

STIEFEL-WHITNEY CLASSES AND TORAL ACTIONS

Janey DACCACH

Universidade Federal de São Carlos

Arthur WASSERMAN

University of Michigan

Received 27 June 1983

Revised 20 January 1984

Certain Stiefel-Whitney classes of manifolds with smooth, effective toral actions are shown to be computable in terms of Poincare duals of fixed point sets of isotropy subgroups. As an application the toral degrees of symmetry of certain Dold manifolds are determined.

AMS Subj. Class.: 57R21, 57S15	
toral actions	Stiefel-Whitney classes
transformation groups	differentiable actions

0. Introduction

The Stiefel-Whitney classes of a manifold were originally defined as primary obstructions to finding frame fields on the manifold (see [6, 9]). A generic collection of vector fields gives rise to a possibly degenerate frame field and the points of degeneracy are Poincare dual to some Stiefel-Whitney class. A smooth action of a Lie group on a manifold gives rise to a collection of vector fields, hence a frame field whose points of degeneracy are just the fixed point sets of the actions. One expects, therefore, a strong connection between fixed point sets of smooth actions and Stiefel-Whitney classes. The purpose of this paper is to show that this is so for toral actions. For $H \subset T^n$ we denote by $F(M, H)$ the fixed point set of H on M and by $D(F(M, H))$ the Poincare dual of $F(M, H)$. Then we have:

Theorem A. *If T^n acts effectively on M^{n+k} then $\omega_i(M) = 0$ for $i > 2k$ and $\omega_{2k}(M) = \sum_{H \cong T^k} D(F(M, H))$.*

Note that the above sum is taken over all subgroups of T^n isomorphic to T^k . This sum may not be finite, but is locally finite, and hence is a well-defined cohomology class.

The action of T^n gives rise to n vector fields on M . One can find a subcollection of $n - k + 1$ fields which is a generic collection, and then the proof of Theorem A

follows quickly. A similar result for Pontrjagin classes of T^n vector bundles and fixed point sets of isotropy subgroups was obtained by Gomez [3] using very different methods.

Theorem B. *If T^n acts smoothly and effectively on M^{n+k} then $\omega_{2k-1}(M) = \sum_{H \approx T^{k-1} \times \mathbb{Z}_2} D(E(M, H))$.*

Here $E(M, H)$ denotes the components of $F(M, H)$ of codimension $2k-1$. The proof of Theorem B is considerably more difficult, since we cannot obtain a generic collection of $n-k+2$ vector fields from the action.

As an application of Theorem B we compute the toral degrees of symmetry of the Dold manifolds $P(4r+1, 2)$.

1. Preliminaries

Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ denote the n -torus. If M is a differentiable manifold with a smooth T^n action and $x \in M$, then T_x^n will denote the isotropy subgroup at x , $T_x^n = \{g \in T^n \mid gx = x\}$, and $F(M, H)$ will denote the fixed point set of H . Let $E(M, H) = \{x \in M \mid T_x^n = H\}$; then $E(M, H)$ is an open subset of $F(M, H)$ and if X is any component of $F(M, H)$, then X is a closed submanifold of M and $X \cap E(M, H)$ is either empty or dense in X (see [1]). Thus $\overline{E(M, H)}$ is a union of disjoint closed submanifolds of M .

If A^a is a closed submanifold of a manifold B^b , then the *Poincare dual* of A , $D(A) \in H^{b-a}(B; \mathbb{Z}_2) = \text{Hom}(H_{b-a}(B; \mathbb{Z}_2); \mathbb{Z}_2)$, is defined by $\langle D(A), C \rangle =$ number of points in $A \pitchfork C$, where \pitchfork indicates the intersection of A with the cycle C in general position [7].

The theorems in this paper have no content if $k > n$, and thus we assume henceforth that $k \leq n$.

If T^n acts effectively on M^{n+k} , then T_x^n must act effectively on the slice at x , and hence T_x^n admits a faithful representation into a space of dimension $\dim M - \dim(T^n/T_x^n) = k + \dim T_x^n$. It is well known that T^r acts effectively and linearly on \mathbb{R}^m only if $m \geq 2r$. Thus if $T_x^n \approx T^k$, T_x^n is maximal and $F(M, T_x^n) = E(M, T_x^n)$ consists of isolated orbits of codimension $2k$; if $T_x^n \approx T^{k-1} \times \mathbb{Z}_2$, then T_x^n admits a faithful representation into \mathbb{R}^{2k-1} by the slice representation; it follows that $\dim E(M, T_x^n)/T^n = 0$ and hence, $E(M, T_x^n) = \overline{E(M, T_x^n)}$ consists of isolated orbits of codimension $2k-1$. Note that $F(M, T_x^n)$ may have components of codimension $2k$ also.

