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INFINITE EULERIAN TESSELLATIONS

J.L. BRENNER and R.C. LYNDON*

10 Phillips Road, Palo Alto, CA 94303, USA and
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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An *Eulerian path* in a graph G is a path π such that (1) π traverses each edge of G exactly once in each direction, and (2) π does not traverse any edge once in one direction and then immediately after in the other direction. A tessellation T of the plane is *Eulerian* if its 1-skeleton G admits an Eulerian path. It is shown that the three regular tessellations of the Euclidean plane are Eulerian. More generally, if T is a tessellation of the plane such that each face has at least p sides and each vertex has degree (number of incident edges) at least q, where $1/p+1/q\leq\frac{1}{2}$, then, except possibly for the case $p=3$ and $q=6$, T is Eulerian. Let T^* be the truncation of T. If every vertex of T has degree 3, then T^* is not Eulerian. If every vertex has degree 4, or degree at least 6, then T is Eulerian.

0. Introduction

Our central result is the following. Let T be the regular tessellation of the Euclidean plane by hexagons, and let G be the graph whose vertices are those of the hexagons and whose edges are the sides that separate pairs of hexagons. Then the graph G possesses an (infinite) Eulerian path π in the following sense:

(0.1) Whenever π enters a vertex v along some side at v, it leaves along a different side at v.

(0.2) π traverses each side in G exactly once in each direction.

It should be noted that we follow the usage (see, e.g., Serre [11, 12]) whereby each edge of a graph is directed, and possesses a unique oppositely directed inverse edge. Under this usage, our definition of an Eulerian path is the natural one, although it differs from that commonly used in the study of graphs with undirected edges.

This result is extended to all combinatorially regular tessellations T of the Euclidean or hyperbolic plane. Let p and q be integers greater than 2. It is well known that a regular tessellation of the plane by p-gons, with q meeting at each vertex, exists if and only if $1/p+1/q\leq\frac{1}{2}$, in which case it can be taken metrically regular, that is, with all p-gons regular and congruent under the Euclidean or hyperbolic metric. For such T, the 1-skeleton G admits an Eulerian path in the

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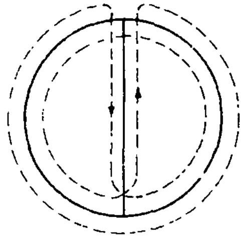


Fig. 1.

sense above. We also extend this result to all but one of the irregular tessellations subject, for the same pairs p and q, to the weaker conditions that each face have at least p sides and that at least q faces meet at each vertex. We obtain a few further results concerning truncated tessellations, and we conclude with a few unsettled problems.

The hexagonal graph G described above is *cubic*, in the sense that exactly three edges emanate from each vertex. Our interest in cubic Eulerian graphs arose from their connection with maximal nonparabolic subgroups of the modular group [2, 3, 4, 6, 14, 15, 16]. It is easy to construct finite cubic Eulerian graphs on any (necessarily) even finite number of vertices; the smallest such graph, on two vertices, is shown in Fig. 1, where an Eulerian path is indicated by a broken line. It seems nonetheless very difficult to obtain any useful enumeration or catalog of all finite cubic Eulerian graphs. It is also easy to construct (uncountably many) infinite cubic Eulerian graphs by piecing together finite graphs. However, the graphs described below are the only infinite Eulerian graphs we know of that could be said to arise naturally.

Despite the simple nature of the hexagonal graph G, the Eulerian paths π provided by our construction are anything but natural, in the sense that our construction permits infinitely many more or less arbitrary choices. It seems fairly clear that an Eulerian path on the graph G cannot have any simple geometric form. Indeed, π must be qualitatively a 'doubled spiral' in the following sense: there is a strictly ascending chain of discs $D_0, D_1, \ldots,$ in the plane such that, for each n, all edges inside D_n are traversed by a segment π_{n+1} of π lying inside D_{n+1}. The possibility remains that π could be chosen with a simple 'local' structure in the following sense: there is a (finite) algorithm (a 'maze threading algorithm'—see Rosenstiehl [10]) and a constant B, such that, given any segment π_1 of π, between vertices v_1 and v_2, the algorithm determines the next edge after v_2 as a function of the part of π_1 lying within a distance B of v_2. But even this seems unlikely.

1. Preliminaries

With minor exceptions, all graphs G considered here will arise from tessellations T of the plane. We accordingly formulate our discussion in terms of tessellations. We define a tessellation T of the plane to be a locally finite 2-complex that subdivides the plane. The 0-cells, 1-cells, and 2-cells will be called *vertices, sides,* and *faces.* The *associated graph* G is the graph whose geometric support is the 1-skeleton of T. The *vertices* of G are those of T. The (directed) *edges* of G come in inverse pairs, e and e^{-1}, obtained by taking each side of T with each of its two orientations. Technically, then, G as abstract graph consists of the set V of vertices, the set E of edges, an involution without fixed element on E carrying each edge to its inverse, and a function assigning to each edge e its initial vertex $\varepsilon(e)$. Note that e runs from $\varepsilon(e)$ to $\varepsilon(e^{-1})$. These graphs G arising from tessellations are *symmetric*, in the sense that to each edge e from a vertex p to a vertex q, there corresponds a unique *inverse edge* $e^{-1} \neq e$ from q to p. *All graphs considered in this paper will be assumed to be symmetric graphs.*

A *path* α can be either *open*, that is, a finite, simply infinite, or doubly infinite sequence of edges, or *closed*, that is, a cyclically ordered sequence of edges. It is required in either case that the edge e' following an edge e begin where e ends. A path α is *reduced* if no edge is followed by its inverse. A path α in G is a *reduced doubly Eulerian path* if it is a reduced path that traverses each (directed) edge of G exactly once. *In this paper the term Eulerian path will be used always as an abbreviation for reduced doubly Eulerian path.*

All *subcomplexes* T_0 of T considered here consist of a finite nonempty set of faces of T, together with all the sides and vertices on the boundaries of these faces. In fact, T_0 will always be connected, and if it is also simply connected, we call it a *disc* in T; in this case its boundary is a simple closed path ∂T_0.

1.1. Definition. A tessellation T is *concentric* if it is the union of an ascending chain of discs $D_0, D_1, \ldots$, satisfying the following conditions:

(1.11) D_0 consists of a single face.

(1.12) For each $n \geq 1$, $A_n = D_n - D_{n-1}$ is an annular chain of faces $F_1, \ldots, F_t$ in cyclic order, where, for each i:

(1.121) F_i and F_{i+1} have a side in common,

(1.122) F_i has at least one side on ∂D_{n-1},

(1.123) F_i has at least two sides on ∂D_n.

Note that (1.123) is a little stronger than might seem natural; its form is dictated by the requirement of the arguments that follow. Note also that these conditions imply that A_n is indeed a combinatorial annulus, that is, its support is homeomorphic to a metric annulus. Its boundary has two disjoint components, each a simple closed curve: an inner component $\delta_{n-1} = \partial D_{n-1}$ and an outer component $\delta_n = \partial D_n$.

