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DOMINATION ALTERATION SETS IN GRAPHS

Douglas BAUER
Stevens Institute of Technology, Hoboken, NJ 07030, USA

Frank HARARY
University of Michigan, Ann Arbor, MI 48109, USA

Juhani NIEMINEN
University of Oulu, 90570 Oulu 57, Finland

Charles L. SUFFEL
Stevens Institute of Technology, Hoboken, NJ 07030, USA

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The domination number $\alpha(G)$ of a graph G is the size of a minimum dominating set, i.e., a set of points with the property that every other point is adjacent to a point of the set. In general $\alpha(G)$ can be made to increase or decrease by the removal of points from G. Our main objective is the study of this phenomenon. For example we show that if T is a tree with at least three points then $\alpha(T-v)>\alpha(T)$ if and only if v is in every minimum dominating set of T. Removal of a set of lines from a graph G cannot decrease the domination number. We obtain some upper bounds on the size of a minimum set of lines which when removed from G increases the domination number.

1. Introduction

We investigate the stability of the domination number of a graph. Let $G=(V,E)$ be the graph and $\mu=\mu(G)$ an arbitrary invariant of G. The *μ-stability* of G is the minimum number of points whose removal changes μ. Some invariants such as the chromatic number of a graph, $\chi(G)$, have the property that removal of any subset $S\subset V$ does not increase the invariant. For other graph invariants there are subsets S_1 and S_2 of V such that $\mu(G-S_1)>\mu(G)$ and $\mu(G-S_2)<\mu(G)$. One example [1, 3] of such an invariant is the point connectivity $\kappa(G)$. The graph G in Fig. 1 has $\kappa=2$; however $\kappa(G-u)=1$ and $\kappa(G-v)=3$.

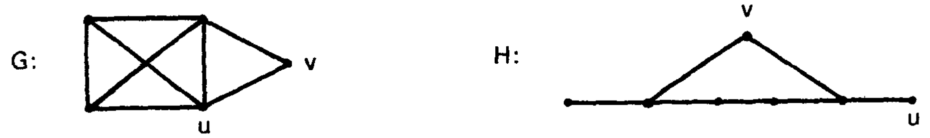

Fig. 1.

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Another example is the diameter $d(G)$. For graph H of Fig. 1, $d(H)=4$, $d(H-u)=3$, and $d(H-v)=5$. We call such invariants *exceptional*. For these invariants we define the *μ^+-stability* to be the minimum number of points whose removal increases μ; to decrease μ we refer to the *μ^--stability*.

The *domination number*[1] of a graph G, denoted $\alpha(G)$, is the minimum number of points of a set $S \subset V$ with the property that each point of $V-S$ is adjacent to some point of S. In the graph G of Fig. 2, we see that $\alpha(G)=2$, $\alpha(G-v)=1$, and $\alpha(G-u)=3$. Thus $\alpha^+(G)=\alpha^-(G)=1$. Note that $\alpha(G)=0$ if G is the discrete graph, hence $\alpha^-(G)=|V|$ if G has a point of full degree. We now concentrate on $\alpha(G)$ and *domination alteration sets*, i.e., sets of points that alter $\alpha(G)$.

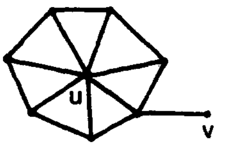

Fig. 2.

In Section 2 we show that if T is a tree with at least three points then $\alpha(T-v)>\alpha(T)$ if and only if v is in every minimum dominating set of T. A surprising result, proved at the end of the section, states that $\alpha^+(G)+\alpha^-(G)$ is a constant whenever G is a sufficiently large path or cycle. In Section 3 we consider the 'line stability' of $\alpha(G)$, i.e., changes in α that result from removing lines from G.

Terminology and notation not introduced here is given in the book by Harary [2].

2. Stability of $\alpha(G)$

The following definitions will be useful. The *neighborhood* of a point v is the set $N(v)$ of all points u which are adjacent to v. The *closed neighborhood* of v is $N[v]=N[v)\cup\{v\}$. For a minimum dominating set A and $v \in A$, let

$$A^*(v)=\{u:\ u\notin A \text{ and } N(u)\cap A=\{v\}\}.$$

In addition let

$$\gamma(G)=\min\{|A^*(v)|: v\in A, \text{ a minimum dominating set}\}.$$

We now give a simple but useful bound for α^-.

