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Geometry of G/P – V

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INTRODUCTION

The purpose of this paper is to give a self-contained exposition of the main results of a "Standard Monomial Theory" partly proved and partly announced in [L-M-S]$_3$. The initial motivation of this work was to prove the announcements (regarding what is called the "mixed case") made in Section 16 of [L-M-S]$_3$, since we found that the proofs we had in mind, and which were briefly sketched in [L-M-S]$_3$, were inadequate. We found a new approach to proving one of the main steps in the proof of the main results (namely, generation by standard monomials), which would also prove these announced results. Then we became aware of a serious gap[1] (pointed out by V. Kac) in the work of Demazure (cf. [D]$_1$), which has been used in [L-M-S]$_3$ in an essential manner. Fortunately, by a suitable modification of this new method, we could avoid the explicit use of this work of Demazure. But this has required an extensive revision of the proofs in [L-M-S]$_3$ and therefore we have taken this occasion to present an exposition which does not make use of [Se]$_1$, [L-S]$_1$, [L-M-S]$_2$, and [L-M-S]$_3$ in any essential manner. We should point out, however, that many of the techniques of this paper are essentially the same as in [L-M-S]$_3$ and

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[1] This has now been set right. See the comments near the end of this section.

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that the cited work of Demazure and the related one (cf. [Se]$_2$) have been of great help.

It should also be pointed out that as a consequence of our main results, we give proof of the main results in [D]$_1$, say, when G is a classical group; in fact, we go farther, since we prove these results in arbitrary characteristic and even over $\mathbf{Z}$.

We recall that the main goal of [Se]$_1$, [L-S]$_1$, [L-M-S]$_2$, and [L-M-S]$_3$ (as well as this paper) is to generalize the classical Hodge–Young theory (cf. [H], [H-P]), which gives a canonical basis for the homogeneous co-ordinate ring of a Schubert variety in the Grassmannian, to the case of a Schubert variety in the flag variety associated to a semi-simple algebraic group. We solve this problem completely when G is a classical group and more generally for a Schubert variety in G/Q, where Q is a parabolic subgroup of "*classical type*" (cf. Definition 2.3). Thanks to the Borel–Weil Theorem (see [B]$_1$, for example), this paper gives, in particular, a canonical basis for any irreducible G-module, when G is a classical group and the base field is of characteristic zero.

We recall that in [Se]$_1$, the case of a Schubert variety in G/P, where P is a maximal parabolic subgroup associated to a *minuscule fundamental weight* (cf. [Se]$_1$ for the definition of a minuscule fundamental weight), was treated. This showed the possibility of generalizing the classical Hodge–Young theory. However, even if G is a classical group and not of type $\mathbf{A}_n$, there exist maximal parabolic subgroups whose associated fundamental weights are not minuscule. Hence the generalization in [Se]$_1$ was not strong enough. In [L-S]$_1$, we arrived at the crucial conjectural formulation of our main results, aided by the work of DeConcini and Procesi on classical invariant theory (cf. [D-P]). In [L-M-S]$_2$ the case of Schubert varieties in G/P, where P is a maximal parabolic subgroup associated to a *quasi-minuscule weight*, was treated; this case figures in the work of G. Kempf (cf. [K]) in his proof of the generalization of the Kodaira vanishing theorem, in arbitrary characteristic, on the flag variety, associated to a semi-simple algebraic group. The results of [L-M-S]$_2$ help in understanding this work of Kempf. In [L-M-S]$_3$, the case of Schubert varieties in G/P, where P is a maximal parabolic subgroup of *classical type* (cf. Definition 2.2), was treated and a sketch of proof was given (in Section 16 of [L-M-S]$_3$) for the case of Schubert varieties in G/Q, where Q is a parabolic subgroup of classical type.

The new aspect of this paper which does not figure in [L-M-S]$_3$ can be briefly summarized as follows: given a Schubert variety X, we choose a nice Schubert variety Y of codimension one in X and we construct a proper birational morphism $\psi: Z \to X$ such that Z is a fiber space over $\mathbf{P}^1$ with fiber Y. The main results are proved by induction on the dimension of X and can therefore be supposed to be true on Y. This allows us to have a control over the geometry of Z and to prove the required theorems on Z. Then one makes the results "go down to X." A similar method was also adopted by Kempf (cf. [K]). It appears likely that this method would be effective in proving the conjectures, on a general standard monomial theory for Schubert varieties in G/B (G being semi-simple algebraic group of *any* type), stated in [L]; in fact, using this method, one gets a standard monomial theory for the case $G = \mathbf{G}_2$ (cf. [L]).

The proof of the main results of this paper can be divided into the following three steps:

(i) Proof of the first basis theorem (cf. Theorem 3.15)—namely, giving a canonical basis for $H^0(X, L)$, where X is a Schubert variety in G/P, where P is a maximal parabolic subgroup of classical type (cf. Definition 2.2) and L is the ample generator of $\mathrm{Pic}(G/P)$.

(ii) The proof of linear independence of "standard monomials" (in the basis elements constructed in (i)) on a Schubert variety in G/Q, where Q is a parabolic subgroup of classical type (cf. Definition 2.3).

(iii) Generation of the space of sections of a line bundle on a Schubert variety, as in (ii), by standard monomials.

The proof of (i) and (ii) runs essentially on the same lines as in [L-M-S]$_3$; however, there is a serious difficulty due to the non-availability of Demazure's results (cf. [D]$_1$). By using the methods of [L-M-S]$_3$ and the constructions in [D]$_1$ (also [Se]$_2$) one arrives at a slighlty weaker assertion[2] than (i).

In the proof of (ii) one cannot completely follow [L-M-S]$_3$, since it makes use of the normality of a Schubert variety (a result in [D]$_1$, in whose proof there is a gap).

The proof of (iii) in [L-M-S]$_3$ is done only for the case of a maximal parabolic subgroup. It is first reduced to the case of standard monomials of degree 2 and then this case is treated in an explicit way be a simple but rather tedious counting argument (cf. Section 4 of [L-M-S]$_3$). The proof presented in this paper is conceptually better but could be called more sophisticated. This is based on the method of taking $\psi: Z \to X$, as described above; this also serves to overcome the difficulties, pointed above, in the proof of (i) and (ii). The main feature in this proof is the following: In [L-M-S]$_3$, we gave, as a consequence of the standard monomial theory, a filtration of a canonical scheme-theoretic hyperplane section of a Schubert variety in G/P, P being a maximal parabolic subgroup of classical type (cf. Theorem 9.3 in [L-M-S]$_3$). If a similar filtration could be established à priori, then it is not difficult to see that the proof of (iii) above would be

[2] We are grateful to C. Musili for helping us to arrive at this step.

achieved. We show (cf. Section 7) that a similar filtration holds on Y as a consequence of our induction hypothesis; this in turn yields a related filtration on Z, using which, the "generation by standard monomials" is proved on Z. Then the required "generation by standard monomials on X" is deduced.

Statement of the Main Results and a Sketch of Its Proof

Let G denote a semi-simple, simply-connected Chevalley group defined over a field k. Let T be a maximal k-split torus, B a Borel subgroup, $T\subset B$, and P a maximal parabolic subgroup, $P\supset B$, with associated fundamental weight ω. Let W (resp. W_p) be the Weyl group of G (resp. P) and $(\ ,\ )$ a W-invariant scalar product on $\mathrm{Hom}(T, \mathbf{G}_{\mathrm{m}})$. We say (cf. Definition 2.2) that ω (or P) is of classical type, if $|\langle\omega, \alpha^*\rangle|$ $(=|2(\omega,\alpha)/(\alpha,\alpha)|)$ $\leqslant 2$ for every root α. For $w\in W$, let $X(w)=\overline{BwP}$ (mod P) with the canonical reduced structure be the Schubert variety in G/P associated to w. Let $[X(w)]$ denote the element of the Chow ring of G/P, determined by $X(w)$. If H denotes the unique codimension one Schubert subvariety in G/P, then it can be shown (cf. [C]) that

$$[X(w)]\cdot[H]=\sum_i d_i[X(w_i)], \qquad d_i>0$$

where $\cdot$ denotes multiplication in the Chow ring of G/P and $X(w_i)$ runs over *all* the Schubert varieties of codimension one in $X(w)$. We call d_i the (intersection) multiplicity of $X(w_i)$ in $[X(w)]\cdot[H]$. A pair (ϕ, w) of Weyl group elements in W/W_p (P being of classical type) is called an *admisible pair* (cf. Definition 2.4) if either $\phi=w$ (in which case it is called a *trivial admissible pair*) or $\phi\neq w$ and there exist $\{\phi_i\}$, $1\leqslant i\leqslant s$, $\phi_i\in W/W_p$, such that

(i) $\phi=\phi_1>\phi_2>\cdots>\phi_s=w$,

(ii) $X(\phi_i)$ is a Schubert divisor in $X(\phi_{i-1})$ with intersection multiplicity 2 in $[X(\phi_{i-1})]\cdot[H]$, $2\leqslant i\leqslant s$.