1.2. Proposition. *The hexagonal tessellation T is concentric.*

Proof. This becomes intuitively clear from inspection of Fig. 2. It also follows from Proposition 3.3 below, for which a precise and explicit proof is given. □

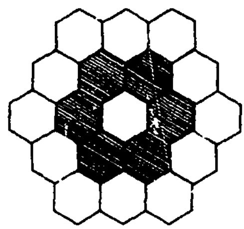


Fig. 2.

2. The hexagonal tessellation

2.1. Theorem. *The hexagonal graph G possesses an Eulerian path.*

In view of Proposition 1.2, this theorem follows from the following one.

2.2. Theorem. *If T is a cubic concentric tessellation of the plane, then the associated graph G admits an Eulerian path.*

We begin the proof with the following definition.

2.3. Definition. If T_0 is a subcomplex of a cubic tessellation T, then an *orientation* ω of T_0 is a function assigning to each vertex v of T_0 a value $+1$ or -1. The *Eulerian system* Σ_ω determined by the orientation ω consists of all paths α on T_0 that satisfy the following rule: if α enters a vertex v along an edge e, then, immediately after, α leaves v along the edge e' immediately following e in the positive or negative cyclic order about v (provided e' belongs to T_0), according as $\omega(v) = +1$ or $\omega(v) = -1$. (See Fig. 3, where a solid dot indicates a positively oriented vertex and a hollow dot a negatively oriented vertex.)

It is immediately clear that if ω is an orientation of T_0, then each edge in the associated graph G_0 occurs exactly once in some path α in Σ_ω.

Our goal is to construct inductively a chain of discs $D_0, D_1, \ldots$, whose union is T, and orientations ω_n of the D_n, where each ω_{n+1} is an extension of ω_n, such that

Fig. 3.

the Eulerian systems $\Sigma_n = \Sigma_{\omega_n}$ have the following properties:

(2.31) Σ_n contains no closed path.

(2.32) For n even, all paths α in Σ_n are contained as segments in one common path π_{n+2} in Σ_{n+2}.

It is clear that under these conditions the union ω of the ω_n is an orientation of T determining a system Σ consisting of a single path π, the union of the π_n, which is therefore an Eulerian path.

For each disc D_n, we define the *fringe* D_n^f to consist of all edges with exactly one end in D_n, and we write $D_n^* = D_n \cup D_n^f$. It is clear that an orientation ω_n on D_n in fact determines an Eulerian system on D_n^*, which we continue to call Σ_n.

We choose the D_n as in (1.1); D_0 is a single face and D_{n+1} is D_n together with all further faces having a side in common with D_n. We choose ω_0 by orienting a single vertex of the face D_0 positively and all the other vertices negatively. The resulting Eulerian system Σ_0 is indicated in Fig. 4.

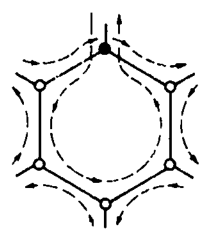


Fig. 4.

For the inductive step we suppose that n is even, $n \geqslant 0$, and that ω_n is given, with the property that ω_n contains no closed path. Then each path α in Σ_n begins in some side of the fringe D_n^f and ends in some side of D_n^f. Moreover, for each side s in D_n^f, there is a unique path in Σ_n beginning in s, and a unique path in Σ_n ending in s.

Let F be any face of the annulus $A_{n+1} = D_{n+1} - D_n$. Then ∂F is described in the negative sense by a simple closed path of the form $e_1 \gamma e_2^{-1} \delta$ where δ is in δ_n, γ is in δ_{n+1}, and e_1 and e_2 are edges running outward along two distinct and adjacent sides s_1 and s_2 in D_n^f. See Fig. 5.

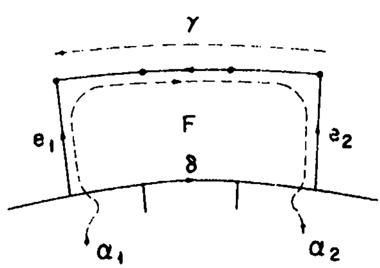


Fig. 5.

Let α_1 be the path in Σ_n that ends with e_1 and α_2 the path that begins with e_2^{-1}. Suppose now that $\alpha_1 \neq \alpha_2$. Let D_n' be the result of adding the face F to D_n^*, and let ω_n' be the extension of ω_n obtained by orienting all vertices on γ positively. Then the resulting Eulerian system Σ_n' on D_n' differs from Σ_n only in that the two paths α_1 and α_2 of Σ_n have been united into a single path $\alpha_1\gamma\alpha_2$ in Σ_n', and that a new path running along γ^{-1} has been added.

We now iterate this construction, adding faces $F_1, F_2, \ldots,$ in A_{n+1} to D_n^* and extending ω_n in the above manner, until no more such faces can be added without producing a closed path in the resulting Eulerian system. We now let D_n' denote the result of this iteration. Now D_n' cannot be all of D_{n+1}; for one thing, if this were the case, all the paths such as γ in Fig. 5 would be united into a single closed path in Σ_n', running along δ_{n+1}.

Now let F be a face in D_{n+1} but not in D_n', and resume the notation of Fig. 5. Suppose again that $\alpha_1 \neq \alpha_2$. Then adding F to D_n' would result in uniting α_1 and α_2 into a single path $\alpha_1\gamma\alpha_2$, which would not be a closed path. The other change in Σ_n' would result from inserting the segment γ^{-1}, joining two (possibly empty) paths γ_1 and γ_2 in Σ_n' running along δ_{n+1}. By the maximality of D_n', the resulting path must be closed, and this could happen only if $\alpha_1 = \alpha_2$, a path running along all of δ_{n+1} except γ. In this case, F must be the only face of D_{n+1} not in D_n', whence s_1 is the only side in D_n' on which any α in Σ_n' ends and s_2 the only side in D_n' on which any α in Σ_n' begins. But this implies that $\alpha_1 = \alpha_2$, contrary to hypothesis. We have shown that, for all faces F as above, $\alpha_1 = \alpha_2$.

Let $F_1, \ldots, F_t$ be all faces of D_{n+1} not in D_n', taken in cyclic order around A_{n+1} in the positive sense. From the argument just given we know that for each F_i there is a path α_i in Σ_n' beginning on the right side of F_i (separating F_i from F_{i-1}) and ending on the left side of F_i (separating F_i from F_{i+1}). Since each path α in Σ_n is contained in some α_i, our goal is to unite all of the paths α_i into a single path π_{n+2} in Σ_{n+2}. We require also that Σ_{n+2} contain no closed path, but, for the moment, we relax this condition and seek to construct an orientation ω^* of D_{n+2} such that all the α_i are united into a single *closed* path π^* in the resulting system Σ^*, and that Σ^* contain no other closed path. For this purpose we propose to link $\alpha_1, \ldots, \alpha_t$ consecutively in cyclic order.

From condition (1.123) in the definition of a concentric tessellation we know that each F_i has a vertex v_i on δ_{n+1} that is joined by a side t_i to a vertex w_i on δ_{n+2}. For each face F_i we choose such v_i, t_i, and w_i. We now extend the orientation ω_n' of D_n' to an orientation ω^* of D_{n+2} as follows:

(2.41) We orient the v_i negatively, and all other new vertices on δ_{n+1} positively.