Proposition 1. *For any graph G*

$$\alpha^-(G)\leqslant\gamma(G)+1.$$

To see that equality does not hold in general consider the graph G of Fig. 3.

[1] Even though this was denoted by $\alpha_{00}(G)$ in [2], we write $\alpha(G)$ for brevity.

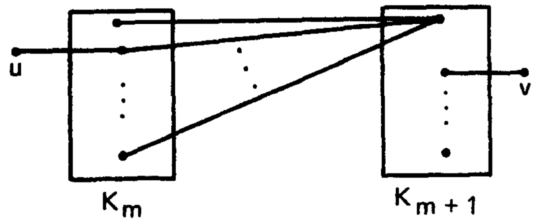

Fig. 3.

In this case $\alpha(G)=2$ and $\gamma(G)=m \geqslant 3$, however $\alpha(G-\{u, v\})=1$.

Corollary 1.1. *For any graph G, $\alpha^-(G)=1$ if and only if $\gamma(G)=0$.*

Proof. If $\gamma(G)=0$, then $\alpha^-(G) \leqslant 1$ by Proposition 1. But $\alpha^-(G)$ must be at least one; hence $\alpha^-(G)=1$. Now suppose $\alpha(G-v)<\alpha(G)$ for some point $v \in G$ and let B be a minimum dominating set for $G-v$. Clearly $A=B \cup \{v\}$ is a minimum dominating set for G with $A^*(v)=\emptyset$, consequently $\gamma(G)=0$. □

If we form a graph H by removing $\alpha^+(G)$ points from G, $\alpha(H)-\alpha(G)$ can be made arbitrarily large, as is easily seen by observing the star $K_{1,n}$. This is not the case if we remove α^- points. By noting that for any graph G, $\alpha(G-v) \geqslant \alpha(G)-1$, we obtain the following result.

Proposition 2. *Let $u_1, \ldots, u_n$ be a minimal point set of G whose removal decreases $\alpha(G)$. Then $\alpha(G-u_1-u_2-\cdots-u_n)=\alpha(G)-1$ and $\alpha(G-U)=\alpha(G)$ for any subset U of $\{u_1, \ldots, u_n\}$ with cardinality $n-1$.*

We note that if U is a minimal set of points whose removal decreases $\alpha(G)$ and if U' is a proper subset of U, it is possible for $\alpha(G-U')$ to exceed $\alpha(G)$. A simple example is given by the star $K_{1,n}$, where $n \geqslant 2$. It is also possible for a minimal set of points whose removal increases α to properly contain a subset of points whose removal decreases α. The graph G shown in Fig. 4 is dominated by $\{v_1, v_2, u_1, u_2\}$. Removing v_1 and v_2 from G increases α to five, however $\alpha(G-v_1)=\alpha(G-v_2)=3$.

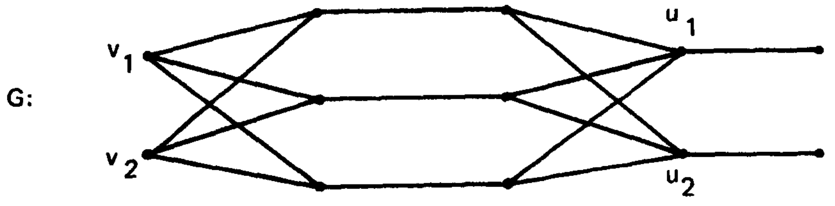

Fig. 4.

Next we state a result which characterizes points whose removal increases α.

Proposition 3. *The removal of a point v from G increases α if and only if*
(1) *v is not isolated and is in every minimum dominating set for G, and*

(2) *there is no dominating set for $G-N[v]$ having α points which also dominates $N(v)$.*

The graphs G and H in Fig. 5 show that neither of the above conditions is sufficient. Clearly v is in every minimum dominating set for G, yet $\alpha(G-v)=\alpha(G)=2$. It is also easy to see that there is no two point dominating set for $H-N[v]$ which dominates $N(v)$; however $\alpha(H-v)=\alpha(H)=2$.

Fig. 5.

For trees we may dispense with the second condition. Before proving this result we note that if a point v is in every minimum dominating set of a tree T, then v is not an endpoint of T.