If (ϕ_1, w_1) and (ϕ_2, w_2) are admissible pairs in W/W_p, we write $(\phi_1, w_1)\geqslant(\phi_2, w_2)$ if $w_1\geqslant\phi_2$ (equivalently $X(w_1)\supseteq X(\phi_2)$).

Now let $Q=\bigcap_{i=1}^r P_i$ be such that P_i is a maximal parabolic subgroup of classical type, $1\leqslant i\leqslant r$ (we refer to such a Q being of *classical type* (cf. Definition 2.3)). We now call (cf. Definition 4.1) a *Young diagram on G/Q of type $a=(a_1,\ldots,a_r)$*, $a_i\geqslant 0$, a pair (θ,δ) where

$$\theta=(\theta_{ij}), \qquad \delta=(\delta_{ij})$$

and

$$(\theta_{ij}, \delta_{ij}) \text{ is an admissible pair in } W/W_{P_i},$$

$1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant a_i$. If $a_t = 0$ for some t, $1 \leqslant t \leqslant r$, we understand that the corresponding admissible pair $((\theta_{t,-}, \delta_{t,-})$ is empty, i.e., does not figure.

We say (cf. Definition 4.2) that a Young diagram (θ, δ) is *standard on* $X(\tau)$, $\tau \in W/W_Q$, if there exists a pair (α, β) which we call a defining pair for (θ, δ) such that

(1) $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$, $\alpha_{ij}, \beta_{ij} \in W/W_Q$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant a_i$,

(2) each α_{ij} (resp. β_{ij}) is a lift in W/W_Q for the element θ_{ij} (resp. δ_{ij}) in W/W_{P_i} under the canonical morphisms $W/W_Q \to W/W_{P_i}$,

(3)

$$\tau \geqslant \alpha_{11} \geqslant \beta_{11} \geqslant \alpha_{12} \geqslant \beta_{12} \geqslant \cdots \geqslant \alpha_{1a_1} \geqslant \beta_{1a_1} \geqslant \alpha_{21} \geqslant \beta_{21} \geqslant \cdots \geqslant \alpha_{ra_r} \geqslant \beta_{ra_r}.$$

Then we have the following (cf. Theorem 9.6).

THEOREM. *Let $\tau \in W/W_Q$ and let L_a (or just L) $= \bigotimes_{i=1}^{r} L_i^{a_i}$ be a positive line bundle on G/Q. Then there exists a basis $\{p(\theta, \delta)\}$ for $H^0(X(\tau), L)$ indexed by Young diagrams (θ, δ), of type a, standard on $X(\tau)$ (the elements $p(\theta, \delta)$ are referred to as standard monomials (cf. Definition* 4.3) *on $X(\tau)$ of type a).*

The linear independence of the standard monomials on $X(\tau)$ is proved using induction on dim $X(\tau)$ and using some *special quadratic* relations on $X(\tau)$ (cf. $(*)$ in Section 5). Proceeding as in [M-S], we in fact prove linear independence of standard monomials on a union of Schubert varieties (cf. Theorem 5.1).

To prove the generation by standard monomials we proceed as follows: Let us fix a maximal parabolic subgroup, say, P_1, containing Q. Let $X(\bar{\tau})$ (resp. $X(\bar{\phi})$) be the projection of $X(\tau)$ (resp. $X(\phi)$) under $G/Q \to G/P_1$. Let e be a generator of the unique B-fixed line in $H^0(G/P_1, L_1)$ (here L_1 is the ample generator of Pic(G/P_1)) and let $p(\bar{\tau})$ (resp. $p(\bar{\phi})$) be the $\bar{\tau}w_0$ (resp. $\bar{\phi}w_0$) translate of e, where w_0 denotes the unique element of largest length in W. Let us denote by $p(\bar{\tau})$ (resp. $p(\bar{\phi})$) itself, the restriction of $p(\bar{\tau})$ (resp. $p(\bar{\phi})$) to $X(\tau)$ (resp. $X(\phi)$). Let $H(\tau)$ (resp. $H(\phi)$) be the zero set of $p(\bar{\tau})$ in $X(\tau)$ (resp. $p(\bar{\phi})$ in $X(\phi)$). Let now α be a simple root such that $s_\alpha \bar{\tau} < \bar{\tau}$, in W/W_{P_1} (and hence $s_\alpha \tau < \tau$, in W), and let $\phi = s_\alpha \tau$. Associated to α, there is a copy of SL_2 in G, which we shall denote by $SL(2, \alpha)$. Let B_α be the Borel subgroup in $SL(2, \alpha)$ given by $B_\alpha = B \cap SL(2, \alpha)$. Now for the canonical action of $SL(2, \alpha)$ on G/Q (induced by the canonical action of G on G/Q)

$X(\tau)$ remains stable; and observing that any Schubert variety in G/Q is stable under the action of B, we set (cf. Section 8; see also $[\mathrm{D}]_1$ and $[\mathrm{Se}]_2$)

$$Z_{\phi,\tau} = SL(2, \alpha) \times^{B_\alpha} X(\phi)$$

$(= P_\alpha \times^B X(\phi)$, where P_α is the rank one parabolic subgroup associated to $\alpha)$, i.e., $Z_{\phi,\tau}$ is the quotient variety modulo the equivalence relation in $SL(2, \alpha) \times X(\phi)$ defined by

$$(g, x) \sim (gb, b^{-1}x); \qquad g \in SL(2, \alpha), b \in B_\alpha, x \in X(\phi).$$

Let p denote the canonical map

$$p\colon Z_{\phi,\tau} \to \mathbf{P}^1 = SL(2, \alpha)/B_\alpha$$

(p is a fibration over $\mathbf{P}^1$ with fibers isomorphic to $X(\phi)$) and let ψ be the canonical map

$$\psi\colon SL(2, \alpha) \times^{B_\alpha} X(\phi) \to X(\tau) \varsubsetneq G/Q.$$

Now for any B_α-object M on $X(\phi)$, we can associate a canonical object $\tilde{M}$ on $Z_{\phi,\tau}$ (namely, $\tilde{M} = SL(2, \alpha) \times^{B_\alpha} M$). Now denoting by $\mathbf{I}(H(\tau))$ (resp. $\mathbf{I}(H(\phi))$) the ideal sheaf of $H(\tau)$ (resp. $H(\phi)$) in $X(\tau)$ (resp. $X(\phi)$), we prove (cf. Lemma 9.2)

$$\psi^*(\mathbf{I}(H(\tau))) \approx \widetilde{\mathbf{I}(H(\phi))} \otimes \mathcal{O}_{\mathbf{P}^1}(-n) \qquad (*)$$

where $n = \langle \phi(\omega_1), \alpha^* \rangle$, ω_1 being the fundamental weight associated to P_1. Now assuming (by induction on $\dim X(\tau)$) that standard monomials on $X(w)$ (where $\dim X(w) \lneqq \dim X(\tau)$) form a basis for $H^0(X(w), L)$, we obtain a filtration (cf. Lemmas 7.3 and 7.4)

$$\mathbf{I}_0 = \mathbf{I}(H(\phi)) \subset \mathbf{I}_1 \subset \cdots \subset \mathbf{I}_d = \mathbf{I}(H(\phi)_{\mathrm{red}})$$

such that

$$\mathbf{I}_s/\mathbf{I}_{s-1} \approx \mathcal{O}_{X(\lambda)}(-1) \qquad (\approx L_1^{-1} \mid X(\lambda))$$

$0 \leqslant s \leqslant d$ (where $\mathbf{I}_{-1} = (0)$), $X(\lambda)$'s being certain Schubert subvarieties of $X(\phi)$. Now using this and $(*)$ above, we prove

$$\begin{aligned} \dim H^0(Z_{\phi,\tau}, \psi^*(L)) = \dim H^0(H(\tau)_{\mathrm{red}}, L) + \dim H^0(X(\tau), L') \\ + \textstyle\sum \dim H^0(X(\lambda), L') \end{aligned} \qquad (**)$$

where $L' = L_{a'}$, $a' = (a_1 - 1, a_2, \ldots, a_r)$, and the summation on the R.H.S. is over all $X(\lambda)$'s such that $X(\lambda)$ is a maximal lift (cf. Remark 4.7) in $X(\tau)$ of

$\bar{\lambda}$, where $(\bar{\tau}, \bar{\lambda})$ is a non-trivial admissible pair on W/W_{P_1}. On the other hand we prove (cf. Proposition 5.8) that

$$\text{R.H.S. of } (**) = \#\{\text{standard monomials on } X(\tau) \text{ of type } a\}.$$

From this it follow that

$$\dim H^0(Z_{\phi,\tau}, \psi^*(L)) = \#\{\text{standard monomials on } X(\tau) \text{ of type } a\}.$$

This together with linear independence of standard monomials on $X(\tau)$ (cf. Theorem 5.1) and the canonical inclusion

$$\psi^*: H^0(X(\tau), L) \subsetneq H^0(Z_{\phi,\tau}, \psi^*(L))$$

implies that standard monomials on $X(\tau)$ of type a form a basis for $H^0(X(\tau), L)$ (in particular it also proves the generation).

