delta \subseteq \Gamma$ there is a tree T such that $T \models \phi$ for all $\phi \in \Delta$, then there is a tree U such that for all $\phi \in \Gamma$, $U \models \phi$.*

Proof. We translate (the semantics of) HML into first-order logic. For each $a \in \Sigma$ let \mathbf{a} be a binary relation symbol, and let \mathbf{k} be a constant symbol. Let L be the first-order language determined by these symbols.

For each $\phi \in \text{HML}$, we define a formula $\phi^* \in L$ with at most one free variable. Let tt^* be some fixed tautological sentence in L , and let $ff^* = \neg(tt^*)$. Further define

$$(\phi \wedge \psi)^* = \phi^* \wedge \psi^*, \quad (\neg\phi)^* = \neg(\phi^*),$$

$$(a\langle\phi\rangle)^* = \exists y (\mathbf{a}(x, y) \wedge \phi^*(y)),$$

where y is the free variable in ϕ^* (if one exists) and x is a new free variable.

For any set Γ of formulas in HML, let

$$\Gamma^* = \{\phi^*(\mathbf{k}) \mid \phi \in \Gamma\}.$$

Now Γ^* is a set of sentences in L , and it is easy to show that Γ^* has a countable model if and only if Γ has a countable tree model. Now Theorem 5.6 follows from the compactness theorem for first-order logic.

We can now prove that $\langle \mathcal{E}, d_w \rangle$ is a complete metric space. Let $\langle T_k \rangle$ be a Cauchy sequence of trees. By passing to a subsequence if necessary, we may assume that for all k , $T_k \equiv_k T_{k+1}$. Now define

$$\Gamma = \{\phi(k, T_k) \mid k \geq 1\},$$

where the $\phi(k, T_k)$ are given by Corollary 5.5. We claim that for any U , if $U \models \phi(k, T_k)$, then for any $j \leq k$, $U \models \phi(j, T_j)$. The proof is by induction on k , and $k = 0$ is trivial. Now if $U \models \phi(k + 1, T_{k+1})$ then by Corollary 5.5, $U \equiv_{k+1} T_{k+1}$. Since $T_{k+1} \equiv_k T_k$, we have $U \equiv_k T_k$. But $|\phi(k, T_k)| \leq k$, so by Lemma 5.1, $U \models \phi(k, T_k)$. The claim follows by induction.

From the claim, if Δ is a finite subset of Γ , then Δ has a tree model. By Theorem 5.6, Γ has a tree model T ; that is, $T \models \phi(k, T_k)$ for all k . By the corollary again, we have $T \equiv_k T_k$ for all k ; that is, $d_w(T, T_k) \rightarrow 0$ as desired.

Finally, we observe that from the compactness of $\langle \mathcal{E}, d_w \rangle$ we can derive the compactness theorem for HML. Let Γ be an arbitrary set of formulas such that every finite subset has a tree model. Enumerate $\Gamma = \{\phi_1, \phi_2, \dots\}$. For each i let Δ_i be the set $\{\phi_1, \dots, \phi_i\}$. Then each Δ_i has a tree model T_i . Since $\langle \mathcal{E}, d_w \rangle$ is compact, the sequence $\langle T_i \rangle$ has a convergent subsequence, say to some tree T . It is easy to see that T is a tree model for Γ . (The compactness of $\langle \mathcal{E}, d_w \rangle$ can be proved directly. One need only show completeness as in the previous sections, and the use the fact that Σ is finite to show that for any ε , a finite number of ε -spheres cover \mathcal{E} .)

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