We define classes $A(M) \in H^{2k}(M; \mathbb{Z}_2)$, $B(M) \in H^{2k-1}(M; \mathbb{Z}_2)$ by $A(M) = \sum_{H \approx T^k} D(F(M, H))$, $B(M) = \sum_{H \approx T^{k-1} \times \mathbb{Z}_2} D(E(M, H))$. Note that these classes are well defined, since homology classes are compactly supported and on any compact subset of M all but a finite number at $F(M, H)$ are empty. We can now state the main results of this paper.

Theorem A. *If T^n acts smoothly and effectively on M^{n+k} , then $A(M) = \omega_{2k}(M)$ and $\omega_i(M) = 0$ for $i > 2k$.*

Theorem B. *If T^n acts smoothly and effectively on M^{n+k} , then $B(M) = \omega_{2k-1}(M)$.*

Here $\omega_i(\zeta)$ is the i th Stiefel-Whitney class of the vector bundle ζ and $\omega_i(M)$ is the i th Stiefel-Whitney class of the tangent bundle of M .

Lemma 1. *It is sufficient to prove theorems A and B for compact manifolds without boundary.*

Proof. If Theorem A (for example) is false then there is a manifold M^{n+k} and a cycle $C \in H_{2k}(M; \mathbb{Z}_2)$ with $\langle A(M), C \rangle \neq \langle \omega_{2k}(M), C \rangle$. Since C is compactly supported there is a compact invariant submanifold $i: M' \subset M$ and $C' \in H_{2k}(M'; \mathbb{Z}_2)$ with $i_*(C') = C$. We may take $M' = f^{-1}([-\eta, \eta])$, where f is a smooth, proper, invariant real-valued function on M , $f(C) \subset (-\eta, \eta)$, and η is a regular value of f . Then $\langle \omega_{2k}(M), C \rangle = \langle \omega_{2k}(M), i_*(C') \rangle = \langle i^* \omega_{2k}(M), C' \rangle = \langle \omega_{2k}(M'), C' \rangle$ and $\langle A(M), C \rangle = \langle A(M), i_*(C') \rangle = \langle i^* A(M), C' \rangle = \langle A(M'), C' \rangle$ by the naturality of the classes $\omega_{2k}(M)$, $A(M)$. Hence it is sufficient to prove the theorems for compact manifolds with boundary. Finally by passing to the double of M' , M'' , we have $j: M' \hookrightarrow M''$ and $\langle A(M''), j_*(C') \rangle = \langle A(M'), C' \rangle$ and $\langle \omega_{2k}(M''), j_*(C') \rangle = \langle \omega_{2k}(M'), C' \rangle$ and hence it is sufficient to prove the theorem for M'' ; i.e. for closed manifolds. \square

Finally, recall that a smooth action of the Lie group G on the manifold M , $\psi: G \times M \rightarrow M$, gives rise to a map ρ from the Lie algebra of G to vector fields on M (cf. [1]) via

$$T_e(G) \times M \subset T_e(G) \times T(M) \subset T(G) \times T(M) \xrightarrow{d\psi} T(M),$$

i.e. for $v \in T_e(G)$, $x \in M$, $\rho(v)(x) = d\psi_{(e, x)}(v, 0)$. Note that $\rho(v)(x) = 0$ if and only if $v \in T_e(G_x)$.

2. Proof of Theorem A

Recall that vector spaces $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^n$ are said to be in *general position* if the dimension of $V \cap W$ is as small as possible; i.e., $V \cap W = 0$ if $\dim V + \dim W \leq n$ and $\dim V \cap W = \dim V + \dim W - n$ if $\dim V + \dim W \geq n$. If $\dim W = 1$ and $W = \text{span}\langle \omega \rangle$, then W is in general position with respect to $V \subsetneq \mathbb{R}^n$ if and only if $\omega \notin V$.