Proposition 4. *For any tree T with at least three points $\alpha(T-v)>\alpha(T)$ if and only if v is in every minimum dominating set for T.*

Proof. By Proposition 3 the necessity of v being in every minimum dominating set for T is immediate. Suppose v is in every minimum dominating set of T. Note that $\alpha(T-v)\geqslant\alpha(T)$, for otherwise a minimum dominating set of $T-v$ could be extended to a dominating set of T which avoids v and has cardinality at most $\alpha(T)$. Let $N(v)=\{v_1, v_2, \ldots, v_m\}$ and T_i be the component of $T-v$ containing v_i. If $\alpha(T-v)=\alpha(T)$, then for each i, v_i is in no minimum dominating set of T_i, for otherwise such a dominating set could be extended to a dominating set of T which avoids v and has cardinality at most $\alpha(T)$. Thus, for each i, $\alpha(T-\bigcup_{j\neq i} T_j)=\alpha(T_i)+1$, and so for any dominating set D of T, $|D\cap V(T_i)|\geqslant\alpha(T_i)$. It follows that $\alpha(T)\geqslant\sum_{i=1}^{n}\alpha(T_i)+1=\alpha(T)+1$, a contradiction. $\square$

With a slight modification of the above proof we can strengthen the result in one direction.

Proposition 5. *If a cutpoint v of G is in every minimum dominating set for G, then $\alpha(G-v)>\alpha(G)$.*

We now extend Proposition 4 by describing the structure of those trees T for which $\alpha^+(T)=2$.

Proposition 6. *Let T be a tree. Then $\alpha^+(T)=2$ if and only if there are points v and u such that*

(1) *every minimum dominating set contains either v or u,*

(2) *v is in every minimum dominating set for $T-u$ and u is in every minimum dominating set for $T-v$, and*

(3) *no point is in every minimum dominating set for T.*

Proof. The necessity of the conditions is clear. Furthermore sufficiency is easily established if we can prove that $\alpha(T-v)=\alpha(T)$, for then condition (2) will serve as the hypothesis for Proposition 4 applied to $T-v$. The fact that $\alpha(T-v)\leq\alpha(T)$ follows from condition (3) and Proposition 4. Suppose $\alpha(T-v)<\alpha(T)$, and let S be a minimum dominating set for T which contains v but not u. Let $v_1, \ldots, v_m$ be the points adjacent to v. Then $S=\{v\}\cup\bigcup_{i=1}^m S_i$ where S_i is a minimum collection of points from T_i which dominates T_i-v_i. Note that if there are two or more values of i for which $\alpha(T_i)=|S_i|+1$ then $\alpha^+(T)=1$, which contradicts condition (3). Suppose there exists one value of i such that $\alpha(T_i)=|S_i|+1$. Then

$$\alpha(T-v)=\sum_{i=1}^{m}\alpha(T_i)=1+\sum_{i=1}^{m}|S_i|=\alpha(T),$$

a contradiction. If $\alpha(T_i)=|S_i|$ for all i, then $\bigcup_{i=1}^m S_i$ is a minimum dominating set for $T-v$ which does not contain u, and we are done. □

For graphs in general, α, α^+ and α^- can be made as large as we wish. In particular, the graph G constructed by joining a point v to one point in each of m distinct copies of K_m has $\alpha(G)=\alpha^+(G)=\alpha^-(G)=m$. However graphs with large α^+ and α^- are constrained to have a large minimum degree, δ.

Proposition 7. *For all graphs G, $\min\{\alpha^+(G), \alpha^-(G)\}\leq\delta(G)+1$.*

It is interesting to note that although α^+ and α^- can be simultaneously large this is not the case for graphs with at least one endpoint.

Proposition 8. *If G is a graph with a point of degree one, then $\alpha^+(G)\geq 2$ implies $\alpha^-(G)\leq 2$. In particular this is true for trees.*

Proof. Let v be a point of T which is adjacent to an endpoint u of T. If $\alpha(T-v)<\alpha(T)$ we are done. If not, since we know $\alpha(T-v)\leq\alpha(T)$, it follows that $\alpha(T-v)=\alpha(T)$. However $T-v=\{u\}\cup T'$, where T' is a subtree of T, and hence $\alpha(T-v)=1+\alpha(T')$. But then $\alpha(T-u-v)=\alpha(T')<\alpha(T-v)=\alpha(T)$ and so $\alpha^-(T)\leq 2$. □

The examples given in Fig. 6 demonstrate that the only restriction on α^+ and α^- for trees is given in the above proposition.