(2.42) We orient the w_i negatively, while every other vertex u on δ_{n+2} is oriented (i) positively if there is a side at u running inward to δ_n, but (ii) negatively if there is a side at u running outward to δ_{n+3}.

We first verify that all of the α_i are joined in a single path π^* in Σ^*. For this it suffices to show that each α_i is joined immediately to α_{i+1}. Reference to Figure 6 and to conditions (2.41) and (2.42) shows that, after leaving α_i, the path π^* in Σ^*

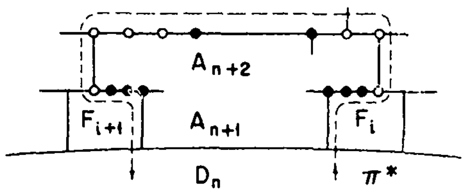


Fig. 6.

containing α_i runs along the part of the boundary of F_i lying on δ_{n+1} until it comes to u_i. Here it turn outward along t_i to the vertex w_i on δ_{n+2}, where, by virtue of (2.42), it turns left again and runs along δ_{n+2} until it encounters w_{i+1}. Here it turns inward along t_{i+1} to u_{i+1}, and then runs along the boundary of F_{i+1} until it meets α_{i+1}, after which it follows α_{i+1}.

We must show also that Σ^* contains no closed path other than π^*. The discussion above shows that every edge in D_n^* is contained in π^*. Thus every path α in Σ^* other than π^* must be contained in A_{n+2}. Further, every edge in δ_{n+1}, taken in the negative sense, already lies in π^*. Thus, if α were a closed path in Σ^* other than π^*, then α would have to contain a segment of δ_{n+1}, traversed in the positive sense, and a segment σ of δ_{n+2} in the negative sense. In particular, σ would have to run in the negative sense along the part τ of the boundary of some face F that lies on δ_{n+2}. By (1.123), τ contains a vertex w at which there is a side t running outward to δ_{n+3}. By (2.42), w is negatively oriented. But this would compel α, or arriving at w, to turn left and hence to terminate on t, in the fringe D_{n+2}^{f} of D_{n+2}. This contradicts the assumption that α is a closed path. Fig. 7 shows typical paths α in Σ^*, other than π^*.

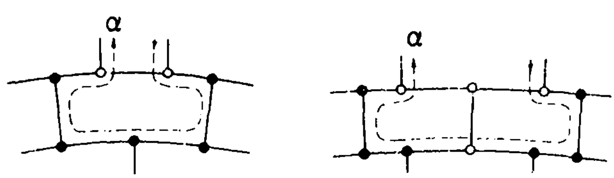


Fig. 7.

We have now constructed ω^*, Σ^*, and π^* with the desired properties. We next obtain Σ_{n+2} from Σ^* by changing the orientation of ω^* at a single vertex. Let F be any face in A_{n+2} lying to the left of a side t_1. Let s be the side of F following t_1, hence running left along δ_{n+2} from w_1 to a vertex w. By (1.123), there must be a side t at w running outward toward δ_{n+3}. By (2.42), ω^* orients w negatively. We obtain ω_{n+2} from ω^* by changing the orientation at this single point w to positive. This has the effect of replacing the path π^* in Σ^* by a path π_{n+2} in Σ_{n+2} that begins and ends on the sides t. The only other effect is to replace the other two paths α and β of Σ^* that pass through w, which are distinct, by a single path γ in Σ_{n+2}. This change is indicated in Fig. 8.

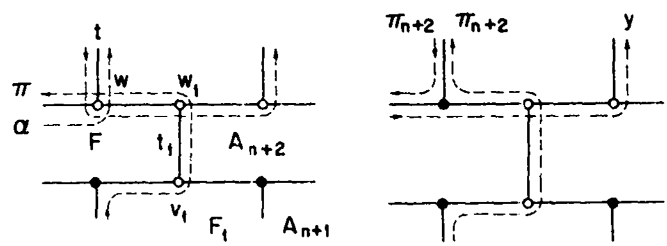


Fig. 8.

Since Σ_{n+2}, with its path π_{n+2}, now satisfies condition (2.32), the inductive step is complete, and Theorem 2.2 is proved. $\square$

3. Hyperhexagonal tessellations

We assume now that T is a cubic tessellation in which each face F has $d(F) \geqslant 6$ sides. We shall show that T is concentric, whence T has an Eulerian path.

3.1. Definition. If T_0 is a finite disc in T, then a *front* e of ∂T_0 is a connected component of some nonempty intersection $\partial F \cap \partial T_0$, where F is a face of T_0. We write $|e|$ for the number of sides in e, and we define $\sigma(T_0) = \sum (|e| - 2)$, summed over all fronts e in ∂T_0.

3.2. Lemma. *If T_0 contains more than a single face, then $\sigma(T_0) \geqslant 6$.*

Proof. Since T is cubic, each front contains at least one side, that is, $|e| \geqslant 1$. We argue by induction on the number $n \geqslant 2$ of faces in T_0. If $n = 2$, then the disc T_0 consists of two faces F_1 and F_2 with a single side s in common, and there are exactly two fronts, $e_1 = \partial F_1 - s$ and $e_2 = \partial F_2 - s$. Since $d(F_1)$, $d(F_2) \geqslant 6$, we have $|e_1|, |e_2| \geqslant 5$, whence $\sigma = \sigma(T_0) \geqslant 2(5-2) = 6$.

Let $n \geqslant 3$. Now T_0 contains some face F_0 such that $\partial F_0 \cap \partial T_0$ is a single front e_0. Let T_0' be obtained from T_0 by removing the face F_0. Then F_0 meets T_0' along an arc $\alpha = \partial F_0 - e_0$. By the induction hypothesis, $\sigma' = \sigma(T_0') \geqslant 6$, and it suffices to show that $\sigma \geqslant \sigma'$.

Let the arc α, from v_1 to v_2, consist of $k \geqslant 1$ consecutive sides $s_1, \ldots, s_k$. Since each of v_1 and v_2 lies on two sides of $\partial T_0'$ and a side of F_0, and has degree 3, each must be interior to some front of $\partial T_0'$. Suppose first that $k = 1$. Then s_1 is interior to a front $e = e's_1e''$ of $\partial T_0'$, where $|e'|, |e''| \geqslant 1$. In passing from $\partial T_0'$ to ∂T_0, the front e is replaced by three fronts e', e'', and e_0. Thus a term

$$|e| - 2 = |e'| + |e''| + 1 - 2 = |e'| + |e''| - 1$$

in σ' is replaced in σ by a sum of three terms,

$$(|e'| - 2) + (|e''| - 2) + (|e_0| - 2) = |e'| + |e''| + (d(F_0) - 1) - 6 \geqslant |e'| + |e''| - 1.$$

Suppose now that $k \geq 2$. Then fronts $e_1' = e_1 s_1, s_2, \ldots, s_{k-1}, e_k' = s_k e_k$ on $\partial T_0'$ are replaced in ∂T_0 by fronts e_1, e_k, and e_0. Thus the sum

$$(|e_1|+1-2)+(k-2)(1-2)+(|e_k|+1-2)=|e_1|+|e_k|-k$$

in σ' is replaced in σ by

$$\begin{aligned}(|e_1|-2)+(|e_k|-2)+(|e_0|-2)&=|e_1|+|e_k|\\&+(d(F_0)-k)-6\geq|e_1|+|e_k|-k. \quad \square\end{aligned}$$

3.3. Proposition. *T is concentric.*

Proof. Choose for D_0 any single face, and suppose, by induction on n, that D_n has been chosen in accordance with (1.12). Define A_{n+1} to consist of all faces not in D_n that have a side on $\delta_n = \partial D_n$, and set $D_{n+1} = D_n \cup A_{n+1}$. We must show that A_{n+1} satisfies (1.12), with $n+1$ in place of n. Now (1.122) holds by virtue of the definition of A_{n+1}.