Lemma 2. *Let $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_r$ be a finite collection of proper vector subspaces of \mathbb{R}^n and $s + 1 \leq n$. Then there is a pair of subspaces $W_1 \subset W_2 \subset \mathbb{R}^n$ with $\dim W_1 = s$, $\dim W_2 = s + 1$ such that W_i is in general position with respect to \mathfrak{h}_j for $1 \leq j \leq r$, $1 \leq i \leq 2$.*

Proof. By induction on s . If $s=0$, choose $\omega \in \mathbb{R}^n - \bigcup_{j=1}^r \mathfrak{h}_j$, $W_1=0$, and $W_2 = \text{span}(\omega)$. If true for $s=k$, then we have subspaces $W_0 \subset W_1$ with $\dim W_1 = k$ and in general position with respect to the \mathfrak{h}_j ; choose $W_2 = \text{span}(W_1, v)$ where v is not in W_1 and not in $\text{span}(\mathfrak{h}_j \cup W_1)$ for $\text{span}(\mathfrak{h}_j \cup W_1) \neq \mathbb{R}^n$. \square

Lemma 3. *If T^n acts smoothly and effectively on the closed manifold M^{n+k} , then there is an equivariant splitting of the tangent bundle of M , $T(M) = \theta^{n-k} \oplus \zeta^{2k}$ where θ^{n-k} is the trivial bundle of dimension $n-k$. Furthermore, there is an equivariant section Y of ζ^{2k} such that $Y(x) = 0$ if and only if $T_x^n \approx T^k$.*

Proof. Since M is compact there are only a finite number of isotropy subgroups, H_1, H_2, \dots, H_r ; let $\mathfrak{h}_i \subset T_e(T^n) = \mathbb{R}^n$ denote the Lie algebra of H_i . By Lemma 2, we can find subspaces $W_1 \subset W_2 \subset T_e(T^n)$ with $\dim W_1 = n-k$, $\dim W_2 = n-k+1$ and both W_1, W_2 in general position with respect to the subspaces \mathfrak{h}_i . Since $\dim \mathfrak{h}_i \leq k$, $W_1 \cap \mathfrak{h}_i = 0$ and hence the map $\rho: T_e(T^n) \times M \rightarrow T(M)$, when restricted to $W_1 \times M$, defines a trivial subbundle θ^{n-k} of $T(M)$, invariant since T^n is abelian. Let ζ^{2k} denote the orthogonal complement of θ^{n-k} in $T(M)$ with respect to some Riemannian metric on M and $\pi: T(M) \rightarrow \zeta$ the orthogonal projection. To construct the desired section of ζ , let $v \in W_2 - W_1$ and set $Y(x) = \pi(\rho(v)(x))$; if $Y(x) = 0$ then $\rho(v)(x) \in \theta^{n-k}$, $\rho(v)(x) = \rho(\omega)(x)$ for some $\omega \in W_1$. Then $\rho(v-\omega)(x) = 0$; i.e. $v-\omega \in T_e(\mathbb{R}^n) = \mathfrak{h}_j$. Since W_2 is in general position with respect to \mathfrak{h}_j , $v-\omega \in W_2 \cap \mathfrak{h}_j \neq 0$, and dimension $W_2 = n-k+1$, we conclude that $\dim \mathfrak{h}_j \geq k$; i.e. $T_x^n \approx T^k$. \square

Finally, we have:

Proof of Theorem A. By Lemma 1 we need only consider the case M closed. If M is closed we have by Lemma 3 that $T(M) = \theta^{n-k} \oplus \zeta^{2k}$ and thus $\omega_i(M) = \omega_i(\zeta)$. Hence $\omega_i(M) = 0$ for $i > 2k$. We next must show that for every class $x \in H_{2k}(M; \mathbb{Z}_2)$ we have $\langle \omega_{2k}(\zeta), x \rangle = \langle A(M), x \rangle$. Since every class x is Steenrod representable [(8)] there is a closed manifold Q^{2k} and a smooth map $f: Q \rightarrow M$ such that $f_*[Q] = x$, where $[Q] \in H_{2k}(Q; \mathbb{Z}_2)$ is the fundamental class; we also choose f to be transversal to $F(M, H_i)$ for $H_i \approx T^k$. Then $\langle A(M), x \rangle = \sum_{H_i \approx T^k} \langle D(F(M, H_i)), x \rangle = \sum_{H_i \approx T^k}$ number of points in $f(Q) \cap F(M, H_i) \bmod 2$ since f is transverse to $F(M, H_i)$. On the other hand, $\langle \omega_{2k}(M), x \rangle = \langle \omega_{2k}(\zeta), f_*[Q] \rangle = \langle \omega_{2k}(f^*\zeta), [Q] \rangle = \langle \chi(f^*\zeta), [Q] \rangle$, where χ denotes the mod 2 Euler class. Thus to compute $\langle \omega_{2k}(M), x \rangle$ it is sufficient to count, mod 2, the number of zeros of generic section, s , of $f^*\zeta$, [(5)]; i.e., a section transverse to the zero section. The section Y of ζ constructed in Lemma 3 pulls back to a section Y^* of $f^*(\zeta)$, and $Y^*(q) = 0$ if and only if $Y(f(q)) = 0$ if and only if $f(q) \in \bigcup_{H_i \approx T^k} F(M, H_i)$ by Lemma 3. Thus we only need to show that Y^* is generic.