Among the several consequences of standard monomial theory we would like to mention the following:

(1) A proof of Demazure's conjecture (cf. [D]$_1$).

(2) A proof of "Vanishing theorems," i.e., the result $H^i(X(w), L) = 0$, $i \geqslant 1$, $L \geqslant 0$ (cf. Theorem 9.6).

(3) Normality of Schubert varieties (cf. Theorem 9.6).

(4) Determination of the Singular Locus of a Schubert variety (cf. [L-S]$_2$).

(5) A character formula for the T-module $H^0(X(\tau), L)$ (cf. Corollary 9.8).

(6) Surjectivity of $H^0(G/Q, L) \to H^0(X(\tau), L)$ (cf. Corollary 9.8).

(7) Behaviour of unions and intersection of Schubert varieties (namely, that unions and intersections of Schubert varieties are reduced) (cf. Theorem 9.6 and Lemma 6.3).

(8) The Cohen–Macaulayness of (the multi-graded ring) $\bigoplus_{(L \geqslant 0)} H^0(X(\tau), L)$ (cf. [D-L], [H-L]$_1$, [H-L]$_2$).

For other methods of construction of "Standard Bases" one may refer to [B-T], [D], [L-T], [T]$_1$, [T]$_2$.

After this work was completed, the gap in Demazure's work, mentioned above, has been set right, just recently, due to the efforts of A Joseph (see his preprint "On the Demazure character formula"), V. Mehta and A. Ramanathan (see their preprint "Forbenius splitting and cohomology vanishing"), and the second author of this paper (see the preliminary version "Normality of Schubert varieties" of a paper to appear in the Bombay Colloquium on "Vector Bundles," 1984). In fact, if X is a Schubert variety in the flag variety G/B, associated to an arbitrary semi-simple, simply-connected algebraic group G (in arbitrary characteristic), and L a line bundle on G/B, associated to a dominant weight, it follows now that:

(i) the Demazure character formula for $H^0(X, L)$ holds,

(ii) X is normal,

(iii) the canonical map $H^0(G/B, L) \to H^0(X, L)$ is surjective,

(iv) $H^i(X, L) = 0$, $i > 0$, and

(v) the Schubert variety in char. 0 "specializes well" to the one in char. p, $p > 0$.

Some of the arguments in this paper could be skipped (as one sees easily) if these results are assumed; to make it very precise, if one assumes (i) above, one may omit Proposition 3.2 in Section 3. Again, if we assume (ii) above, then we obtain that the "special quadratic relations" (cf. $(*)$ of Section 5) hold on any X; the proof of this fact is on the same lines as that of claim 1 in the body of the proof of Theorem 9.6, wherein it is proved that assuming normality of Schubert varieties of dimension $\lneqq \dim X$, the special quadratic relations hold on X. For the same reasons, one need not have the additional assumption that the "special quadratic relations" hold on Schubert varieties for the results in Section 6, Lemmas 7.3, 7.4 and 9.2 through 9.5.

The sections are organized as follows.

In Section 1 we recall some generalities on Schubert schemes.

In Section 2 we define fundamental weights, admissible pairs, etc., and prove some lemmas on admissible pairs.

In Section 3 we prove the first basis theorem (which is, in essence, about the construction of a basis for $H^0(X(\tau), L_\omega)$, ω being a fundamental weight of classical type).

In Section 4 we define Young diagrams and standard monomials.

In Section 5 we prove linear independence of standard monomials.

In Sections 6 and 7 we discuss the consequences of the induction hypothesis; i.e., assuming that standard monomial theory holds for Schubert varieties of dimension $< \dim X(\tau)$, we show that we get a standard monomials theory for unions (and intersections) of Schubert varieties also and as a consequence we obtain that unions and intersections of Schubert varieties are reduced. Further, as a consequence of the induction hypothesis, we also obtain the filtration for $\mathrm{I}(H(\phi)_{\mathrm{red}})$, described above.

In Section 8 we define the variety $Z_{\phi,\tau}$ (or Z_ϕ) and prove some results relating to Z_ϕ (required for our purpose).

In Section 9 we prove the main theorem and its consequences.

In Section 10 we explicitly write down the ideal sheaf of $X(\tau)$ in G/Q and state a conjecture regarding the defining equations of a Schubert variety.

1. Notations and Preliminaries

Let $G_{\mathbf{Z}}$ denote a semi-simple, simply-connected, Chevalley group scheme over the ring of integers $\mathbf{Z}$ (for many basic facts on Chevalley groups see [St]). We fix a maximal torus subgroup scheme $T_{\mathbf{Z}}$ and a Borel subgroup scheme $B_{\mathbf{Z}}$ containing $T_{\mathbf{Z}}$. We talk of roots, weights, etc., with respect to $T_{\mathbf{Z}}$ and $B_{\mathbf{Z}}$. The Weyl group scheme $N(T_{\mathbf{Z}})/T_{\mathbf{Z}}$ ($N(T_{\mathbf{Z}})=$ normalizer of $T_{\mathbf{Z}}$) is a constant group scheme and hence we talk of the Weyl group W of $G_{\mathbf{Z}}$.

If A is any ring, we denote the objects obtained by the base change $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}$ with the suffix A (unless otherwise stated), e.g., G_A, B_A, T_A, etc. In the sequel, when A is a field k, we will often drop the subscript k, e.g., $G_K = G$, $T_K = T$, etc; further, we shall be mostly concerned with $\mathbf{Z}$ or a field k.

Let $Q_{\mathbf{Z}}$ be a parabolic subgroup scheme of $G_{\mathbf{Z}}$ containing $B_{\mathbf{Z}}$, *associated* to a subset of the set of simple roots or equivalently a subset of the set of fundamental weights. Let λ be a weight of the form

$$\lambda = \sum_{i=1}^{r} a_i \omega_i,$$

where the set $\{\omega_i\}$, $1 \leqslant i \leqslant r$, is the *complement* of the set of fundamental weights associated to $Q_{\mathbf{Z}}$. Then λ gives rise to a line bundle on G_A/Q_A for every ring A and we denote this by $L_{\lambda,A}$. One knows that this line bundle is *ample* on G_A/Q_A (relative to A) if and only if $a_i > 0$ for $1 \leqslant i \leqslant r$. One knows that if $L_{\lambda,A}$ is ample on G_A/Q_A, it is in fact *very ample* as a consequence of a lemma of Deodhar (cf. Lemma 5.8 of [L-M-S]$_3$). One knows tht $\operatorname{Pic} G_{\mathbf{Z}}/Q_{\mathbf{Z}}$ (resp. $\operatorname{Pic} G_k/Q_k$, k a field) is isomporphic to $\mathbf{Z}_r$; in fact $\{L_{\omega_i,\mathbf{Z}}\}$ (resp. $\{L_{\omega_i,k}\}$) constitutes a basis.

Let $\tau \in W$. Then if A is any ring, we see that τ determines an A-valued point of G_A/B_A, which we denote by the same letter τ. More generally, if W_Q denotes the Weyl group of a parabolic subgroup Q_A of G_A (the Weyl group scheme of Q_A is a constant group scheme and we talk of the Weyl group W_Q of Q_A for any ring A) and $\tau \in W/W_Q$, for any ring A, τ determines an A-valued point of G_A/Q_A, which we denote by the same letter. Let now k be a field and $\tau \in W/W_Q$. We denote by $X_k(\tau)$ the *Schubert variety* in G_k/Q_k associated to τ, i.e., we define $X_k(\tau)$ to be the (Zariski) closure of $B_k\tau$ (the B_k orbit through τ) in G_k/Q_k, endowed with the canonical reduced structure. Similarly, we define the *Schubert subscheme* $X_{\mathbf{Z}}(\tau)$, $\tau \in W/W_Q$, as the Zariski closure of $B_{\mathbf{Z}}\tau$ in $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$, endowed with the canonical structure of a closed reduced subscheme of $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$. We note that $X_{\mathbf{Z}}(\tau)$ is the *flat closure* of $X_{\mathbf{Q}}(\tau)$, ($\mathbf{Q}=$ field of rational numbers) in $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$, i.e., the canonical morphism $X_{\mathbf{Z}}(\tau) \to \operatorname{Spec} \mathbf{Z}$ is $\mathbf{Z}$-flat and its generic fibre is $X_{\mathbf{Q}}(\tau)$. It is *not* clear that the base change of $X_{\mathbf{Z}}(\tau)$ by

$\operatorname{Spec} k \to \operatorname{Spec} \mathbf{Z}$ coincides with $X_k(\tau)$ for any field k. In fact, we get this as a consequence of the main results of this paper (cf. Corollary 9.8 (d)).