Suppose that (1.121) fails. Then either (i) some F_i in A_{n+1} has $\partial F_i \cap \delta_n$ not connected, or (ii) some pair of faces F_i and F_j in A_{n+1}, which are not successive in the sense that $\partial F_i \cap \delta_n$ and $\partial F_j \cap \delta_n$ are disjoint, nonetheless have a point (not on δ_n) in common. We treat first the slightly harder case (ii). Then, as shown in Fig. 9a, there are arcs α_i, α_j, and α of ∂F_i, ∂F_j, and δ_n that form a curvilinear triangle enclosing a disc T_0. We write v_i, v_j, v for the vertices of this triangle opposite these three arcs. If T_0 consisted of a single face F, then, by virtue of (1.123) for faces in A_n, the face F could have at most two sides on α. But these, together with the two sides α_i and α_j, would give $d(F) \leq 4$, contrary to hypothesis. Thus T_0 has more than one face and (3.2) applies. A front e of ∂T_0 that is contained in α_i or α_j must have $|e| = 1$, while, by (1.123) again, a front contained in α must have $|e| \leq 2$. Thus the only positive contribution to $\sigma = \sigma(T_0)$ must come from fronts containing one or more of the vertices v_i, v_j, v. Since each vertex contributes at most 1, this implies that $\sigma \leq 3$, contrary to (3.2).

The case (i), shown in Fig. 9b, differs only in that T_0 is now bounded by an arc α_i of ∂F_i together with a segment α of δ_n. The same reasoning in this case gives $\sigma \leq 2$, again contrary to (3.2).

It remains to prove (1.123), that each F in A_{n+1} has at least two sides on δ_{n+1}. By the induction hypothesis, each face of A_n has at least two sides on δ_n, which implies that between two successive sides running inward from δ_n, there must be at least one side running outward. But this implies that between two successive

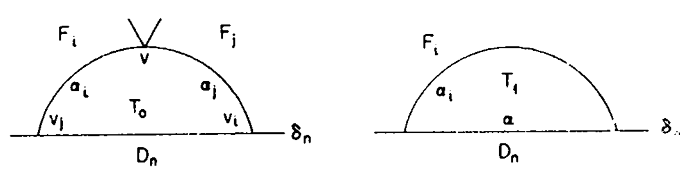


Fig. 9.

sides running outward from δ_n, there cannot be more than one side running inward. This last implies that a face F of A_{n+1} can have at most two sides on δ_n. These, together with the two outward running sides of F, account for at most four sides of F, whence, since $d(F) \geq 6$, F must have at least two more sides on δ_{n+1}. □

3.4. Proposition. *If T is a cubic tessellation in which each face has at least six sides, then T admits an Eulerian path.*

Proof. This follows directly from (3.3) and (2.2). □

According to our definition, each vertex v of a tessellation T has degree $d(v) \geq 3$. Our next step will be to replace the condition that t be cubic, that is, that all $d(v) = 3$, by the weaker, and tacit, condition that all $d(v) \geq 3$.

3.5. Definition. Let T_1 and T_2 be tessellations of the plane, and let Θ be a family of disjoint finite trees in the 1-skeleton of T_1. Let ϕ be a continuous map from the plane onto itself, carrying T_1 to T_2, which is injective on the complement of the trees θ in Θ, and which maps each θ in Θ to a single vertex v_θ of T_2, where distinct θ in Θ have distinct images v_θ. Then ϕ is a *retraction* and T_2 is a *retract* of T_1. We also say that T_1 is obtained from T_2 by *separation of vertices*.

For the applications, it is more natural to start with T_2 and construct T_1. About each vertex v of T_2 we choose a closed disc B_v meeting only the $d = d(v)$ sides at v, with ∂B_v meeting these sides at points $p_1, \ldots, p_n$. We obtain T_1 by replacing each B_v by a closed disc B'_v containing a finite tree θ_v with n ends, at the points $p_1, \ldots, p_n$. See Fig. 10.

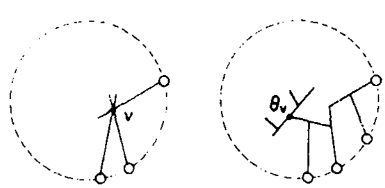


Fig. 10.

3.6. Proposition. *If T_1 is obtained from $T = T_2$ by separation of vertices and T_1 admits an Eulerian path, then T admits an Eulerian path.*

Proof. Let π be an Eulerian path on T_1 and ϕ a retraction from T_1 to T. Then $\pi\phi$ clearly defines a path π' on T. We show that π' is an Eulerian path on T. Let E_1 be the set of all edges of T_1 not contained in any θ in Θ. Then ϕ is bijective from E_1 to the set E of all edges of T. Thus each edge of T occurs exactly once in π'. It remains to show that π' is reduced. Let $e\phi$ and $e'\phi$ be successive edges in π'. If e and e' are successive edges in T_1, then $e' \neq e^{-1}$, whence $e'\phi \neq (e\phi)^{-1}$. Otherwise e and e' occur in π separated by an arc α all of whose edges lie in trees

θ in Θ. Since the trees θ in Θ are disjoint, all the edges in α lie in a single tree θ in Θ. Since α is a nontrivial reduced path in the tree θ, its initial vertex v and its terminal vertex v' must be different. Since e ends at v and e'^{-1} ends at v', we again have $e' \neq e^{-1}$, whence $e'\phi \neq (e\phi)^{-1}$. □

3.7. Theorem. *Let T be a tessellation of the plane in which each face has at least six sides. Then T admits an Eulerian path.*

Proof. In the notation above, let v be a vertex of T of degree $d(v) = d \geq 3$, with the d sides at v meeting the boundary of B_v at points $p_1, \ldots, p_d$. It is trivial to replace B_v by B'_v containing a cubic tree θ_v with d ends at the points $p_1, \ldots, p_d$. Thus we obtain T as a retract of a cubic graph T_1. But it is also immediately clear that ϕ is injective on the faces of T, with $d(F\phi) \leq d(F)$. Since $d(F\phi) \geq 6$ for each face $F\phi$ of T, it follows that $d(F) \geq 6$ for each face F of T_1. By (3.4), T_1 has an Eulerian path π. By (3.6), T also has an Eulerian path.

4. Regular tessellations

A *regular tessellation* T of the plane, of *type* (p, q), is one in which, for certain integers $p, q \geq 3$, each face F has exactly $d(F) = p$ sides and in which there are the same number $d(v) = q$ of sides at each vertex v. It is well known that such a tessellation T exists if and only if $1/p + 1/q \leq \frac{1}{2}$, and that T can be realized by a metrically regular tessellation of the Euclidean plane if $1/p + 1/q = \frac{1}{2}$, and by a metrically regular tessellation of the hyperbolic plane if $1/p + 1/q < \frac{1}{2}$.