At a zero of Y^* the map f is an immersion (since codimension $F(M, H_i) = 2k = \text{dimension } Q$) and we may assume that f maps a neighborhood of $q \in Q$ onto a linear slice $S_{f(q)} = S_p$ at $f(q)$, and hence we only need to examine the section Y on

the slice S_p . Set $H = T_p^n$, $\mathfrak{h} = \text{Lie algebra of } H$ and choose $0 \neq z \in \mathfrak{h} \cap W_2$. Then $z = av + \omega$ for some $\omega \in W_1$ and $a \neq 0$ since W_1 is in general position with respect to \mathfrak{h} ; i.e., $W_1 \cap \mathfrak{h} = 0$ and thus, by choosing a multiple of z we can take $a = 1$; i.e., $v + \omega = h \in \mathfrak{h}$. It is crucial to note that h is *not* in $\mathfrak{h}' \subsetneq \mathfrak{h}$ since W_2 is in general position with respect to \mathfrak{h}' and thus $W_2 \cap \mathfrak{h}' = 0$. Applying $\pi \circ \rho$ we have $\pi\rho(v) + \pi\rho(\omega) = \pi\rho(h)$ or $Y = \pi\rho(h)$. We now claim that the map π is an isomorphism when restricted to the tangent space of the slice S_p , and hence we need only show that $\rho(h)$ is generic. At p we have $\theta_p = \rho(\omega_1)(p)$ is $n - k$ dimensional as is, the orbit of p . Hence $\zeta_p = T_p(S_p)$ and $\pi: T(S_p) \rightarrow \zeta$ is an isomorphism in a neighborhood of p . Thus it is sufficient to show that $\rho(h): S_p \rightarrow T(S_p)$ is generic. To study the section $\rho(h)$, choose an isomorphism $\alpha: H \rightarrow T^k$ and a diffeomorphism $\psi: S_p \rightarrow \mathbb{C}^k$ such that the action of H on S_p corresponds to the action of T^k on \mathbb{C}^k given by $(t_1, \dots, t_k) \cdot (z_1, \dots, z_k) = (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_k} z_k)$. Note that $\rho(\partial/\partial t_j)(z_1, \dots, z_k) = (0, \dots, 0, 2\pi i z_j, 0, \dots, 0)$ and since $d\alpha(h) = \sum a_j \partial/\partial t_j$, $\rho d\alpha(h) = d\alpha(\rho(h))$ is the linear vector field $Z \rightarrow 2\pi i \text{diag}(a_j)Z$. This complex linear map has 0 as a regular value if and only if all a_j 's are non-zero; but if $a_i = 0$, for example, $d\alpha(h) \in T_e(T^{k-1})$ where $T^{k-1} = \{(0, t_2, \dots, t_k) \in T^k\}$ and $T_e(T^{k-1}) = d\alpha(\mathfrak{h}')$; i.e., $h \in \mathfrak{h}' \subset \mathfrak{h}$ which is impossible. Hence $d\alpha(\rho(h))$ is generic, and consequently Y^* is also. \square

3. Proof of Theorem B

We start with a simplifying lemma:

Lemma 4. *It is sufficient to prove Theorem B for closed manifolds M such that $F(M, H) = \emptyset$ for $H \approx T^k$.*