Recall that we have a canonical *partial order* in W/W_Q (W/W_Q as above), which is defined as follows: For τ_1, τ_2 in W/W_Q, we say that $\tau_1 \geqslant \tau_2$ if for some field k, $X_k(\tau_1) \supseteq X_k(\tau_2)$ (this partial order in W/W_Q can also be defined in a combinatorial manner). Recall that $\dim X_k(\tau) = l(\tau)$, where l denotes the *length function* on W/W_Q (k a field).

DEFINITION 1.1. Let $\tau, \phi \in W/W_Q$, $Q_{\mathbf{Z}}$ being a parabolic subgroup scheme of $G_{\mathbf{Z}}$ as above. Then we say that $X_{\mathbf{Z}}(\phi)$ (or $X_k(\phi)$, k a field) is a *moving divisor* in $X_{\mathbf{Z}}(\tau)$ (or $X_k(\tau)$), *moved by a simple root* α, if $\phi = s_\alpha \tau$ (in W/W_Q), s_α being the reflection associated to the simple root α, and $X_{\mathbf{Z}}(\phi)$ (or $X_k(\phi)$) is of codimension one in $X_{\mathbf{Z}}(\tau)$ (or $X_k(\tau)$).

We shall now recall some simple lemmas which are of basic importance.

LEMMA 1.2. *Let $P_{\mathbf{Z}}$ be a maximal parabolic subgroup scheme of $G_{\mathbf{Z}}$ associated to a fundamental weight ω. Let α be a simple root and $\phi \in W/W_p$. Then we have the following*:

(i) *$X_{\mathbf{Z}}(\phi)$ is a moving divisor in $X_{\mathbf{Z}}(s_\alpha \phi)$ moved by α (see Definition* 1.1), *if and only if*

$$\langle \phi(\omega), \alpha^* \rangle = \frac{2(\phi(\omega), \alpha)}{(\alpha, \alpha)} > 0$$

(*here* (,) *denotes the W-invariant scalar product in* $\operatorname{Hom}(T_k, \mathbf{G}_m)$).

(ii) *$X_{\mathbf{Z}}(s_\alpha \phi)$ is a moving divisor in $X_{\mathbf{Z}}(\phi)$ moved by α, if and only if*

$$\langle \phi(\omega), \alpha^* \rangle < 0.$$

(iii) *$X_{\mathbf{Z}}(s_\alpha \phi)$ is a moving divisor in $X_{\mathbf{Z}}(\phi)$ moved by α, if and only if $X_{\mathbf{Z}}(\phi)$ is stable under the unipotent subgroup scheme of $G_{\mathbf{Z}}$, denoted by $G_{-\alpha,\mathbf{Z}}$, and isomorphic to the additive group scheme $\mathbf{G}_a$, canonically associated to the root $-\alpha$ (then $X_{\mathbf{Z}}(\phi)$ is in fact stable under the $SL(2)$ in $G_{\mathbf{Z}}$ canonically associated to α).*

These results follow essentially from the following

LEMMA 1.3. *Let $w \in W$. Then the following are equivalent.*

(a) *The closed Bruhat cell $\overline{B_{\mathbf{Z}} w B_{\mathbf{Z}}}$ in $G_{\mathbf{Z}}$ is stable for multiplication on the left (resp. right) by $G_{-\alpha,\mathbf{Z}}$, equivalently by the minimal parabolic subgroup scheme $P_{\alpha,\mathbf{Z}}$.*

(b) *$l(s_\alpha w) < l(w)$ (resp. $l(w s_\alpha) < l(w)$).*

(c) *$w^{-1}(\alpha) < 0$ (resp. $w(\alpha) > 0$).*

A proof of Lemma 1.3 may be found in $[\text{L-M-S}]_1$ (cf. Proposition 1.4 in $[\text{L-M-S}]_1$). One considers the canonical map $\pi\colon G_{\mathbf{Z}}/B_{\mathbf{Z}} \to P_{\alpha,\mathbf{Z}}/B_{\mathbf{Z}}$ and expresses the condition for $X_{\mathbf{Z}}(w)$ to be saturated for π to prove stability for multiplication on the right by $P_{\alpha,\mathbf{Z}}$, etc.

Remark 1.4. Note that the assertion (iii) of Lemma 1.2 is valid when $P_{\mathbf{Z}}$ is any parabolic subgroup scheme of $G_{\mathbf{Z}}$ instead of being a maximal parabolic one.

LEMMA 1.5. *Let $Q_{\mathbf{Z}}$ be a parabolic subgroup scheme of $G_{\mathbf{Z}}$ and $\phi \in W/W_Q$. Suppose that $X_{\mathbf{Z}}(\phi')$ is a Schubert divisor in $X_{\mathbf{Z}}(\phi)$ moved by a simple root α. Then any Schubert subscheme $X_{\mathbf{Z}}(w)$ of $X_{\mathbf{Z}}(\phi)$ is of the form*

(i) *either $X_{\mathbf{Z}}(w) \subseteq X_{\mathbf{Z}}(\phi')$, or*

(ii) *$X_{\mathbf{Z}}(w) = X_{\mathbf{Z}}(s_\alpha w')$ for some $X_{\mathbf{Z}}(w') \subseteq X_{\mathbf{Z}}(\phi')$.*

In the latter case, $X_{\mathbf{Z}}(w)$ is obtained by (moving by the same root α) a suitable Schubert subscheme $X_{\mathbf{Z}}(w')$ of $X_{\mathbf{Z}}(\phi')$. We then say that $X_{\mathbf{Z}}(w')$ is *moved outside of* $X_{\mathbf{Z}}(\phi')$ by α. (Note again that the above lemma could have been stated for a more general base than $\mathbf{Z}$.)

Proof. For any reduced expression

$$\phi' \equiv s_{\beta_1} \cdots s_{\beta_r} \qquad (\text{mod } W_Q),\ \beta_i \in S,\ S = \text{set of simple roots}$$

ϕ has the reduced expression

$$\phi \equiv s_{\beta_0} s_{\beta_1} \cdots s_{\beta_r} \qquad (\text{mod } W_Q),\ \beta_0 = \alpha.$$

Now suppose that $X_{\mathbf{Z}}(w)$ is any Schubert subscheme of $X_{\mathbf{Z}}(\phi)$. We know that w has a reduced expression of the form

$$w \equiv s_{\beta_{i_1}} \cdots s_{\beta_{i_s}} \qquad (\text{mod } W_Q);\ 0 \leqslant i_1 < \cdots < i_s \leqslant r.$$

It is clear that if $w \not\leqslant \phi'$ in W/W_Q (i.e., $X_{\mathbf{Z}}(w) \not\subseteq X_{\mathbf{Z}}(\phi')$), we must have $i_1 = 0$, i.e., $w \equiv s_\alpha(w')$ (mod W_Q) for some $w' \leqslant \phi'$ in W/W_Q. But then $X_{\mathbf{Z}}(w')$ is a moving divisor in $X_{\mathbf{Z}}(w)$, moved by the same root α.

This completes the proof of Lemma 1.5.

Remark 1.6. Note that Lemmas 1.2, 1.3, and 1.5 and Remark 1.4 hold over a general base rather than Spec $\mathbf{Z}$, i.e., we could have taken $X_A(\phi)$, etc., instead of $X_{\mathbf{Z}}(\phi)$, etc., A being any ring.

