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Semi-boundedness of Systems of Differential Operators

LI-YENG SUNG

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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1. INTRODUCTION

Let $P(x, D) = A(x)\, D^2 + B(x) D + C(x)$ be a system of second order differential operators defined on an open interval (a, b) of $\mathbb{R}$. $A(x)$, $B(x)$ and $C(x)$ are N by N matrices whose components belong to $C^\infty(a, b)$ and

$$D = \frac{1}{\sqrt{-1}} \frac{d}{dx}.$$

The main result of this paper is the following theorem.

THEOREM 1.1. *If $p(x, \xi) = A(x)\, \xi^2 + B(x)\, \xi + C(x)$ is positive semi-definite for any $x \in (a, b)$ and $\xi \in \mathbb{R}$, then given a compact subset K of (a, b) we can find a constant $M_K > 0$ such that*

$$\operatorname{Re} \int_a^b (P(x, D)\, u) \cdot \bar{u}\, dx \geqslant -M_K \int_a^b |u|^2\, dx, \tag{1.1}$$

where $u = (u_1, ..., u_n)^T$ and $u_i \in C^\infty(K)$ for $1 \leqslant i \leqslant n$.

The proof of Theorem 1.1 can easily be generalized to prove that if $P(x, D)$ is a system of differential operators of order m whose symbol is positive semi-definite, then

$$\operatorname{Re} \int_a^b (P(x, D)\, u) \cdot \bar{u}\, dx \geqslant -M_K\, \|u\|^2_{(m-2)/2}. \tag{1.2}$$

An inequality of the type in (1.2) with $(m-2)/2$ replaced by $(m-1)/2$ was proved by Hörmander [2] for pseudo-differential operators on $\mathbb{R}^n$. His result was generalized by Lax and Nirenberg [3] to systems of pseudo-differential operators on $\mathbb{R}^n$ (still with $(m-2)/2$ replaced by $(m-1)/2$). Fefferman and Phong [1] proved (1.2) for a single pseudo-differential operator on $\mathbb{R}^n$.

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2. Fourier Series Representation

In this section we will transform (1.1) into an inequality for infinite quadratic forms.

Since the assumption and conclusion of Theorem 1.1 are invariant under any affine transformation of the real line, we may assume that $K \subseteq (0, 2\pi) \subseteq (a, b)$.

Let $\phi(x) \geqslant 0$ be a C^∞ function with compact support in $(0, 2\pi)$ such that $\phi(x) = 1$ on a neighborhood of K, then $\int_a^b (P(x, D)\, u) \cdot \bar{u}\, dx = \int_0^{2\pi} (\phi(x)\, P(x, D)\, u) \cdot \bar{u}\, dx$. We may therefore assume that the supports of $A(x)$, $B(x)$ and $C(x)$ are contained in $(0, 2\pi)$.

Instead of taking real parts all the time, we can simplify in the following way. Let $Q(x, D)\, u = D(A(x)\, Du) + \frac{1}{2}[B(x)\, Du + D(B(x)\, u)] + C(x)\, u$, then $P(x, D)\, u - Q(x, D)\, u = S_1 u + S_2 u$, where $S_1 u = (\sqrt{-1}/2)[A'(x)\, Du + D(A'(x)\, u)]$ and $S_2 u = [-\frac{1}{2} A''(x) + (\sqrt{-1}/2)\, B'(x)]\, u$. Since $p(x, \xi) = A(x)\, \xi^2 + B(x)\, \xi + C(x)$ is positive semi-definite for any $\xi \in \mathbb{R}$, it follows that $A(x)$, $B(x)$ and $C(x)$ are Hermitian matrices. Therefore, $\int_0^{2\pi} (s_1 u) \cdot \bar{u}\, dx$ is purely imaginary and $\operatorname{Re} \int_0^{2\pi} (P(x, D) u) \cdot \bar{u}\, dx = \int_0^{2\pi} (Q(x, D)\, u) \cdot \bar{u}\, dx + \operatorname{Re} \int_0^{2\pi} (S_2 u) \cdot \bar{u}\, dx$. We can obviously find $\alpha > 0$ such that $|\int_0^{2\pi} (S_2 u) \cdot \bar{u}\, dx| \leqslant \alpha \int_0^{2\pi} |u|^2\, dx$. Therefore (1.1) is equivalent to

$$\int_0^{2\pi} (Q(x, D)\, u) \cdot \bar{u}\, dx \geqslant -M_K \int_0^{2\pi} |u|^2\, dx. \tag{2.1}$$

From now on K is fixed and we can drop the subscript K.