Proof. By Lemma 1 it is sufficient to prove Theorem B for closed T^n manifolds. Let M_1 be any closed T^n manifold and $x \in H_{2k-1}(M_1; \mathbb{Z})$. Choose $f: Q^{2k-1} \rightarrow M_1$ with $f_*[Q] = x$ and f transverse to $F(M_1, H_i)$ for $H_i \approx T^k$; we use the fact that there are only a finite number of H_i since M_1 is compact. Since $\dim Q = 2k - 1 < 2k = \text{codimension } F(M_1, H_i)$ for $H_i \approx T^k$, we have $f(Q) \cap F(M_1, H_i) = \emptyset$ for $H_i \approx T^k$. Let $j: M_2 \subset M_1$ be an invariant manifold with boundary such that M_2 is a neighborhood of $f(Q)$, $M_2 \cap F(M_1, H_i) = \emptyset$ for $H_i \approx T^k$, and $\tilde{f}: Q \rightarrow M_2$. Let $M = \text{double of } M_2$ and $k: M_2 \subset M$. We may take $M_2 = g^{-1}([- \eta, \eta])$ where g is a smooth, proper, invariant real-valued function on M_1 , $g(f(Q)) = 0$, $\eta > 0$ is a regular value of g , and $g(x) > \eta$ for $x \in F(M_1, H_i)$ and all i . Then $\langle \omega_{2k-1}(M_1), x \rangle = \langle \omega_{2k-1}(M_1), f_*[Q] \rangle = \langle \omega_{2k-1}(M_1), j_* \tilde{f}_*[Q] \rangle = \langle j^* \omega_{2k-1}(M_1), \tilde{f}_*[Q] \rangle = \langle \omega_{2k-1}(M_2), \tilde{f}_*[Q] \rangle = \langle k^* \omega_{2k-1}(M), \tilde{f}_*[Q] \rangle = \langle \omega_{2k-1}(M), k_* \tilde{f}_*[Q] \rangle$, and similarly $\langle B(M_1), x \rangle = \langle B(M), k_* \tilde{f}_*[Q] \rangle$. Hence, it is sufficient to prove the theorem for M , a closed manifold with $F(M, H_i) = \emptyset$ for $H_i \approx T^k$. \square

Lemma 5. *If M^{n+k} is a closed manifold with a smooth effective T^n action and*

$F(M, H) = \emptyset$ for $H \approx T^k$, then $T(M)$ splits equivariantly as $T(M) = \theta^{n-k+1} \oplus \zeta^{2k-1}$, where θ^{n-k+1} is a trivial bundle of dimension $n-k+1$. Furthermore, ζ^{2k-1} admits a section Y such that $Y(x) = 0$ if and only if $T_x^n \approx T^{k-1} \times \mathbb{Z}_2$.

Proof. Let H_1, \dots, H_r be the isotropy subgroups in M and $\mathfrak{h}_i = T_e(H_i)$. By Lemma 2 we can choose subspaces $W_1 \subset W_2 \subset \mathbb{R}^n = T_e(T^n)$ with $\dim W_1 = n-k+1$, $\dim W_2 = n-k+2$ and W_1, W_2 in general position with respect to $\mathfrak{h}_1, \dots, \mathfrak{h}_r$. In particular, since $\dim \mathfrak{h}_i < k$ for all i and $\dim W_1 = n-k+1$, $W_1 \cap \mathfrak{h}_i = 0$. Then, as in the proof of Lemma 3, the map $\rho: W_1 \times M \rightarrow T(M)$ defines an invariant trivial subbundle θ^{n-k+1} of $T(M)$. Let ζ^{2k-1} denote the orthogonal complement of θ^{n-k+1} with respect to some invariant Riemannian metric and let $\pi: T(M) \rightarrow \zeta^{2k-1}$ denote the projection. Choose $v \in W_2 - W_1$ and set $Z = \pi\rho(v)$. Then $Z(x) = 0$ if and only if $\rho(v)(x) - \rho(\omega)(x) = 0$ for some $\omega \in W_1$; i.e. if and only if $v - \omega = \mathfrak{h} \in T_e(T_x^n)$. Since $T_x^n = H_i$ for some i , W_2 is in general position with respect to \mathfrak{h}_i , and $W_2 \cap \mathfrak{h}_i = \mathfrak{h} \neq 0$, we conclude that $\dim \mathfrak{h}_i \geq k-1$. Since $\dim \mathfrak{h}_i \leq k-1$ by assumption, we conclude that $\dim H_i = k-1$ and $H_i \approx T^{k-1} \times \mathbb{Z}_2$ or $H_i = T^{k-1}$. We must now modify Z to eliminate the second possibility.