2. Classical Fundamental Weights and Admissible Pairs

Let $P_{\mathbf{Z}}$ be a *maximal* parabolic subgroup of $G_{\mathbf{Z}}$ containing $B_{\mathbf{Z}}$, whose associated fundamental weight is ω. For $\tau \in W/W_P$, let us denote by $[X_k(\tau)]$ (k a field) the element of the Chow ring, $\mathrm{Ch}(G_k/P_k)$ of G_k/P_k, determined by the Schubert variety $X_k(\tau)$ in G_k/P_k. Let H_k denote the unique codimention one Schubert subvariety of G_k/P_k. It can be shown that

$$[X_k(\tau)] \cdot [H_k] = \sum_i d_i [X_k(\phi_i)], \qquad d_i > 0 \tag{1}$$

where $\cdot$ denotes multiplication in G_k/P_k and the summation on R.H.S. runs over the set of *all* Schubert subvarieties of $X_k(\tau)$ of codimension one. By a formula of Chevalley (cf. [C]), the d_i is expressed in the form

$$d_i = |\langle \omega, \alpha_i^* \rangle| = \left| \frac{2(\omega, \alpha_i)}{(\alpha_i, \alpha_i)} \right|$$

where (,) stands for the usual W-invariant scalar product in $\mathrm{Hom}(T_k, \mathbf{G}_m)$ and α_i is given by $\phi_i = \tau s_{\alpha_i}$.

Definition 2.1. We call d_i the *multiplicity* of $X_k(\phi_i)$ in $X_k(\tau)$ and denote it by $m(\phi_i, \tau)$.

Definition 2.2. *A fundamental weight ω (or equivalently the associated maximal parabolic subgroup scheme $P_{\mathbf{Z}}$* or *P_k) is said to be of classical type if*

$$|\langle \omega, \alpha^* \rangle| = \left| \frac{2(\omega, \alpha)}{(\alpha, \alpha)} \right| \leqslant 2 \qquad \forall \text{ root } \alpha.$$

If ω is of classical type, by the formula of Chevalley referred to above, it follows that $d_i \leqslant 2$ in (1) above. It can be seen easily that conversely, if $d_i \leqslant 2$ for every $\tau \in W/W_P$, then P is of classical type. Note that if G_k is a *classical group*, every maximal parabolic subgroup of G_k is of classical type (and conversely). Note that for an arbitrary G_k there exists always a maximal parabolic subgroup which is of classical type. For a *simple* G_k, a list of all the fundamental weights of classical type is easily written down. Recall also (cf. $[\mathrm{Se}]_1$) that the property $d_i \leqslant 1$ for all $\tau \in W/W_P$ is equivalent to ω being *minuscule* (i.e., $|\langle \omega, \alpha^* \rangle| \leqslant 1 \ \forall$ root α).

Definition 2.3. A parabolic subgroup scheme $Q_{\mathbf{Z}}$ of $G_{\mathbf{Z}}$ is said to be of *classical type* if every maximal parabolic subgroup scheme $P_{\mathbf{Z}}$ containing $Q_{\mathbf{Z}}$ is of classical type.

DEFINITION 2.4. Let $P_{\mathbf{Z}}$ be a maximal parabolic subgroup scheme of $G_{\mathbf{Z}}$ of classical type. Then a pair of elements (τ, ϕ) in W/W_P is said to be an *admissible pair*, if either $\tau = \phi$ (in which case it is called a *trivial admissible pair*) or there exists $\{\tau_i\}$, $1 \leqslant i \leqslant s$, $\tau_i \in W/W_P$, such that

(i) $\tau = \tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_s = \phi$,

(ii) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$ and the multiplicity of $X(\tau_i)$ in $X(\tau_{i-1})$ is exactly 2 (for $2 \leqslant i \leqslant s$) (note that in the minuscule case, every admissible pair is trivial).

In the rest of this section we prove some results on admissible pairs (to be used later).

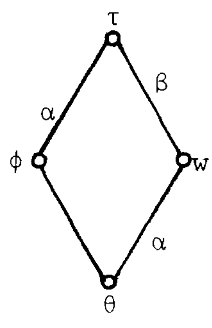
Given an admissible pair (τ, ϕ), a chain

$$\mu_0 = \tau > \mu_1 > \cdots > \mu_n = \phi$$

with $X(\mu_i)$ being a double divisor in $X(\mu_{i-1})$, $1 \leqslant i \leqslant n$, shall be referred to as *an admissible chain for the admissible pair* (τ, ϕ).

LEMMA 2.5. *Let $X_{\mathbf{Z}}(\phi)$ be a moving divisor in $X_{\mathbf{Z}}(\tau)$ (in G/P) moved by the simple root α and $X_{\mathbf{Z}}(w)$ be any other divisor in $X_{\mathbf{Z}}(\tau)$. Let $X_{\mathbf{Z}}(\theta)$ be the divisor in $X_{\mathbf{Z}}(\phi)$ which is moved out to $X_{\mathbf{Z}}(w)$ by α (cf. Lemma 1.5). Then multiplicity of $X_{\mathbf{Z}}(w)$ in $X_{\mathbf{Z}}(\tau)$ is equal to that of $X_{\mathbf{Z}}(\theta)$ in $X_{\mathbf{Z}}(\phi)$.*

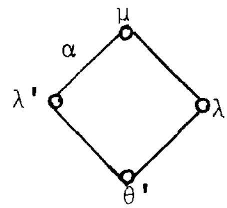
Proof. $X_{\mathbf{Z}}(w)$ being a divisor in $X_{\mathbf{Z}}(\tau)$, there exists a root β such that $w = s_\beta \tau$ (cf. [D]$_1$).



Now $m(w, \tau)$, the multiplicity of $X_{\mathbf{Z}}(w)$ in $X_{\mathbf{Z}}(\tau)$ is given by (cf. [C]) $m(w, \tau) = |\langle \tau(w), \beta^* \rangle|$. Also $\phi = s_\alpha \tau$ and $\theta = s_\alpha w$ imply that $\phi = s_\alpha s_\beta s_\alpha \theta$ and hence $m(\theta, \phi) = |\langle \phi(\omega), s_\alpha(\beta)^* \rangle| = |\langle s_\alpha \phi(\omega), \beta^* \rangle| = |\langle \tau(\omega), \beta^* \rangle|$ (since $\tau = s_\alpha \phi) = m(w, \tau)$.

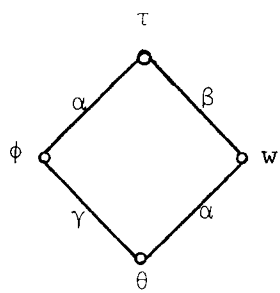
LEMMA 2.6. *Every double divisor is a moving divisor.*

Proof. Let $X_{\mathbf{Z}}(w)$ be a double divisor in $X_{\mathbf{Z}}(\tau)$. We shall prove the result by induction on $\dim X_{\mathbf{Z}}(\tau)$. Let $X_{\mathbf{Z}}(\mu)$ be of least dimension such that $X_{\mathbf{Z}}(\mu)$ has a double divisor $X_{\mathbf{Z}}(\lambda)$. If $X_{\mathbf{Z}}(\lambda)$ is not a moving divisor, let us fix a moving divisor $X_{\mathbf{Z}}(\lambda')$ in $X_{\mathbf{Z}}(\mu)$, moved out by α, say. Let $X_{\mathbf{Z}}(\theta')$ be the divisor in $X_{\mathbf{Z}}(\lambda')$ moved out by α (cf. Lemma 1.5).



Then by Lemma 2.5, we obtain that $X_{\mathbf{Z}}(\theta')$ is a double divisor in $X_{\mathbf{Z}}(\lambda')$ contradicting the minimality assumption on dimension of $X_{\mathbf{Z}}(\mu)$.

Now fix a moving divisor $X_{\mathbf{Z}}(\phi)$ in $X_{\mathbf{Z}}(\tau)$, moved out by α, say. Let $X_{\mathbf{Z}}(\theta)$ be the divisor in $X_{\mathbf{Z}}(\phi)$ moved out to $X_{\mathbf{Z}}(w)$ by α.



Now in view of Lemma 2.5, we have

$$m(\theta, \phi) = m(w, \tau) = 2.$$

Hence $X_{\mathbf{Z}}(\theta)$ is a double divisor in $X_{\mathbf{Z}}(\phi)$ and hence by induction hypothesis $X_{\mathbf{Z}}(\theta)$ is moving in $X_{\mathbf{Z}}(\phi)$, say, $\phi = s_\gamma \theta$ where γ is simple. Let $m(\phi, \tau) = p$ and $m(\theta, w) = q$. Then we have

$$2\gamma + p\alpha = 2\beta + q\alpha; \qquad \text{i.e.,} \qquad 2\beta = 2\gamma + (p - q)\,\alpha.$$

Hence we obtain $p \equiv q \pmod 2$ and hence in fact $p = q$ (since both p and q are $\leqslant 2$). Now this yields that $\beta = \gamma$, proving that $X_{\mathbf{Z}}(w)$ is a moving divisor in $X_{\mathbf{Z}}(\tau)$.