We shall show that all of these regular tessellations, as well as various related irregular tessellations, admit Eulerian paths. This follows from (3.7) for regular tessellations of types (p, q) whenever $p \geq 6$. These include the hexagonal type (6, 3), and we next give a direct treatment of the two remaining Euclidean types (3, 6) and (4, 4).

4.1. Theorem. *In the Euclidean plane, the regular tessellation (3, 6) by equilateral triangles, and the regular tessellation (4, 4) by squares, both admit Eulerian paths.*

Proof. In both cases we are able to separate vertices to obtain a cubic tessellation T in a manner that is translationally uniform, that is, such that ϕ commutes with the translation group of T. This is shown in Figs. 11a, b and 12a, b. It remains to

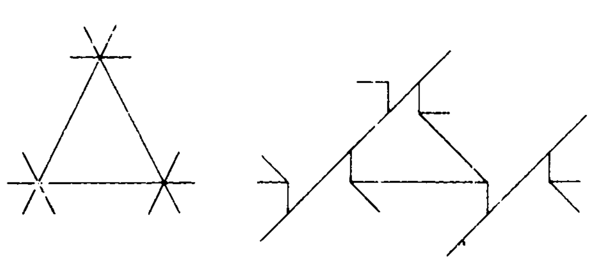


Fig. 11.

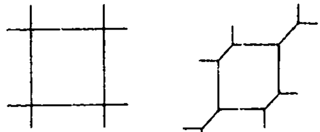


Fig. 12.

verify by inspection that, in both cases, each face of T has exactly six sides. The conclusion now follows as before from (3.4) and (3.6). □

We note that it is possible to construct an Eulerian path on the tessellation (4, 4) directly, in a similar manner, but simpler than that used above for cubic concentric tessellations, in particular, for the tessellation (6, 3).

We say that a tessellation T of the plane is of type (p^*, q^*) if each face F has degree $d(F) \geq p$ and each vertex v has degree $d(v) \geq q$. In this terminology, Theorem 3.7 says that every tessellation of type $(6^*, 3^*)$ has an Eulerian path. Now the three types $(6^*, 3^*)$, $(4^*, 4^*)$, $(3^*, 6^*)$, corresponding to the three regular types of Euclidean tessellations (6, 3), (4, 4), (3, 6), play a central role in small cancellation theory (see [5]). This theory grew out of Dehn's solution of the word problem for orientable surface groups, which can be construed as resting on the fact that the corresponding tessellations of the hyperbolic plane are of type $(6^*, 3^*)$. Indeed, Lemma 3.2, leading to the proofs of Theorems 3.3 and 3.4, is essentially a version of the case $(6^*, 3^*)$ of the Curvature Formula of small cancellation theory. One can therefore reasonably expect analogous results for the cases $(4^*, 4^*)$ and $(3^*, 6^*)$; note that this would yield all the remaining types of regular hyperbolic tessellations, $(4, q)$ for $q \geq 5$ and $(3, q)$ for $q \geq 7$. However, instead of trying to obtain these cases by arguments parallel to those used for the case $(6^*, 3^*)$, we seek to derive them from the results already obtained. In fact, we shall obtain the case $(4^*, 4^*)$, but we fall short of $(3^*, 6^*)$, obtaining only $(3^*, 7^*)$.

We begin with the case $(4^*, 4^*)$. The uniform treatment used for the type (4, 4) is no longer available, and, to establish the ideas, we begin with a different treatment of the case (4, 4) that we are able to extend to the case $(4^*, 4^*)$. As before, we take D_0 to be a single face and, by induction on n, given D_n we define $D_{n+1} = D_n \cup A_{n+1}$, where A_{n+1} consists of all faces not in D_n but with a side on the boundary δ_n of D_n. It is clear in the case (4, 4), and will follow in the course of our argument in the general case $(4^*, 4^*)$, that each D_n is a disc. The complexes A_{n+1} are not strictly annuli, but rather each is a cyclic chain of faces, each face having either a single vertex or a single side in common with its two neighbors, and otherwise disjoint. In particular, δ_n and δ_{n+1} may have a point v in common, which will then lie on faces of A_n and A_{n+2}, as well as of A_{n+1} and possibly other A_m.

Let $(v_1, \ldots, v_t)$ be the vertices on δ_n in cyclic order; and let $\kappa_n = (k_1, \ldots, k_t)$, where k_i is the number of outward sides at v_i, that is, the number of sides at v_i

that are not contained in D_n. For the tessellation (4, 4), since all $d(v) = 4$ and each v_i has two sides on δ_n, it follows that k_i has one of the values 0, 1, 2. Now, given κ_n, it is routine to calculate κ_{n+1}. We represent a segment of δ_n by a horizontal line segment with the vertices v_i labeled by the numbers k_i. We then draw in the faces of A_{n+1}, with sides on this segment; the remaining sides of these faces, on the upper boundary of A_{n+1}, form a segment of κ_{n+1}, and we label the vertices v_j on this segment with the corresponding k_j in κ_{n+1}.

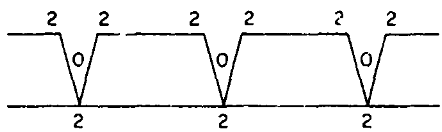


Fig. 13.

Evidently $\kappa_0 = (2, 2, 2, 2) = (2)^4$. Fig. 13 shows the construction described above, beginning with a segment of κ_0 and ending with a segment of κ_1. From this we read off that $\kappa_1 = (2, 2, 0)^4$. A segment [2, 2, 0, 2] of κ_n gives rise to a segment [2, 2, 0, 2, 0, 2] of κ_{n+1}. We conclude inductively that $\kappa_n = (2, (2, 0)^n)^4$; this can also be verified directly from inspection of the D_n, which are roughly diamond shaped configurations of squares (see Fig. 14).

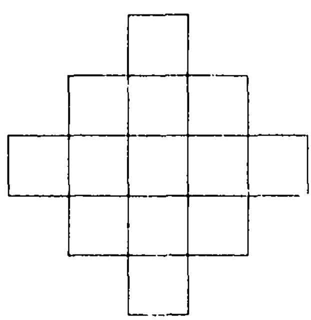


Fig. 14. D_2 for (4, 4).

We next consider the case $(4, 4^*)$, where each face F has $d(F) = 4$ sides and each vertex v has degree $d(v) \geqslant 4$. In describing the κ_n we use the symbol k ambiguously to indicate any integer $k \geqslant 2$. Thus we write $\kappa_0 = (k, k, k, k)$ to indicate that κ_0 has the form $\kappa_0 = (k_1, k_2, k_3, k_4)$ with all $k_i \geqslant 2$. In passing from κ_n to κ_{n+1}, we note that each k in κ_n is decreased by 2; thus a segment of the form $[k, \ldots, k]$ in κ_n goes into a segment of the form $[k-2, \ldots, k-2]$ in κ_{n+1}. Specifically, Fig. 15a shows that $[k, k]$ goes to $[k-2, k, k, k-2.]$ Likewise, Figs. 15b and 15c show that $[k, 0, k]$ goes to $[k-2, k, k-2]$ and that $[k, 1, k]$ goes to $[k-2, k, k-1, k, k-2]$. We see inductively that no κ_n contains two consecutive

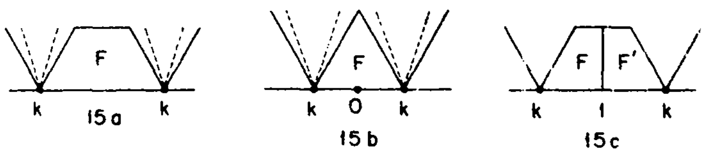


Fig. 15.

terms both less than 2, whence all faces F in A_{n+1} enter in one of the configurations shown in Figs. 15a, b, c. From this it follows by induction that each D_{n+1} is a disc.