Let $H_1 \approx T^{k-1}$. We shall first show that $F(M, H_1)/T^n$ is a one-manifold (possibly with boundary). Let $x \in F(M, H_1)$. If $T_x^n = H_1$, then the slice at x , S_x , has dimension $2k-1$. We may choose a diffeomorphism $S_x \approx \mathbb{C}^{k-1} \times \mathbb{R}$ and an isomorphism $H_1 \approx T^{k-1}$ such that the action $H \times S_x \rightarrow S_x$ is equivariantly diffeomorphic to $T^{k-1} \times \mathbb{C}^{k-1} \times \mathbb{R} \rightarrow \mathbb{C}^{k-1} \times \mathbb{R}$, $(t_1, \dots, t_{k-1})(z_1, \dots, z_{k-1}, y) \rightarrow (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_{k-1}} z_{k-1}, y)$. Hence $F(S_x, H_1) = \{(0, \dots, 0, y) \in \mathbb{C}^{k-1} \times \mathbb{R}\}$. Thus, in a sliced neighborhood U of x we have $F(U, H_1)/T^n \approx \mathbb{R}$. If $T_x^n \neq H_1$, then $T_x^n \approx H_1 \times \mathbb{Z}_2$ and dimension S_x is still $2k-1$; but the action $T_x^n \times S_x \rightarrow S_x$ is equivalent to $T^{k-1} \times \mathbb{Z}_2 \times \mathbb{C}^{k-1} \times \mathbb{R} \rightarrow \mathbb{C}^{k-1} \times \mathbb{R}$, $(t_1, \dots, t_{k-1}, g)(z_1, \dots, z_{k-1}, y) \rightarrow (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_{k-1}} z_{k-1}, -y)$, and again $F(S_x, H_1) = \{(0, 0, \dots, 0, y) \in \mathbb{C}^{k-1} \times \mathbb{R}\}$. Thus, in a sliced neighborhood U at x we have that $F(U, H_1)/T^n$ is diffeomorphic to $\mathbb{R}/\mathbb{Z}_2 \approx [0, \infty)$ and the map $F(U, H_1) \rightarrow F(U_1, H_1)/T^n$ given on S_x by $(0, \dots, 0, y) \mapsto \pm y^2$ is smooth. Thus $F(M, H_1)/T^n$ is a manifold, and the boundary points of this manifold correspond to orbits with isotropy subgroup isomorphic to $T^{k-1} \times \mathbb{Z}_2$.

The section Z of ζ^{2k-1} has as zero set the manifold $F = \bigcup_{H_i \approx T^{k-1}} F(M, H_i)$; the section Y will agree with Z outside of an invariant tubular neighborhood of F . Let A be a component of F , $A \subset F(M, H_1)$ say, with U an invariant tubular neighborhood of A and $p: U \rightarrow A \rightarrow A/T^n$ the projection.

Consider first the case that $A/T^n \approx S^1$. The map $p: U \rightarrow A/T^n$ is constant on orbits, thus $dp \circ \rho = 0$. In particular $dp(\theta^{n-k+1}) = 0$. But p is the projection of a smooth equivariant fibre bundle. Hence, there is a section X of ζ^{2k-1} such that $dp(X) = \partial/\partial\theta \in T(S^1)$. We define the desired section Y in U by $Y(x) = Z(x) + b(x)X(x)$, where $b: M \rightarrow [0, 1]$ is any smooth invariant function with $b(A) = \{1\}$ and support $b \subset U$. Note that $Z(x) = \pi\rho(v)(x)$ and hence $dp_x Y(x) = dp_x(b(x)X(x)) = b(x) \partial/\partial\theta$. Thus if $b(x) \neq 0$, $Y(x) \neq 0$, and if $b(x) = 0$, $Y(x) = Z(x)$ which does not vanish on $U - A$. Hence $Y|_U$ is a nonzero section of ζ^{2k-1} .