LEMMA 2.7. *Let (τ, μ) be an admissible pair and $\tau = \mu_n > \mu_{n-1} > \cdots > \mu_0 = \mu$ be an admissible chain, i.e., μ_{i-1} is a double divisor in μ_i,*

$1 \leqslant i \leqslant n$; *further let* $\mu_i = s_{\beta_i}\mu_{i+1}$, $0 \leqslant i \leqslant n-1$, *where* β_i *is simple* (*cf. Lemma* 2.6). *Then*

(a) *If for any simple root* α, $\langle \mu(\omega), \alpha^* \rangle = 1$, *then* $\langle \mu_i(\omega), \alpha^* \rangle = 1$, $1 \leqslant i \leqslant n$.

(b) *If for any simple root* α, $\langle \tau(\omega), \alpha^* \rangle = -1$, *then* $\langle \mu_i(\omega), \alpha^* \rangle = -1$, $0 \leqslant i \leqslant n-1$.

Proof. Note that (b) follows from (a).

(a) Now $\langle \mu_1(\omega), \alpha^* \rangle = 1 - 2 < \beta_0, \alpha^* \rangle$ (since $\mu_1(\omega) = \mu_0(\omega) - 2\beta_0$), from which it follows that $\langle \beta_0, \alpha^* \rangle = 0$. Thus $\langle \mu_1(\omega), \alpha^* \rangle = 1$ and the same argument yields that $\langle \mu_i(\omega), \alpha^* \rangle = 1$, $1 \leqslant i \leqslant n$.

LEMMA 2.8. *Let* $X_{\mathbf{Z}}(\phi)$ *be a moving divisor in* $X_{\mathbf{Z}}(\tau)$ *moved by* α; *further let* $\langle \phi(\omega), \alpha^* \rangle = 2$.

(a) *Let* (ϕ, λ) *be a non-trivial admissible pair such that* $\mu = s_\alpha \lambda > \lambda$. *Then* (τ, λ), (τ, μ) *are again admissible pairs.*

(b) *Let* (ϕ, λ) *be an admissible pair such that* $s_\alpha \lambda = \lambda$. *Then* (τ, λ) *is an admissible pair* (*note that the proof of* (b) *is trivial*).

Proof. (a) The proof of the assertion that (τ, λ) is an admissible pair is trivial (since (τ, ϕ) and (ϕ, λ) are admissible pairs). It remains to prove that $(\tau, s_\alpha \lambda)$ is an admissible pair. The facts that

$$\langle \phi(\omega), \alpha^* \rangle = 2 \qquad \text{and} \qquad \langle \lambda(\omega), \alpha^* \rangle > 0$$

(since $s_\alpha \lambda > \lambda$) imply that $\langle \lambda(\omega), \alpha^* \rangle = 2$. Let $\phi = \lambda_n > \lambda_{n-1} > \cdots > \lambda_0 = \lambda$ be an admissible chain defining the admissible pair (ϕ, λ) and let $\lambda_1 = s_{\beta_1}\lambda$, $\lambda_2 = s_{\beta_2}\lambda_1$, etc. Suppose $\beta_1 = \alpha$, then $\lambda_1 = s_\alpha \lambda$ and hence $(\phi, s_\alpha \lambda)$ is an admissible pair. From this, it follows that $(\tau, s_\alpha \lambda)$ is an admissible pair (since (τ, ϕ) and $(\phi, s_\alpha \lambda)$ are admissible pairs).

If $\beta_1 \neq \alpha$, then

$$\begin{aligned} \langle \lambda_1(\omega), \alpha^* \rangle &= \langle s_{\beta_1}\lambda(\omega), \alpha^* \rangle \\ &= \langle \lambda(\omega) - 2\beta_1, \alpha^* \rangle \qquad (\text{note that } \langle \lambda(\omega), \beta_1^* \rangle = 2) \\ &= 2 - 2\langle \beta_1, \alpha^* \rangle \qquad (\text{since } \langle \lambda(\omega), \alpha^* \rangle = 2). \end{aligned}$$

Hence, if $\langle \beta_1, \alpha^* \rangle \neq 0$, then $\langle \lambda_1(\omega), \alpha^* \rangle \geqslant 4$ (note that $\langle \beta_1, \alpha^* \rangle \leqslant 0$, since they are both simple roots). This is not possible, since ω is of classical type (cf. Definition 2.2). Thus $\langle \lambda_1(\omega), \alpha^* \rangle = 2$. To complete the proof, we use induction on the length of the admissible chain. If the length of the admissible chain is 1 i.e., if $X_{\mathbf{Z}}(\lambda)$ is a double divisor in $X_{\mathbf{Z}}(\phi)$, then we obtain that $X_{\mathbf{Z}}(s_\alpha \lambda)$ is a double divisor in $X_{\mathbf{Z}}(\tau)$ (cf. Lemma 2.5) and the result follows in this case. Let then the length of the chain >1. By the induction hypothesis, we obtain that $(\tau, s_\alpha \lambda_1)$ is an admissible pair. Also $(s_\alpha \lambda_1, s_\alpha \lambda)$ is an admissible pair (cf. Lemma 2.5). Hence $(\tau, s_\alpha \lambda)$ is an admissible pair, as asserted. This completes the proof of Lemma 2.8.

LEMMA 2.9. *Let $X_{\mathbf{Z}}(\phi)$ be a moving divisor in $X_{\mathbf{Z}}(\tau)$ moved by α; further, let $\langle \phi(\omega), \alpha^* \rangle = 1$. Let (ϕ, λ) be a non-trivial admissible pair such that $\mu = s_\alpha \lambda > \lambda$. Then (τ, μ) is an admissible pair.*

Proof. Let $\phi = \lambda_n > \lambda_{n-1} > \cdots > \lambda_0 = \lambda$ be an admissible chain defining the admissible pair (ϕ, λ). Now the facts that $\langle \phi(\omega), \alpha^* \rangle = 1$ and $\langle \lambda(\omega), \alpha^* \rangle > 0$ (since $s_\alpha \lambda > \lambda$) imply that $\langle \lambda(\omega), \alpha^* \rangle = 1$ and hence $\langle \lambda_i(\omega), \alpha^* \rangle = 1$, $0 \leqslant i \leqslant n$ (cf. Lemma 2.7). Hence $\lambda_i < s_\alpha \lambda_i$; further $X_{\mathbf{Z}}(s_\alpha \lambda_{i-1})$ is a double divisor in $X_{\mathbf{Z}}(s_\alpha \lambda_i)$ (cf. Lemma 2.5). Hence we obtain $(s_\alpha \phi, s_\alpha \lambda)$, i.e., (τ, μ) is an admissible pair.

COROLLARY 2.10. *Let (ϕ, λ) be an admissible pair such that $\langle \phi(\omega) + \lambda(\omega), \alpha^* \rangle > 0$. Then $(s_\alpha \phi, s_\alpha \lambda)$ is again an admissible pair. If $\langle 1/2(\phi)(\omega) + \lambda(\omega)), \alpha^* \rangle = 2$, then $(s_\alpha \phi, \lambda)$ is also an admissible pair.*

Proof. The hypothesis that $\langle \phi(\omega) + \lambda(\omega), \alpha^* \rangle > 0$ implies that $\langle \phi(\omega), \alpha^* \rangle$ and $\langle \lambda(\omega), \alpha^* \rangle$ are both $\geqslant 0$ (since ω is of classical type). If $\langle \phi(\omega), \alpha^* \rangle > 0$, then the result follows from Lemmas 2.7 and 2.8. If $\langle \phi(\omega), \alpha^* \rangle = 0$, then this implies that $\langle \lambda(\omega), \alpha^* \rangle = 2$ and we proceed as in the proof of Lemma 2.8 to conclude that $(s_\alpha \phi, s_\alpha \lambda)$ is an admissible pair (note that if $\langle \phi(\omega), \alpha^* \rangle = 0$, then $s_\alpha \phi = \phi$).

COROLLARY 2.11. *Let (ϕ, λ) be an admissible pair such that $\langle \phi(\omega) + \lambda(\omega), \alpha^* \rangle < 0$. Then $(s_\alpha \phi, s_\alpha \lambda)$ is again an admissible pair.*

Proof. We first claim that

> (ϕ, λ) is an admissible pair if and only if $(w_0 \lambda, w_0 \phi)$ is an admissible pair, w_0 being the element of largest length in W. $\quad (*)$

To prove the claim, it obviously suffices to prove it when $X(\lambda)$ is a double divisor in $X(\phi)$. Let them $\phi = s_\gamma \lambda$, where γ is simple (cf. Lemma 2.6). Now

$$w_0 \phi = s_\delta w_0 \lambda$$

where $\delta = i(\gamma)$, $i = -w_0$, is the Weyl involution. Further,

$$|\langle w_0 \phi(\omega), \delta^* \rangle| = |\langle \phi(\omega), \gamma^* \rangle| = 2.$$

Thus $X(w_0 \phi)$ is a double divisor in $X(w_0 \lambda)$. This completes the proof of the claim (*) above. Now, denoting $\beta = i(\alpha)$, $i = -w_0$, the Weyl involution, we have

$$\langle w_0\lambda(\omega) + w_0\phi(\omega), \beta^* \rangle = -\langle \phi(\omega) + \lambda(\omega), \alpha^* \rangle > 0.$$

Hence (in view of Corollary 2.10) we obtain that $(s_\beta w_0\lambda, s_\beta w_0\phi)$ is an admissible pair i.e., $(w_0 s_\alpha\lambda, w_0 s_\alpha\phi)$ is an admissible pair. Hence $(s_\alpha\phi, s_\alpha\lambda)$ is an admissible pair (cf. (*) above). This completes the proof of Corollary 2.11.