We can represent u, A, B and C by their Fourier series, i.e., $u(x) = \sum_{n=-\infty}^{\infty} e^{inx} u_n$, $A(x) = \sum_{n=-\infty}^{\infty} e^{inx} A_n$, $B(x) = \sum_{n=-\infty}^{\infty} e^{inx} B_n$ and $C(x) = \sum_{n=-\infty}^{\infty} e^{inx} C_n$. Here $u_n = (1/2\pi) \int_0^{2\pi} e^{-inx} u(x)\, dx$ and $A_n = (1/2\pi) \int_0^{2\pi} e^{-inx} A(x)\, dx$, etc.

Since A, B, C and u are smooth and have compact supports in $(0, 2\pi)$, we have

$$\max(|u_n|, |A_n|, |B_n|, |C_n|) = O(n^{-\beta}) \qquad \text{for } \beta = 1, 2, \ldots, \tag{2.2}$$

where $|\cdot|$ is any norm for vectors or matrices.

A simple calculation shows that (2.1) is equivalent to

$$\sum_{m, n \in \mathbb{Z}} \left[mnA_{n-m} + \frac{m+n}{2} B_{n-m} + C_{n-m} \right] u_m \cdot \bar{u}_n \geqslant -M \sum_{n \in \mathbb{Z}} |u_n|^2. \tag{2.3}$$

Let $m, n \in \mathbb{Z}$ and $|m-n| \geqslant |n|/2$, then we also have $|m-n| \geqslant \frac{1}{3}|m|$. By

(2.2), we have

$$\sum_{\substack{m,n\in\mathbb{Z}\\ |m-n|\geqslant|n|/2}} \left[|m|\,|n|\,|A_{n-m}| + \frac{|m|+|n|}{2}\,|B_{n-m}| + |C_{n-m}|\right] |u_m|\,|u_n|$$

$$\leqslant |c_0|\,|u_0|^2 + \sum_{\substack{m,n\in\mathbb{Z}\\ |m-n|\geqslant\max(1,|n|/2)}} \frac{c}{|m-n|^2}\,|u_m|\,|u_n|$$

for some $c>0$. But

$$\sum_{\substack{m,n\in\mathbb{Z}\\ |m-n|\geqslant\max(1,|n|/2)}} \frac{c}{|m-n|^2}\,|u_m|\,|u_n|$$

$$\leqslant \sum_{|m-n|\geqslant 1} \frac{c}{|m-n|^2}\,|u_m|\,|u_n|$$

$$= \sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{c}{k^2}\,|u_m|\,|u_{m+k}| + \sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{c}{k^2}\,|u_m|\,|u_{m-k}|$$

$$\leqslant \frac{c\pi^2}{3}\sum_{m=-\infty}^{\infty} |u_m|^2.$$

Therefore in order to prove (2.3), it suffices to prove

$$\sum_{m,n\in\mathbb{Z}}{}' \left[mnA_{n-m} + \frac{m+n}{2}\,B_{n-m} + C_{n-m}\right] u_m\cdot\bar{u}_n$$

$$\geqslant -M\sum_{m=-\infty}^{\infty} |u_m|^2, \tag{2.4}$$

where $\sum'$ means the summation is taken over m, n such that $|m-n|<|n|/2$. In particular, m and n are either both positive or both negative. We shall prove (2.4) for $m, n\in\mathbb{Z}^+$, the case where $m, n\in\mathbb{Z}^-$ can be proved similarly.