Consider finally the case $A/T^n = [0, 1]$. The map $p: U \rightarrow A \rightarrow A/T^n \approx [0, 1]$ is not the projection of a fibre bundle in this case. Hence we define X_1 , a section of $T(U)$, by $X_1 = \text{grad } p$ (with respect to some convenient, invariant Riemannian metric) and define the section X of ζ by $X(q) = \pi b(q)X_1$ where $b: M \rightarrow [0, 1]$ satisfies $b(A) = 1$ and support $B \subset U$, and $\pi: T(M) \rightarrow \zeta$. Finally, set $Y = X + Z$ and note that if $b(q) = 0$, $Y(q) = Z(q) \neq 0$ since $Z(q)$ vanishes only on A and $b(A) = 1$; if $b(q) \neq 0$, then $dp(Y)(q) = dp(X)(q) = b(q) dp(\pi \text{ grad } p) = b(q) dp(\text{grad } p)$ since $dp(\theta) = 0$; thus $dp(Y)(q) = b(q) dp(\text{grad } p) = b(q) \langle \text{grad } p, \text{grad } p \rangle \partial/\partial t \neq 0$ except at the minimum and maximum of p ; i.e., at $F(A, H)$, $H \approx T^{k-1} \times \mathbb{Z}_2$. \square

Finally we have:

Proof of Theorem B. We may assume by Lemma 1 that T^n acts smoothly and effectively on the closed manifold M^{n+k} , and that $F(M, H) = \emptyset$ for $H \approx T^k$ (by Lemma 4). Thus, by Lemma 5, we assume that $T(M) = \theta^{n-k+1} \oplus \zeta^{2k-1}$ and $\omega_i(M) = \omega_i(\zeta)$. Let $x \in H_{2k-1}(M, \mathbb{Z}_2)$ be represented by a map $f: Q^{2k-1} \rightarrow M$ of the closed manifold Q with f transversal to $F(M, H_i)$ for each isotropy subgroup H_i . As in the proof of Theorem A, $\langle x, B(M) \rangle = \sum_{H_i \approx T^{k-1} \times \mathbb{Z}_2} \langle f_*[Q], D(F(M, H_i)) \rangle =$ number of points (mod 2) in $f(Q) \cap F(M, H_i)$ for $H_i \approx T^{k-1} \times \mathbb{Z}_2$ since f is transverse to $F(M, H_i)$. Similarly, $\langle x, \omega_{2k-1}(M) \rangle = \langle f_*[Q], \omega_{2k-1}(\zeta) \rangle = \langle [Q], \omega_{2k-1}(f^*\zeta) \rangle = \langle [Q], \chi(f^*\zeta) \rangle$ is the number of zeros of a generic section of $f^*\zeta$. We claim that Y^* , the pullback of the section Y , is a generic section and clearly Y^* vanishes at p if and only if $f(p) \in F(M, H_i)$, $H_i \approx T^{k-1} \times \mathbb{Z}_2$. Thus the theorem will be proven when we show that Y^* is generic; i.e., transverse to the zero section of $f^*\zeta$. If $Y^*(p) = 0$ then $Y(f(p)) = 0$; i.e., $T_{f(p)}^n \approx T^{k-1} \times \mathbb{Z}_2$ by Lemma 5. Let $H = T_{f(p)}^n$; then $f(p) \in F(M, H)$, $\text{codim } F(M, H) = 2k - 1 = \dim Q$ and f is transverse to $F(M, H)$. Thus there is a neighborhood \mathcal{O} of p such that $f|_{\mathcal{O}}$ is an imbedding: we may choose f so that $f(\mathcal{O})$ is a slice S at $f(p)$. Thus we need only examine the section Y on S . Choosing coordinates on S centered at $f(p)$, and identifying $T_q(S)$ with ζ_q via π as before, we have $S = \{(z_1, \dots, z_{k-1}, y)\}$, $Z(q)$ is the linear vector field $q = (z_1, \dots, z_{k-1}, y) \rightarrow 2\pi i(a_1 z_1, \dots, a_{k-1} z_{k-1}, 0)$ with $a_i \neq 0$ as in Theorem A and $X(q)$ is given by $(0, 0, \dots, 0, \pm b(q)v)$ up to a positive constant. Here we have chosen the Riemannian metric in U to be an invariant bundle metric. Thus computing $dY_{f(p)}$ we have $dY_{f(p)} = \mathcal{R}(2\pi i \text{diag}(a_1, \dots, a_{k-1})) \oplus [+1]$ where \mathcal{R} denotes the real form of a complex matrix, \oplus denotes direct sum matrices, and we have used the fact that $b(f(p)) = 1$ and $db_{f(p)} = 0$ since b has a maximum on A . Thus Y is generic. \square

Corollary 6. *If T^n acts smoothly and effectively on the oriented manifold M^{n+k} then $\omega_{2k-1}(M) = 0$.*