COROLLARY 2.12. *Let $X_{\mathbf{Z}}(w_1)$ be a moving divisor in $X_{\mathbf{Z}}(w)$ moved by the simple root α. Let (θ, ρ) be an admissible pair on $X_{\mathbf{Z}}(w_1)$ such that*

$$s_\alpha\theta > \theta, \tag{1}$$

$$\langle \theta(\omega) + \rho(\omega), \alpha^* \rangle > 0. \tag{2}$$

Then $(s_\alpha\theta, s_\alpha\rho)$ is an admissible pair on $X_{\mathbf{Z}}(w)$; further, if $\langle \frac{1}{2}(\theta(\omega) + \rho(\omega)), \alpha^* \rangle = 2$, then $(s_\alpha\theta, \rho)$ is also an admissible pair on $X_{\mathbf{Z}}(w)$.

Proof. Case 1. $\langle \frac{1}{2}(\theta(\omega) + \rho(\omega)), \alpha^* \rangle = 2$. This implies $\langle \theta(\omega), \alpha^* \rangle = 2 = \langle \rho(\omega), \alpha^* \rangle$. Now, we have $s_\alpha\rho > \rho$; hence taking $\phi = \theta$ and $\lambda = \rho$, in (a) of Lemma 2.8, we obtain that $(s_\alpha\theta, \rho)$ and $(s_\alpha\theta, s_\alpha\rho)$ are admissible pairs; further, they are admissible pairs on $X_{\mathbf{Z}}(w)$ (since $w_1 \geqslant \theta$, we have $w \geqslant s_\alpha\theta$).

Case 2. $\langle \frac{1}{2}(\theta(\omega) + \rho(\omega)), \alpha^* \rangle = 1$. In this case, we have the following two possibilities:

$$\langle \theta(\omega), \alpha^* \rangle = 2, \qquad \langle \rho(\omega), \alpha^* \rangle = 0$$

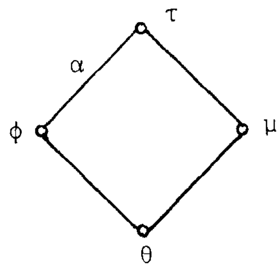
(or)

$$\langle \theta(\omega), \alpha^* \rangle = 1 = \langle \rho(\omega), \alpha^* \rangle.$$

If the former possibility holds, then taking $\phi = \theta$ and $\lambda = \rho$, in (b) of Lemma 2.8, the result follows. If the latter possibility holds, then taking $\phi = \theta$ and $\lambda = \rho$ in Lemma 2.9 (note that $s_\alpha\rho > \rho$), the result follows. This completes the proof of Corollary 2.12.

LEMMA 2.13. *Let $X_{\mathbf{Z}}(\phi)$ be a moving divisor in $X_{\mathbf{Z}}(\tau)$ moved by α. Then for any admissible pair (τ, μ), we have (ϕ, λ) is an admissible pair, where λ is the smaller of $\{\mu, s_\alpha\mu\}$ (note that $\phi \geqslant \lambda$ (cf. Lemma 1.5)).*

Proof. (By induction on $d = \operatorname{codim}$ of $X_{\mathbf{Z}}(\mu)$ in $X_{\mathbf{Z}}(\tau)$.) If $d = 1$, then denoting by $X_{\mathbf{Z}}(\theta)$ the divisor in $X_{\mathbf{Z}}(\phi)$ moved out to $X_{\mathbf{Z}}(\mu)$ by α, we obtain (cf. Lemma 2.5) that $X_{\mathbf{Z}}(\theta)$ is a double divisor in $X_{\mathbf{Z}}(\phi)$ (since $m(\theta, \phi) = m(\mu, \tau) = 2$). Thus $\phi > s_\alpha \mu$ and $(\phi, s_\alpha \mu)$ is an admissible pair.



Let now $d > 1$. Since $\langle \tau(\omega), \alpha^* \rangle < 0$, we conclude that $\langle \mu(\omega), \alpha^* \rangle \neq 1$ (cf. Lemma 2.7). Now we discuss the remaining possibilities for $\langle \mu(\omega), \alpha^* \rangle$.

Case 1: $\langle \mu(\omega), \alpha^* \rangle = 2$. Let us fix an admissible chain $\mu_n = \tau > \mu_{n-1} > \cdots > \mu_0 = \mu$, further let $\mu_i = s_{\beta_i} \mu_{i+1}$, $0 \leqslant i \leqslant n-1$. Now $\langle \mu(\omega), \alpha^* \rangle = 2$ implies that $\langle \tau(\omega), \alpha^* \rangle$ is even and is in fact -2 (since $\langle \tau(\omega), \alpha^* \rangle < 0$).

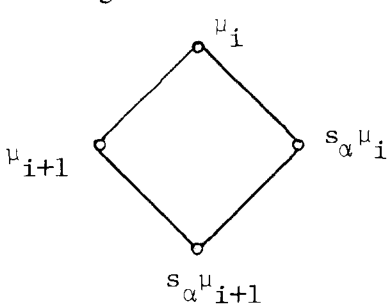
Now

$$\langle \mu_1(\omega), \alpha^* \rangle = 2 - 2\langle \beta_0, \alpha^* \rangle.$$

Hence we obtain that either $\beta_0 = \alpha$ in which case $\langle \mu_1(\omega), \alpha^* \rangle = -2$ or $\langle \beta_0, \alpha^* \rangle = 0$. Now since $\langle \tau(\omega), \alpha^* \rangle = -2$, we find that $\beta_i = \alpha$ for at least one i, $0 \leqslant i \leqslant n-1$. Hence we obtain $\mu_{i+1} = s_\alpha \mu_i$ (for that particular i). Now applying induction hypothesis to (τ, μ_{i+1}), we conclude that (ϕ, μ_i) is an admissible pair (note that $\phi \geqslant \mu_i$) and hence also is (ϕ, μ).

Case 2: $\langle \mu(\omega), \alpha^* \rangle = -2$. Now (τ, μ) and $(\mu, s_\alpha \mu)$ are admissible pairs and hence $(\tau, s_\alpha \mu)$ is an admissible pair with $\langle s_\alpha \mu(\omega), \alpha^* \rangle = 2$. Hence by Case 1, we obtain that $(\phi, s_\alpha \mu)$ is an admissible pair.

Case 3: $\langle \mu(\omega), \alpha^* \rangle = -1$. This implies $\langle \tau(\omega), \alpha^* \rangle = -1$ and hence (fixing an admissible chain as above) $\langle \mu_i(\omega), \alpha^* \rangle = -1$, $\forall 0 \leqslant i \leqslant n$ (cf. Lemma 2.7). Now considering



we have $m(s_\alpha\mu_{i+1}, s_\alpha\mu_i) = m(\mu_{i+1}, \mu_i)$ (cf. Lemma 2.5). Thus $\{s_\alpha\mu_n, s_\alpha\mu_{n-1}, ..., s_\alpha\mu\}$ is an admissible chain and in particular $(\phi, s_\alpha\mu)$ is an admissible pair.

Case 4: $\langle\mu(\omega), \alpha^*\rangle = 0$. Now, in this case $\phi \geqslant \mu$ (necessarily). Fixing an admissible chain as above, we have

$$\langle\mu_1(\omega), \alpha^*\rangle = -2(\beta, \alpha^*) = 0 \text{ or } 2.$$

If $\langle\mu_1(\omega), \alpha^*\rangle = 0$, then applying induction hypothesis to (τ, μ_1), we obtain that (ϕ, μ_1) (and hence also (ϕ, μ)) is an admissible pair). If $\langle\mu_1(\omega), \alpha^*\rangle = 2$, again induction hypothesis applied to (τ, μ_1) gives that (ϕ, μ_1) (and hence also (ϕ, μ)) is an admissible pair.

This completes the proof of Lemma 2.13.