3. Consequences of the Symbol Being Positive Semi-definite

If $A(x)\,\xi^2 + B(x)\,\xi + C(x)$ is positive semi-definite, so is $\frac{1}{2}A(x)\,\xi^2 + B(x)\,\xi + 2C(x)$. Let v_n, $n\geqslant 1$, be a rapidly decreasing (i. e., satisfying (2.2)) sequence of N-vectors and $v=\sum_{n\geqslant 1} e^{inx}v_n$. Then we have $\int_0^{2\pi}[\frac{1}{2}A(x)\,\xi^2 + B(x)\,\xi + 2C(x)]\,v\cdot\bar{v}\,dx\geqslant 0$ and hence

$$\sum_{m,n\geqslant 1}\left[\frac{\xi^2}{2}A_{n-m} + \xi B_{n-m} + 2C_{n-m}\right] v_m\cdot\bar{v}_n\geqslant 0. \tag{3.1}$$

It follows that we have

$$\sum_{m,\, n \geqslant 1} \left[\frac{\xi^2}{2} A_{n-m} + \xi B_{n-m} + 2C_{n-m}\right] e^{-(\xi-m)^2/4m^2} \frac{v_m}{\sqrt{m}}$$

$$\times e^{-(\xi-n)^2/4n^2} \frac{\bar{v}_n}{\sqrt{n}} \geqslant 0. \tag{3.2}$$

If we integrate (3.2), we get

$$\int_{-\infty}^{\infty} \sum_{m,\, n \geqslant 1} e^{\Psi(m,\, n,\, \xi)} \frac{[(\xi^2/2) A_{n-m} + \xi B_{n-m} + 2C_{n-m}]}{\sqrt{mn}} v_m \cdot \bar{v}_n \geqslant 0, \tag{3.3}$$

where $\Psi(m, n, \xi) = -(\xi - m)^2/4m^2 - (\xi - n)^2/4n^2$.

It follows easily from the analysis below that we can interchange integration and summation to obtain

$$\sum_{m,\, n \geqslant 1} [\lambda_{mn} A_{n-m} + \mu_{mn} B_{n-m} + \delta_{mn} C_{n-m}]\, v_m \cdot \bar{v}_n \geqslant 0, \tag{3.4}$$

where

$$\lambda_{mn} = \frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} e^{\Psi(m,\, n,\, \xi)} \frac{\xi^2}{2}\, d\xi, \qquad \mu_{mn} = \frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} e^{\Psi(m,\, n,\, \xi)}\, \xi\, d\xi$$

and

$$\delta_{mn} = \frac{2}{\sqrt{mn}} \int_{-\infty}^{\infty} e^{\Psi(m,\, n,\, \xi)}\, d\xi.$$

Now

$$\begin{aligned}
|\lambda_{mn}| &\leqslant \frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} e^{-(\xi-m)^2/4m^2} \frac{\xi^2}{2}\, d\xi \\
&= \frac{1}{2} \cdot \sqrt{\frac{m}{n}} \int_{-\infty}^{\infty} \left[m^2 \frac{(\xi - m)^2}{m^2} + 2m^2 \frac{(\xi - m)}{m} + m^2\right] e^{-(\xi-m)^2/4m^2} \frac{1}{m}\, d\xi \\
&= \frac{1}{2} \cdot \sqrt{\frac{m}{n}} \int_{-\infty}^{\infty} m^2[\xi^2 + 1]\, e^{-\xi^2/4}\, d\xi.
\end{aligned}$$

Similarly, we can show that $|\mu_{mn}| \leqslant \sqrt{(m/n)} \int_{-\infty}^{\infty} m[|\xi| + 1]\, e^{-\xi^2/4}\, d\xi$ and $|\delta_{mn}| \leqslant 2 \cdot \sqrt{(m/n)} \int_{-\infty}^{\infty} e^{-\xi^2/4}\, d\xi$.

Therefore we have

$$\lambda_{mn} = O\left(m^2 \cdot \sqrt{\frac{m}{n}}\right), \qquad \mu_{mn} = O\left(m \cdot \sqrt{\frac{m}{n}}\right), \qquad \delta_{mn} = O\left(\sqrt{\frac{m}{n}}\right). \tag{3.5}$$

It follows from (3.5) that $\sum_{m,n\geqslant 1}[|\lambda_{mn}|\,|A_{n-m}|+|\mu_{mn}|\,|B_{n-m}|+|\delta_{mn}|\,|C_{n-m}|]\,|v_m|\,|\bar{v}_n|$ converges, which justifies (3.4).