We now construct T_1 by separation of vertices. A vertex v will be separated only if it has at most one outward edge or at most one inward edge. The case of a vertex v of degree $d(v)=4$, with one outward edge and one inward edge is treated as in the case of the tessellation (4, 4); this is shown in Fig. 16a. The case of a vertex v of degree $d(v)\geq 5$ with at most one inward edge is treated as shown in Fig. 16b. The case of $d(v)\geq 5$ with at most one outward edge is treated symmetrically, as shown in Fig. 16c.

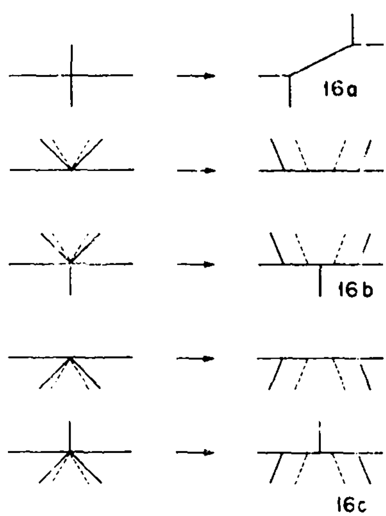


Fig. 16.

Inspection shows that, if F is as in Fig. 16a, then in passing from T to T_1, F gains at least one new side at each of its outer vertices. For F as in Fig. 16b, F gains at least one side at its outer vertex and another at its middle inner vertex. For two faces F and F' as in Fig. 16c, F gains a side at the inner vertex on the side joining them, and F' at the outer vertex on this side, while each gains at least one more side at its remaining outer vertex. We see thus that on passing to T_1, each face is replaced by a face with degree at least 6. Since T_1 is of type $(6^*, 3)$, it follows as before that T_1 has an Eulerian path, whence T has an Eulerian path.

For the general case $(4^*, 4^*)$, the argument remains essentially unchanged. The only effect on the κ_n of having faces with $d(F) \geqslant 5$ is that an entry k is replaced by a sequence of terms k. The faces in Figs. 16a, b, c are altered only in that there may be additional vertices, and the separation, as before, ensures that each face is replaced by one in T_1 that has at least 6 sides.

We next examine the case $(3, 7^*)$, where each face is a triangle, $d(F) = 3$, and each vertex has degree $d(v) \geqslant 7$. We now use the symbol k to denote ambiguously any integer $k \geqslant 5$. Here $\kappa_0 = (k, k, k)$. For $h \geqslant 2$, a segment $[k, h, k]$ of κ_n goes to a segment $[k-2, k, h-2, k, k-2]$ of κ_{n+1}, while a segment $[k, 1, k]$ goes to $[k-2, k-3, k-2]$ and $[k, 0, k]$ goes to $[k-2, k-2]$. Since $[k, 0, k]$ can arise only from $[k, 3, k]$, which goes to $[k-2, k, 1, k, k-2]$, and $[k, 1, k]$ can arise only from $[k, 2, k]$, which goes to $[k-2, k, 0, k, k-2]$, no κ_n can contain a part $[a, b, c]$ with more than one of a, b, c less than 2.

From this it follows that every face in A_{n+1} arises in one of the configurations shown in Figs. 17a, b, c, as one of the faces F, F', or F'', and that no face occurs as part of two such configurations. It follows that all the D_n are discs.

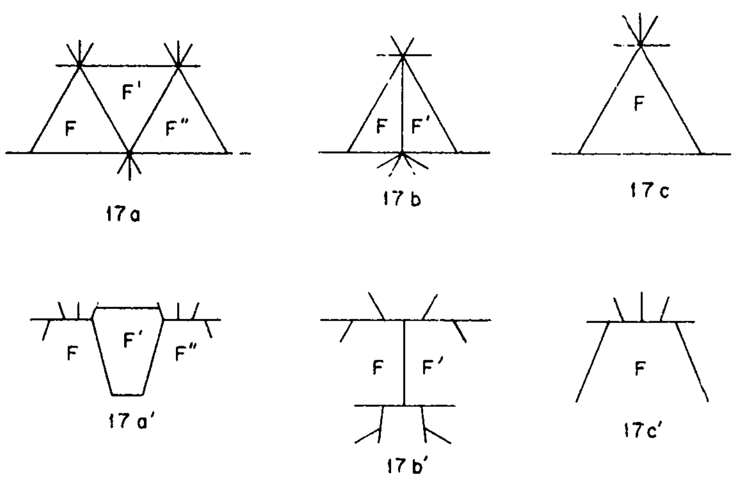


Fig. 17. Note that at each top vertex v we have indicated only 7 of the $d(v) \geqslant 7$ sides.

We call a vertex v of a face F in A_{n+1} a *top vertex* of F if v is not in δ_n. Every face has a top vertex except a face with two sides on δ_n. (Also, no face has more than one top vertex, but we make no use of this observation, which will no longer remain valid when we pass to the case $(3^*, 7^*)$.) Further, no vertex is the top vertex of more than one face except as shown in Fig. 17b, where F and F' have a common top vertex.

We now pass to a tessellation T_1 by separating all top vertices. This separation is indicated in Figs. 17a', b', c', where only the separated parts are shown. (Here the rule given for case 17c is to be applied only to vertices that do not fall under cases 17a or 17b.) Note that in Fig. 17a′ the vertex at the bottom of F' will be the top vertex of one or two faces in D_n, but that the indicated separation, giving F' an additional side at this vertex, is in accordance with the rules we have given for separating vertices in the three cases.

It is immediate by inspection that every face of T_1 has at least six sides, whence it follows as before that T_1 and so also T has an Eulerian path. If we now pass to the case $(3^*, 7^*)$, admitting faces with more than three sides, the only change is that now faces may have additional top vertices, and the same rules for separating vertices again lead to T_1 of type $(6^*, 3^*)$, whence again T has an Eulerian path.

4.2. Theorem. *Every tessellation of the plane of type* $(4^*, 4^*)$ *or* $(3^*, 7^*)$ *has an Eulerian path.*

4.3. Theorem. *Every regular tessellation of the plane has an Eulerian path.*

Proof. By Theorem 4.1, the regular tessellation (3, 6) has an Eulerian path. By Theorem 3.7, every tessellation of type $(6^*, 3^*)$ has an Eulerian path. Since every regular tessellation except (3, 6) is of one of the types $(6^*, 3^*)$, $(4^*, 4^*)$, or $(3^*, 7^*)$, it has an Eulerian path. □

4.4. Remark. It seems virtually certain that every tessellation of type $(3^*, 6^*)$ has an Eulerian path, but we have not succeeded in adapting the rather ad hoc method of separation of vertices to this case.