We now show that one can select and remove a point from a tree without changing the domination number.

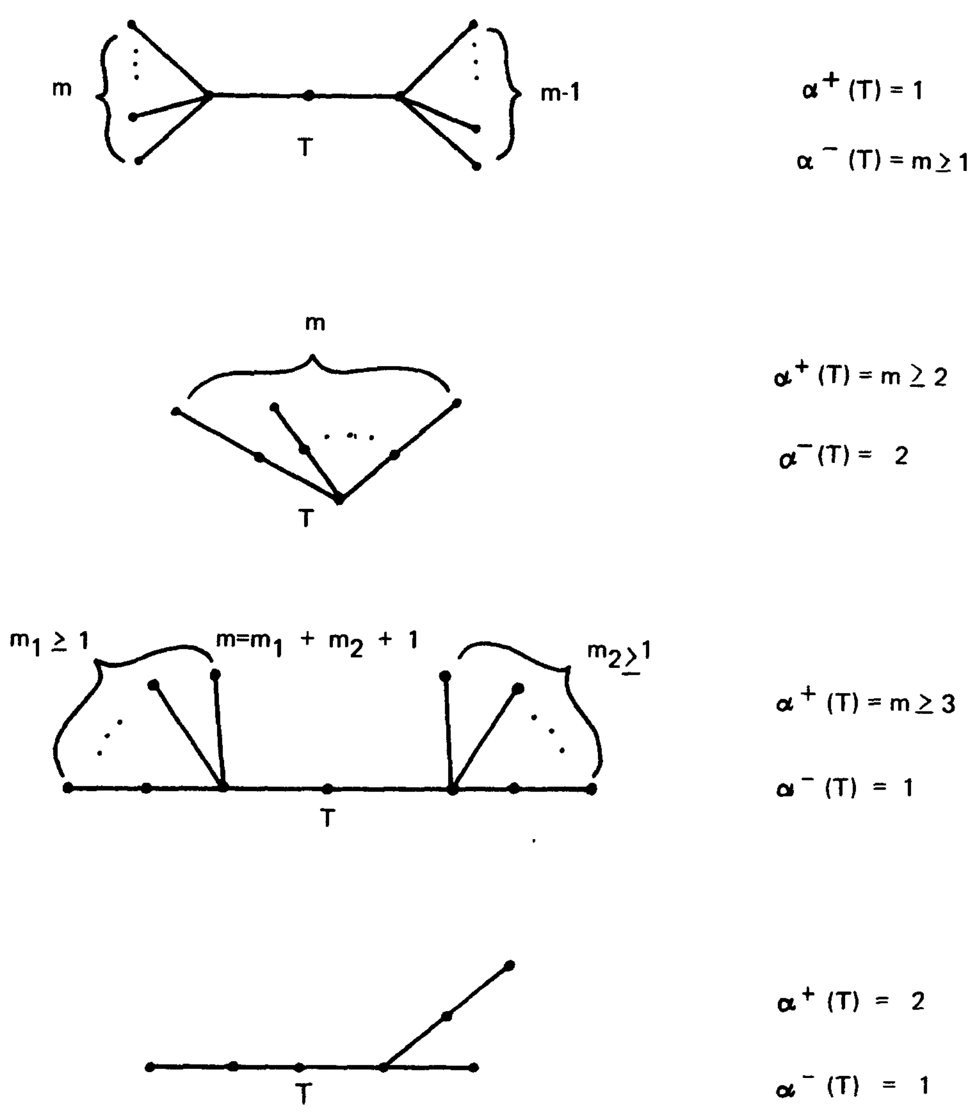

Fig. 6.

Proposition 9. *For every tree T there exists a point $v \in T$ such that $\alpha(T-v)=\alpha(T)$.*

Proof. We first note that if there is a point $v \in T$ which is adjacent to two (or more) endpoints u_1 and u_2 of T then v is in every minimum dominating set for T and $\alpha(T-u_1)=\alpha(T)$. If not, then T contains a point w of degree two which is adjacent to an endpoint u.