If we note that $|m-n|\geqslant n/2$ implies $|m-n|\geqslant m/3$, then we have

$$\sum_{\substack{m,n\geqslant 1\\ |m-n|\geqslant n/2}} [|\lambda_{mn}|\,|A_{n-m}|+|\mu_{mn}|\,|B_{n-m}|+|\delta_{mn}|\,|C_{n-m}|]\,|v_m|\,|\bar{v}_n|$$

$$\leqslant c \sum_{\substack{m,n\geqslant 1\\ |m-n|\geqslant n/2}} \frac{1}{(m-n)^2}|v_m|\,|v_n| \leqslant c \sum_{\substack{m,n\geqslant 1\\ |m-n|\geqslant 1}} \frac{1}{(m-n)^2}|v_m|\,|v_n|$$

for some $c>0$, by (2.2). But

$$\sum_{\substack{m,n\geqslant 1\\ |m-n|\geqslant 1}} \frac{|v_m|\,|v_n|}{(m-n)^2}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sum_{m=1}^{\infty}|v_m|\,|v_{m+k}|+\sum_{m=1}^{\infty}|v_m|\,|v_{m-k}|\right)$$

$$\leqslant\frac{\pi^2}{3}\sum_{m=1}^{\infty}|v_m|^2$$

(where we let $v_j=0$ when $j\leqslant 0$).

Therefore, it follows from (3.4) that

$$\sum_{m,n\geqslant 1}{}' [\lambda_{mn}A_{n-m}+\mu_{mn}B_{n-m}+\delta_{mn}C_{n-m}]\,v_m\cdot v_n\geqslant -M_1\sum_{m=1}^{\infty}|v_m|^2 \tag{3.6}$$

for some $M_1>0$, where $\sum'$ means the summation is taken over m, n such that $|m-n|<n/2$.

4. Proof of Theorem 1.1

First of all, observe that $|m-n|<n/2$ implies that $|m-n|<m$. Under the assumption $|m-n|<n/2$, we want to give an estimate better than (3.5).

LEMMA 4.1. $\lambda_{mn}=\sqrt{2\pi}\,mn+O[(m-n)^2]$, *if* $|m-n|<n/2$.

Proof. $\Psi(m,n,\xi)=\Psi_1(m,n,\xi)+\Psi_2(m,n,\xi)+(m-n)^2/8mn$, where

$$\Psi_1(m,n,\xi)=-\frac{1}{2mn}\left(\xi-\frac{m+n}{2}\right)^2$$

and

$$\Psi_2(m, n, \xi) = -\frac{(m-n)^2}{4m^2n^2}\,\xi^2,$$

$$\begin{aligned}\lambda_{mn} &= \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1+\Psi_2+(m-n)^2/8mn}\,\xi^2\,d\xi \\ &= \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1+\Psi_2}\,\xi^2\,d\xi \\ &\quad - \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1+\Psi_2}\,\xi^2(1-e^{(m-n)^2/8mn})\,d\xi.\end{aligned}$$

We can estimate the second integral in the following way. For some $c>0$,

$$\begin{aligned}&\frac{1}{2\sqrt{mn}}\left|\int_{-\infty}^{\infty} e^{\Psi_1+\Psi_2}\,\xi^2(1-e^{(m-n)^2/8mn})\,d\xi\right| \\ &\leqslant \frac{c}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1}\,\xi^2\,\frac{(m-n)^2}{8mn}\,d\xi \\ &= \frac{c(m-n)^2}{16}\int_{-\infty}^{\infty}\left(\xi^2+\frac{m+n}{\sqrt{mn}}\,\xi+\frac{(m+n)^2}{4mn}\right)e^{-\xi^2/2}\,d\xi \\ &= O[(m-n)^2].\end{aligned}$$