5. Truncated tessellations

If T is any tessellation of the plane, we define the truncation T^* of T as follows. We draw circles c_v about the vertices v of T small enough so that the circles c_v are disjoint and that c_v meets no sides except those at v. At a vertex v, of degree $d = d(v)$, let $v_1, \ldots, v_d$ be the points at which the sides $s_1, \ldots, s_d$ at v, in cyclic order, meet c_v. We take these vertices $v_1, \ldots, v_d$ as the vertices of T^*. The sides of T^* are of two sorts. First, we take as sides of T^* all the arcs of the circles c_v between consecutive vertices v_i and v_{i+1}. Second, if s is a side of T, between vertices v and v' of T, we take as a side of T^* the segment of s between the points v_i and v'_j where s meets the circles c_v and $c_{v'}$. See Fig. 18.

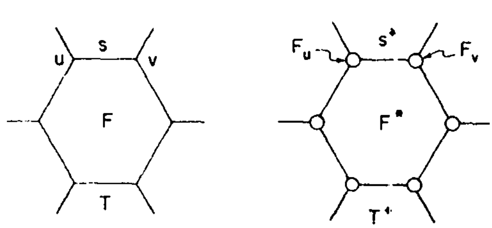


Fig. 18.

The truncation T^* of every tessellation is cubic; thus truncations provide a natural source of cubic tessellations. The faces of T^* are of two sorts. First, for every vertex v, of degree $d(v)$, the circle c_v is the boundary of a face F_v of T^* of degree $d(F_v) = d(v)$. Second, every face F of T is replaced in T^* by a truncated face F^* of degree $d(F^*) = 2d(F)$, as shown in Fig. 18.

5.1. Theorem. *If T is any tessellation of the plane in which there are at least six sides at each vertex, then the truncation T^* of T admits an Eulerian path.*

Proof. First, T^* is cubic. Next, if F is any face of T, then $d(F) \geq 3$, whence the face F^* of T^* has degree $d(F^*) \geq 6$. Again, if v is any vertex of T, then the face F_v of T^* has degree $d(F_v) = d(v) \geq 6$. Thus T^* is of type $(6^*, 3)$ and the conclusion follows by Theorem 3.7. □

The next theorem contrasts with (5.1).

5.2. Theorem. *Let T be an infinite cubic tessellation. Then the truncation T^* of T admits no Eulerian path.*

Proof. Let Σ be an Eulerian system on T^* determined by an orientation ω. We shall show that Σ must contain more than a single path. If v is any vertex of T, then the corresponding face F_v of T^* is a triangle with vertices v_1, v_2, v_3. Let s_1, s_2, s_3 be the sides of T^* at v_1, v_2, v_3 other than those of $\partial F_v = c_v$. If each time a path α in Σ enters F_v along one of the sides s_i it next leaves along a side s_j different from s_i, say along s_{i+1} (subscripts modulo 3), then all of v_1, v_2, v_3 must be oriented in the same way. But this implies that Σ contains also a closed path α' running around c_v.

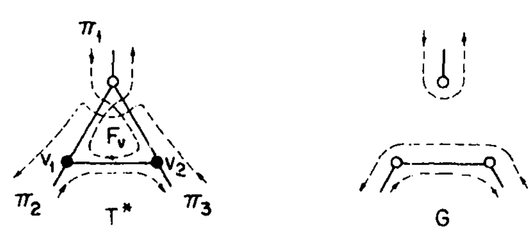


Fig. 19.

We may now suppose that, on each F_v, one of the vertices v_i is oriented differently from the other two, say v_1 is oriented differently from v_2 and v_3. This implies that some path in Σ enters F_v along s_1 and, after running around c_v, leaves again along s_1. Whatever path enters along s_2 leaves along s_3, and whatever path enters along s_3 leaves along s_2. We say that the first path 'turns around' at F_v, while the other two 'run parallel' (but oppositely) through F_v. See Fig. 19b.

Now the two branches of a path α in Σ that turns around at some F_v must either run parallel forever, or join by turning around at some other $F_{v'}$. If they join at some $F_{v'}$, then α is finite and hence not the only path in Σ. If they run parallel forever, then α must be distinct from the path α' in Σ turning around at $F_{v'}$, for v' any vertex of T different from v. □

We digress to state another result that is proved by a very similar line of reasoning.

5.3. Theorem. *Let G be any graph, finite or infinite, that contains a subgraph G_0 isomorphic to that shown in Fig. 20, where it is understood that the two sides labeled s_1 and s_2 are the only sides joining G_0 to the rest of G. Then G admits no Eulerian path.*

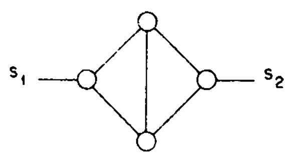


Fig. 20.

Proof. As before, if an Eulerian system Σ on G does not contain a closed path running around one of the circles c_v in G_0, then some path turns around at each of the four circles in G_0. Since at most two of these pairs of branches of paths can escape from G_0, running parallel out of G_0 along s_1 and s_2, two of them must be united to form a closed path in Σ, contained entirely within G_0. □

Remark. It is obvious that, in the statement of the theorem, the subgraph G_0 can be replaced by various more complicated subgraphs.

Our next theorem contrasts in turn with (5.2).

5.4. Theorem. *Let T be any quartic tessellation, that is, having four sides at each vertex. Then the truncation T^* of T admits an Eulerian path.*

Proof. We begin by showing that T^* is the union of an ascending chain D_0, $D_1, \ldots$, of discs such that D_0 has a single face and that each $A_{n+1} = D_{n+1} - D_n$ is an annular chain of faces that are alternately of the types F_v and F^*. We begin by choosing D_0 to consist of any single F-*face* F_0, that is, a face of type F^*. Then the faces, other than F_0, having a side in common with F_0 form an annulus A_1 of the required kind, and we define $D_1 = D_0 \cup A_1$. For the inductive construction, suppose D_n given, $n \geq 1$, and define A_{n+1} to consist of all faces not in D_n but having a side on the boundary of D_n. Since (i) one v-*face*, that is, face of type F_v, cannot meet another, (ii) a v-face in A_n can meet only one further F-face in

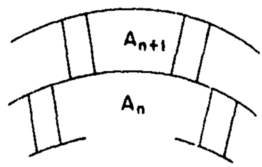


Fig. 21.

addition to the three it meets in D_n, and (iii) the faces abutting on any F-face must be alternately F-faces and v-faces, we conclude that A_{n+1} is made up alternately of F-faces and v-faces. See Fig. 21.

To define an Eulerian path π on T^* it suffices, as before, to choose a suitable function ω assigning an orientation to each vertex of T^*. Now each vertex lies on exactly one v-face, and each A_n, $n \geq 1$, contains some v-faces. We choose one v-face in each A_n as *special* and orient its vertices as shown in Fig. 22a; the remaining *nonspecial* v-faces in A_n have their vertices oriented as in Fig. 22b.