Let $T'=T-w-u$. Now for any graph G, if $\deg v=1$, then $\alpha(G-v)\leqslant\alpha(G)$. Hence $\alpha(T')\leqslant\alpha(T-u)\leqslant\alpha(T)$. However clearly $\alpha(T')\geqslant\alpha(T)-1$. Now if $\alpha(T')=\alpha(T)-1$, then $\alpha(T)=\alpha(T-w)$. Otherwise $\alpha(T')=\alpha(T)=\alpha(T-u)$. $\square$

We conclude this section by proving that for sufficiently large $n, \alpha^+ + \alpha^-$ is a constant for paths P_n and cycles C_n. First note that $\alpha(P_n)=\alpha(C_n)=\lceil\frac{1}{3}n\rceil$ if $n\geqslant 3$.

Proposition 10. *For $n\geqslant 7$, $\alpha^+(P_n)+\alpha^-(P_n)=4$.*

Proof. Let path $P_n = v_1, v_2, \ldots, v_n$. We show that $\alpha^+(P_n)+\alpha^-(P_n)=4$ by proving this separately for $n \equiv 0, 1,$ and 2 (mod 3).

Case 1: $n \equiv 0 \pmod 3$. Clearly v_2 is in every minimum dominating set, hence by Proposition 4 $\alpha^+(P_n)=1$. To see that $\alpha^-(P_n)=3$ first note that $\alpha(P_{n-3})=\alpha(P_n)-1$; hence $\alpha^-(P_n)\leqslant 3$. Since $\alpha(P_{n-1})=\alpha(P_{n-2})=\alpha(P_n)$ the only way to lower the domination number of P_n by removing either one or two points is to disconnect P_n. Suppose we create two components, A and B, containing a and b points respectively, by removing either one or two points from P_n. Let $k=\frac{1}{3}n$. Then

$$\alpha(A)+\alpha(B)=\lceil \tfrac{1}{3}a\rceil+\lceil \tfrac{1}{3}b\rceil \geqslant \tfrac{1}{3}a+\tfrac{1}{3}b \geqslant k-\tfrac{2}{3}$$

and so $\alpha(A)+\alpha(B)\geqslant k$. The last possibility, namely removing two points from P_n and creating three components, is immediate and we omit the details.

Case 2: $n \equiv 1 \pmod 3$. Now $\alpha(P_{n-1})=\alpha(P_n)-1$ and hence $\alpha^-(P_n)=1$. If we remove $\{v_2, v_4, v_6\}$ from P_n we obtain three isolated points and P_{n-6}. Since $\alpha(P_{n-6})=\alpha(P_n)-2$ we conclude that $\alpha^+(P_n)\leqslant 3$. Now note that no point of P_n is in every minimum dominating set of P_n. In fact the only pairs of points satisfying condition (1) of Proposition 6 are $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$. However in either case condition (2) is not satisfied. Hence by Propositions 4 and 6, $\alpha^+(P_n)=3$.

Case 3: $n \equiv 2 \pmod 3$. Here v_2 and v_{n-1} satisfy the hypothesis of Proposition 6 and thus $\alpha^+(P_n)=2$. Now by Proposition 8 $\alpha^-(P_n)\leqslant 2$. To see that $\alpha^-(P_n)\neq 1$ we appeal to an argument similar to that used in Case 1. □

Proposition 11. *For* $n \geqslant 8$, $\alpha^+(C_n)+\alpha^-(C_n)=6$.

Proof. It suffices to show that for $n \equiv 0, 1,$ and 2 (mod 3), we have respectively $\alpha^+(C_n)=\alpha^-(C_n)=3$, $\alpha^+(C_n)=5$ and $\alpha^-(C_n)=1$, and $\alpha^+(C_n)=4$, $\alpha^-(C_n)=2$. We indicate how to prove that $\alpha^+(C_n)=5$ when $n \equiv 1 \pmod 3$. The remaining cases follow easily from the proof of Proposition 10.