The last estimate follows from the fact that m and n are comparable. For the first integral, we have

$$\begin{aligned}&\frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1+\Psi_2}\,\xi^2\,d\xi \\ &\quad = \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1}\,\xi^2\,d\xi + \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1}(e^{\Psi_2}-1)\,\xi^2\,d\xi.\end{aligned}$$

Now

$$\begin{aligned}&\frac{1}{2\sqrt{mn}}\left|\int_{-\infty}^{\infty} e^{\Psi_1}(e^{\Psi_2}-1)\,\xi^2\,d\xi\right| \\ &\leqslant \frac{1}{2\sqrt{mn}}\int_{-\infty}^{\infty} e^{\Psi_1}|\Psi_2|\,\xi^2\,d\xi \\ &= \frac{(m-n)^2}{2}\int_{-\infty}^{\infty}\left[\frac{\xi^4}{4}+\frac{m+n}{2\sqrt{mn}}\,\xi^3+\frac{3}{8}\frac{(m+n)^2}{mn}\,\xi^2\right. \\ &\quad \left.+\frac{1}{8}\frac{(m+n)^3}{mn\sqrt{mn}}\,\xi+\frac{1}{64}\frac{(m+n)^4}{m^2n^2}\right]e^{-\xi^2/2}\,d\xi \\ &= O[m-n)^2].\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{1}{2\sqrt{mn}} \int_{-\infty}^{\infty} e^{\Psi_1 \xi^2}\, d\xi \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \left[mn\xi^2 + \sqrt{mn}\,(m+n)\,\xi + \frac{(m+n)^2}{2} \right] e^{-\xi^2/2}\, d\xi \\
&= \frac{1}{2} mn \sqrt{2\pi} + \frac{(m+n)^2}{8} \sqrt{2\pi} \\
&= \sqrt{2\pi}\, mn + \frac{\sqrt{2\pi}}{8} (m-n)^2 \qquad \text{Q.E.D.}
\end{aligned}$$

Similarly, we can prove the following lemma:

LEMMA 4.2. *$\mu_{mn} = \sqrt{2\pi}((m+n)/2) + O[(m-n)^2]$ and $\delta_{mn} = 2\sqrt{2\pi} + O[(m-n)^2]$, if $|m-n| < n/2$.*

Now

$$\begin{aligned}
&\sum_{m,n \geqslant 1}' (m-n)^2(|A_{n-m}| + |B_{n-m}| + |C_{n-m}|)\, |v_m|\, |v_n| \\
&\qquad \leqslant c \sum_{\substack{m,n \geqslant 1 \\ |m-n| \geqslant 1}} \frac{|v_m|\, |v_n|}{(m-n)^2}
\end{aligned}$$

for some $c > 0$, by (2.2). We have already proved in Section 3 that

$$\sum_{\substack{m,n \geqslant 1 \\ |m-n| \geqslant 1}} \frac{|v_m|\, |v_n|}{(m-n)^2}$$

is bounded by a multiple of $\sum_{m=1}^{\infty} |v_m|^2$. Therefore (3.6) and Lemmas 4.1 and 4.2 imply that

$$\begin{aligned}
&\sum_{m,n \geqslant 1}' \left[mnA_{n-m} + \frac{m+n}{2} B_{n-m} + 2C_{n-m} \right] v_m \cdot \bar{v}_n \\
&\qquad \geqslant -M_2 \sum_{m=1}^{\infty} |v_m|^2, \qquad (4.1)
\end{aligned}$$

for some $M_2 > 0$.

Since $\sum_{m,n \geqslant 1}' |C_{n-m}|\, |v_m| \cdot |\bar{v}_n|$ is obviously bounded by some multiple of $\sum_{m=1}^{\infty} |v_m|^2$, we have

$$\sum_{m,\, n \geqslant 1}' \left[mnA_{n-m} + \frac{m+n}{2} B_{n-m} + C_{n-m} \right] v_m \cdot \bar{v}_n$$

$$\geqslant -M \sum_{m=1}^{\infty} |v_m|^2, \tag{4.2}$$

for some $M > 0$.

Equation (2.4) therefore follows from (4.2) if we let $v_n = u_n$ and the proof of Theorem 1.1 is complete.

References

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