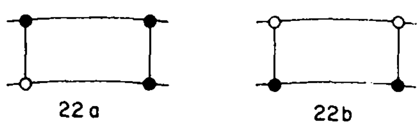


Fig. 22.

This defines an Eulerian system Σ on T^*, and it remains to show that Σ contains only a single path π, which must then be an Eulerian path. Fig. 23 shows that if any nonspecial v-face is deleted, in the sense shown in the figure, to yield a new tessellation T', and if Σ' is the system on T' defined by the resctiction ω' of the orientation ω to vertices of T', then Σ' has the same number of paths as Σ. We now define T'' by successively deleting in this way all nonspecial v-faces. (We note

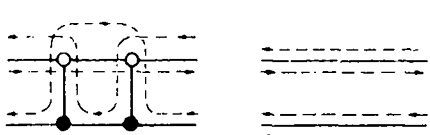


Fig. 23.

that T'' is not strictly speaking a tessellation in our previous sense, since a pair of faces may abut along two disjoint sides; but this does not affect our argument.) The resulting 'tessellation' T'' is shown in Fig. 24, together with the resulting system Σ'', which can be seen to consist of only a single path π''. It follows that Σ contains only a single path π, which is therefore an Eulerian path on T^*. □

5.5. Remark. We leave unsettled the question: If T is a quintic tessellation, does T^* admit an Eulerian path?

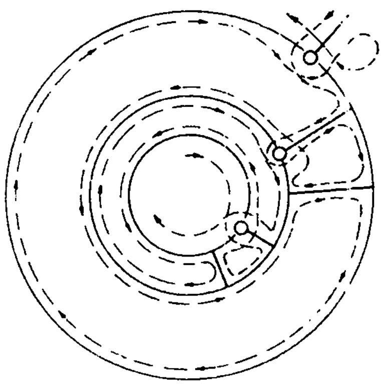


Fig. 24.

6. Problems

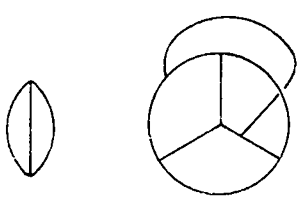
6.1. Problem. We have shown that every tessellation T of one of the types $(6^*, 3^*)$, $(4^*, 4^*)$, $(3^*, 7^*)$, or $(3, 6)$ has an Eulerian path. Does every tessellation T of type $(3^*, 6^*)$ have an Eulerian path? In particular, if T is of type $(3, 6^*)$, with all faces triangular, and with 6 sides at some vertices and more than 6 at others, does T have an Eulerian path? We believe the answer is yes.

6.2. Problem. We have shown that if T is an infinite cubic graph, that is, with all vertices v of degree $d(v) = 3$, then its truncation T^* has no Eulerian path, while if T is a quartic tessellation of the plane, with all $d(v) = 4$, then T has an Eulerian path, and that if T is a planar tessellation with all $d(v) \geqslant 6$, then T^* has an Eulerian path. Does the truncation T^* of every tessellation T of type $(3^*, 4^*)$ have an Eulerian path? In particular, if T is a quintic tessellation, does T^* have an Eulerian path? We believe the answer is yes.

6.3. Problem. Let T be given as an abstract 2-complex. Then the conditions that T be locally finite and that T be a tessellation of some 2-manifold (without boundary) may reasonably be described as 'local', while the condition that T be a tessellation of the plane is global. The conditions that T be of one of the types (p, q) or (p^*, q^*) may again be regarded as local, imposing a limitation on the stars of single faces and of single vertices. The condition that T does not contain any of the 'bad' subcomplexes illustrated by the subcomplex G_0 in Theorem (5.3) is again local. Does there exist any set of local conditions, excluding certain types of finite subcomplexes, which is necessary and sufficient for a planar tessellation T to have an Eulerian path? This seems unlikely. Do there exist reasonably simple conditions, weaker that the conditions (p^*, q^*), which imply the existence of an Eulerian path? This question is suggested by the work of Perraud [7, 8, 9] in small cancellation theory, who has shown that a certain natural condition on subcomplexes consisting of the star of a face together with all its abutting faces, although not implying the usual conditions (p^*, q^*) of small cancellation theory, nonetheless implies the validity of Dehn's algorithm for the solution of the word problem.

6.4. Problem. What can be said about tessellations T of manifolds M other than the plane? Saul Stahl (personal communication; see [13]) has indicated how the existence of an Eulerian path on a finite graph G can be used to exhibit G as the 1-skeleton of a decomposition T of a closed surface M. Let the Eulerian path π be represented by the closed path $e_1 \cdots e_n$. Let Δ be an n-gon with sides labeled $e_1, \ldots, e_n$, in cyclic order, and let M be the surface obtained by identifying pairs of sides with labels e_i and $e_j = e_i^{-1}$, in the usual manner. Then G is the image of $\partial\Delta$, and is the 1-skeleton of a decomposition T of M in which there is only a single face, the image of Δ. If G has v vertices and s sides (undirected edges, whence $n = 2s$), then M has Euler characteristic $\chi = 1 - s + v$ and genus $g = \frac{1}{2}(s - v + 1)$. This is clearly the largest possible genus for a surface M with a decomposition T whose 1-skeleton is isomorphic to G.

If G is a finite cubic graph, with Eulerian path, then $n = 3v$, whence $4g = v + 2$, or $v = 2(2g - 1)$. For example, if G is the cubic graph with $v = 2$ vertices, shown in Fig. 25a, then the construction above yields G as the 1-skeleton of a decomposition of the torus. Since it is clear that G is a planar graph, it can also be obtained as the 1-skeleton of a decomposition of the sphere. Again, if G is now the nonplanar Kuratowski 'utilities' graph shown in Fig. 25b, then G has an Eulerian path. Since G has $v = 6$ vertices, G can be obtained as the 1-skeleton of a decomposition T of the orientable surface M of genus 2. It is easy to see that G can be obtained also from a decomposition of the torus, but not, of course, from one of the sphere.



25a 25b

Fig. 25.

Next let a graph G with Eulerian path be embedded in the plane, as the 1-skeleton of a finite or infinite, locally finite, connected and simply connected 2-complex K contained in the plane, M. Suppose that the complement of K in the plane M had two or more components. Then an arc γ joining two of these components, and avoiding the vertices of G, could be chosen to meet only finitely many sides of G. Since the Eulerian path π on G must now cross γ an even number of times, one of the two parts into which γ divides G must be finite. This is a contradiction. One concludes that G cannot be the 1-skeleton of a decomposition T of the closed strip $0 \leq \mathrm{Im}\, z \leq 1$, although the case of a decomposition of a closed half plane, $\mathrm{Im}\, z \geq 0$, is not ruled out. What is the situation for other manifolds with boundary?

6.5. Problem. With minor exceptions, our results here all concern graphs given as 1-skeletons of tessellations of the plane. Are there any purely 1-dimensional graph theoretic conditions for an abstract graph G to have an Eulerian path?

6.6. Problem. We have considered Eulerian paths on the 1-skeletons of tessellations of the plane. Are there similar results for tessellations of higher dimensional Euclidean space? Does the 1-skeleton G of the regular ressellation T of Euclidean 3-space by cubes admit an Eulerian path? (*Added in proof.* We have answered the last question affirmatively, and the analog for $n > 3$.)

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