Suppose $n \equiv 1 \pmod 3$ and let $k=\lceil \frac{1}{3}n\rceil$. If we denote C_n by $v_0\, v_1 \cdots v_n = v_0$, then removal of the set of points $\{v_0, v_2, v_4, v_6, v_8\}$ leaves four isolated points and P_{n-9}. However $\alpha(P_{n-9})=\alpha(P_n)-3=\alpha(C_n)-3$ and thus $\alpha^+(C_n)\leqslant 5$. If we remove only a single point from C_n, we obtain P_{n-1} and since $\alpha(P_{n-1})=k-1$, we know $\alpha^+(C_n)\geqslant 2$. It remains to show that removal of fewer than four points from P_{n-1} will not cause the domination number to exceed k. Suppose three points are removed from P_{n-1} leaving four components A_i, $1\leqslant i\leqslant 4$, containing a_i points respectively, and that $\sum_{i=1}^{4}\alpha(A_i)\geqslant k+1$. Since $a_i \geqslant 3\alpha(A_i)-2$ we have

$$\sum_{i=1}^{4} a_i \geqslant \left[3\sum_{i=1}^{4}\alpha(A_i)\right]-8 \geqslant 3(k+1)-8=3k-5.$$

However,

$$\sum_{i=1}^{4} a_i = 3(k-1)-3=3k-6,$$

a contradiction. Analogous arguments will show that if less than four components are formed as a result of removing fewer than four points from P_{n-1} the domination number will never exceed k. □

In the next section we begin a discussion of α-line-stability, i.e., we examine the effect of removing lines from a graph G on the domination number of G.

3. Line stability of $\alpha(G)$

For any graph invariant μ we define the *μ-line-stability* of a graph to be the minimum number of lines whose removal changes μ. The minimum number of lines which when removed from G increases μ is denoted by $\mu^{+\prime}(G)$; $\mu^{-\prime}(G)$ is the minimum number of lines the removal of which decreases μ.

We now present some elementary results concerning the α-line-stability of a graph. First note that when lines are removed from G, $\alpha(G)$ can only increase.

The following proposition, stated without proof, establishes a relation between $\alpha^{+\prime}(G)$ and the maximum degree Δ of G.

Proposition 12. *If there is at least one point $v \in G$ such that $\alpha(G-v) \geq \alpha(G)$, then $\alpha^{+\prime}(G) \leq \Delta$.*

To see that the hypothesis is required note that $\Delta(C_{3n+1}) = 2$ and $\alpha^{+\prime}(C_{3n+1}) = 3$. We now show that for trees the bound can be sharpened.

Proposition 13. *If T is a tree with at least two points, then $\alpha^{+\prime}(T) \leq 2$.*

Proof. If T contains a point v which is adjacent to at least two endpoints u_1 and u_2 then v is in every minimum dominating set for T. However both u_1 and either v or another endpoint adjacent to v will be in every dominating set for $T' = T - e$, where $e = \{u_1, v\}$. Hence $\alpha^{+\prime}(T) = 1$.

If no point of T is adjacent to two or more endpoints then T must have an endpoint u which is adjacent to a point w of degree two. Now remove a line from T such that the line $\{u, w\}$ forms a component in the resulting forest F. If $\alpha(F) > \alpha(T)$, we are done. If not, removing $\{u, w\}$ from F yields $\alpha(F - \{u, w\}) > \alpha(T)$ and we are done. □

The following result, analogous to that of Proposition 12, concerns another bound on $\alpha^{+\prime}$ for graphs. Define the *degree of an edge* $\{u, v\}$ of G, $d_e(\{u, v\})$, to be $\deg u + \deg v$ and set

$$\delta'(G) = \min\{d_e(\{u, v\}) \mid \{u, v\} \text{ is a line of } G\}.$$

We may now state the following inequality.

Proposition 14. *For any graph G,* $\alpha^{+\prime}(G) \leqslant \delta' - 1$.

Finally, we note that Sumner [4] has worked on a closely related problem. A graph G is *k-domination critical* if $\alpha(G) = k$ and $\alpha(G)$ decreases whenever any line from $\bar{G}$ is added to G. Sumner characterized 2-domination critical graphs and investigated k-critical graphs for $k \geqslant 3$. As an interesting dual concept we define the connected graph G to be $\alpha^{+\prime}$-critical if for each edge e of G, $\alpha(G-e) > \alpha(G)$. These graphs can be characterized as follows.

Proposition 15. *A graph G is* $\alpha^{+\prime}$*-critical if and only if it is the union of stars* $K_{1,n}$.

Proof. The sufficiency is clear. Suppose D is a minimum dominating set for G. First note that every point of degree at least two must be in D. However no two vertices in D can be adjacent. Hence G is a union of stars. □

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