Proof. Since M is orientable and T^n is connected, the T^n action must preserve the orientation on M . If $x \in M$ with $T_x^n = T^{k-1} \times \mathbb{Z}_2$, then $g = (1, h) \in T^{k-1} \times \mathbb{Z}_2$ with $h \neq 1$ reverses the orientation on the slice at x , $S_x \approx \mathbb{C}^{k-1} \times \mathbb{R}$, $g(z_1, \dots, z_{k-1}, y) =$

$(z_1, \dots, z_{k-1}, -y)$ and obviously preserves the orientation on the orbit, hence is orientation reversing. Thus $E(M, H) = \emptyset$ for $H \approx T^{k-1} \times \mathbb{Z}_2$, the class $B(M) = 0$, and, by Theorem B, $\omega_{2k-1}(M) = 0$. \square

As an application of the result above we shall compute the smooth toral degree at symmetry of the Dold manifold $P(4r+1, 2)$. Recall that the smooth toral degree of symmetry of a manifold, M , is the dimension of the largest torus that acts smoothly and effectively on M (see [4]).

The Dold manifold $P(m, n)$ is the orbit space of $S^m \times \mathbb{C}P^n$ by the involution $(x, z_0, \dots, z_n) \rightarrow (-x, \bar{z}_0, \dots, \bar{z}_n)$, thus $\dim P(m, n) = m + 2n$. Also $H^*(P(m, n); \mathbb{Z}_2)$ is generated by classes c, d with degree $c = 1$, degree $d = 2$, with only the relations $c^{m+1} = 0, d^{n+1} = 0$. We now have:

Proposition 7. *The toral degree of symmetry of $P(4r+1, 2)$ is $2r+2$.*

Proof. We first construct an effective action of T^{2r+2} on $S^{4r+1} \times \mathbb{C}P^2$ that commutes with the involution. Regard $S^{4r+1} \subset \mathbb{C}^{2r+1}$ and let $\psi: T^{2r+2} \times S^{4r+1} \times \mathbb{C}P^2 \rightarrow S^{4r+1} \times \mathbb{C}P^2$ be given by $(t_1, \dots, t_{2r+2})(z_1, \dots, z_{2r+1}, \omega_0, \omega_1, \omega_2) = (e^{2i\pi t_1} z_1, \dots, e^{2i\pi t_{2r+1}} z_{2r+1}, \omega_0, \omega_1 \cos 2\pi t_{2r+2} + \omega_2 \sin 2\pi t_{2r+2} - \omega_1 \sin 2\pi t_{2r+2} + \omega_2 \cos 2\pi t_{2r+2})$. Clearly ψ descends to an effective action on $P(4r+1, 2)$ and hence the toral degree at symmetry is $\geq 2r+2$.

To prove that toral degree of symmetry is $2r+2$, we suppose T^{2r+3} acts smoothly and effectively on $P(4r+1, 2)$. Then $\omega_1(P(4r+1, 2)) = 0$ and $\omega_{4r+3}(P(4r+1, 2)) = c^{4r+1}d \neq 0$ since the total Stiefel-Whitney class of $P(m, n)$ is given by $W(P(m, n)) = (1+c)^m(1+c+d)^{n+1}$ ([2]). But by Corollary 6, $\omega_{4r+3} = 0$ if $P(4r+1, 2)$ admits an action of T^{2r+3} . Thus the smooth toral degree of symmetry of $P(4r+1, 2)$ is $2r+2$. \square

References

- [1] G. Bredon, Introduction to Compact Transformation Groups (Academic Press, New York, 1972).
- [2] A. Dold, Erzeugende du Thomschen Algebra \mathcal{N}_* , Math. Z. 65 (1956) 25-35.
- [3] F. Gomez, A residue formula for characteristic classes, Topology 21(1) (1982).
- [4] Wu Yi Hsiang, Cohomological Theory of Topological Transformation Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85.
- [5] J.W. Milnor and J.D. Stasheff, Characteristic Classes, Annals of Math. Studies 76 (Princeton University Press, Princeton, NJ, 1974).
- [6] E. Stiefel, Richtungsfelder und Fernparallelismus in Mannigfaltigkeiten, Comment. Math. Helv. 8 (1936) 3-51.
- [7] M. Seifert and W. Threlfall, A Textbook of Topology (Academic Press, New York, 1980).
- [8] R. Thom, Quelques proprietes globales des varietes differentiables, Comment. Math. Helv. 28 (1954) 17-86.
- [9] H. Whitney, Sphere spaces, Proc. Nat. Acad. Sci. 21 (1935) 462-468.