LEMMA 2.14. *Let ϕ, τ be as in Lemma* 2.8. *Any admissible pair (τ, μ) on $X_{\mathbf{Z}}(\tau)$ is uniquely of the form* (a) *or* (b) *of Lemma* 2.8.

Proof. Let $\lambda =$ the smaller of $\{\mu, s_\alpha\mu\}$. Then by Lemma 2.13, (ϕ, λ) is an admissible pair. If $s_\alpha\lambda = \lambda$, so that $\mu = \lambda$, then (τ, μ) is the admissible pair as given by (b) of Lemma 2.8. If $s_\alpha\lambda > \lambda$, then (τ, μ) is the admissible pair as given by (a) of Lemma 2.8.

LEMMA 2.15. *Let ϕ, τ be as in Lemma* 2.9. *Any admissible pair (τ, μ) on $X_{\mathbf{Z}}(\tau)$ is uniquely of the form as given in Lemma* 2.9.

Now $\langle\tau(\omega), \alpha^*\rangle = -1$ implies that $\langle\mu(\omega), \alpha^*\rangle = -1$ (cf. Lemma 2.7). Hence $\mu > s_\alpha\mu$ and Lemma 2.13 implies that $(\phi, s_\alpha\mu)$ is an admissible pair. Thus (τ, μ) is the admissible pair as given by Lemma 2.9.

LEMMA 2.16. *Let $X_{\mathbf{Z}}(w_1)$ be a moving divisor in $X_{\mathbf{Z}}(w)$ moved by α. The admissible pairs on $X_{\mathbf{Z}}(w)$ are those on $X_{\mathbf{Z}}(w_1)$ and those of the form $(s_\alpha\theta, s_\alpha\rho)$ or $(s_\alpha\theta, \rho)$ for (θ, ρ) as given in Corollary* 2.12 *with $w_1 \not\geqslant s_\alpha\theta$.*

Proof. We have only to prove that if (η, σ) is an admissible pair on $X_{\mathbf{Z}}(w)$ with $w_1 \not\geqslant \eta$, then (η, σ) is as mentioned in the Lemma. Now $w_1 \not\geqslant \eta$ implies that $w_1 \geqslant s_\alpha\eta$ (cf. Lemma 1.5) and $\eta > s_\alpha\eta$. The result now follows from Lemmas 2.14 and 2.15. (One takes $\phi = s_\alpha\eta$, etc.)

Remark 2.17. Given an admissible pair (λ, μ), any chain

$$\lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_n = \mu$$

where $X(\lambda_i)$ is a Schubert divisor in $X(\lambda_{i-1})$, $1 \leqslant i \leqslant n$, is an admissible chain, i.e., $X(\lambda_i)$ is a double divisor in $X(\lambda_{i-1})$, $1 \leqslant i \leqslant n$. For a proof of this, the reader may refer to [D-L].

3. First Basis Theorem

In this section, we prove the first basis theorem. Notations being as in Section 1, let U denote the enveloping algebra of Lie $G_{\mathbf{Q}}$—the Lie algebra of $G_{\mathbf{Q}}$. Let $U_{\mathbf{Z}}$ (resp. $U_{\mathbf{Z}}^{+}$, resp. $U_{\mathbf{Z}}^{-}$) denote the canonical $\mathbf{Z}$-form of U, i.e., the $\mathbf{Z}$-subalgebra of U spanned by $X_{\alpha}^{n}/n!$, α a root (resp. α, a positive root, resp. a negative root) where X_{α} denotes the element in the Chevalley basis of Lie $G_{\mathbf{Q}}$ corresponding to α. Let U_{α} (resp. $U_{\alpha,\mathbf{Z}}$) denote the $\mathbf{Q}$-vector subspace (resp. $\mathbf{Z}$-submodule) of U (resp. $U_{\mathbf{Z}}$) generated by X_{α}^{n} (resp. $X_{\alpha}^{n}/n!$). Let $G_{\alpha,\mathbf{Q}}$ or just G_{α} (resp. $G_{\alpha,\mathbf{Z}}$) denote the unipotent subgroup scheme isomorphic to $\mathbf{G}_{a,\mathbf{Q}}$ (resp. $\mathbf{G}_{a,\mathbf{Z}}$) of $B_{\mathbf{Q}}$ (resp. $B_{\mathbf{Z}}$) corresponding to α. We see that Lie $G_{\alpha,\mathbf{Q}} \approx \mathbf{Q}X_{\alpha}$.

Let V be a finite-dimensional $\mathbf{Q}$-vector space which is also a $G_{\mathbf{Q}}$-module. Then a lattice $V_{\mathbf{Z}}$ in V is said to be an *admissible* $\mathbf{Z}$*-form* if any of the following three equivalent conditions are satisfied.

(i) $V_{\mathbf{Z}}$ is $U_{\mathbf{Z}}$-stable.

(ii) $V_{\mathbf{Z}}$ is a $G_{\mathbf{Z}}$-$\mathbf{Z}$ module, i.e., for every commutative ring D (with 1) $V_{\mathbf{Z}} \otimes_{\mathbf{Z}} D$ has a $G_{\mathbf{Z}}(D)$-module structure (here $G_{\mathbf{Z}}(D) =$ group of D-valued points of $G_{\mathbf{Z}}$) which is functorial in D.

(iii) $V_{\mathbf{Z}}$ is stable under $X_{\pm\alpha}^{n}/n!$, α a simple root and $n \in \mathbf{N}$.

We observe that if V, D, etc., are as above, then for $d \in D$,

$$\exp(dX_{\alpha}) = \sum d^{n} \frac{X_{\alpha}^{n}}{n!}$$

defines an automorphism of the D-module $V_{\mathbf{Z}} \otimes_{\mathbf{Z}} D$. When we identify $G_{\alpha,\mathbf{Z}}(D)$ with D, the action of d on $V_{\mathbf{Z}} \otimes D$ is given by $\exp(dX_{\alpha})$. If A is any ring, we set

$$U_{A} = U_{\mathbf{Z}} \otimes_{\mathbf{Z}} A, \qquad U_{\alpha,A} = U_{\alpha,\mathbf{Z}} \otimes_{\mathbf{Z}} A$$

and the above definitions made for $\mathbf{Z}$ can be generalized to A.

For a dominant weight ω, let V_{ω} or just V denote the finite-dimensional $\mathbf{Q}$-vector space which is the irreducible G-module with highest weight ω. Fix a highest weight vector $e = e_{\omega}$ in V_{ω} (determined uniquely up to a non-zero factor in $\mathbf{Q}$). For $\tau \in W/W_{P}$, we write

$$V_{\mathbf{Z}}(\tau) = U_{\mathbf{Z}}^{+} e_{\tau}, \qquad \text{where } e_{\tau} = \tau \cdot e$$

(τ can be represented by a $\mathbf{Z}$-valued point of $N(T_{\mathbf{Z}})$ and we see that e_{τ} is well-determined up to a factor ± 1). We write

$$V_{\mathbf{Z}}(w_{0}) = V_{\mathbf{Z}}$$

w_0 being the element of W of maximal length. One knows that

$$V_{\mathbf{Z}} \otimes \mathbf{Q} = V$$

and that $V_{\mathbf{Z}}$ is a $U_{\mathbf{Z}}$-stable $\mathbf{Z}$-submodule of V or, equivalently, a $G_{\mathbf{Z}} - \mathbf{Z}$ module. We see that

$$V_{\mathbf{Z}} = U_{\mathbf{Z}}^{-} e = U_{\mathbf{Z}}^{+} e_{w_0}.$$

If A is any ring we define

$$V_A(\tau) = V_{\mathbf{Z}}(\tau) \otimes_{\mathbf{Z}} A; \qquad V_A = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} A.$$

We observe that $V_{\mathbf{Q}} = V$ and that V_A is a G_A-A module. Also, e is a primitive element in $V_{\mathbf{Z}}$, i.e., $\mathbf{Z}e$ is a direct summand in $V_{\mathbf{Z}}$. Consequently, every e_τ, $\tau \in W$, is a primitive element in $V_{\mathbf{Z}}$. For the rest of this section, we shall assume that *ω is a fundamental weight of classical type* (cf. Definition 2.2).

Let $\mathbf{Z}[\mathrm{N}]$ denote the group ring of the multiplicative group $\exp N$, where

$$\exp N = \{\exp \lambda / \lambda \in N\},\ N = \mathrm{Hom}(T, \mathbf{G_m})\}.$$

For a simple root α, let L_{s_α} be the linear operator $L_{s_\alpha}: \mathbf{Z}[\mathrm{N}] \to \mathbf{Z}[\mathrm{N}]$ defined by

$$L_{s_\alpha}(\exp \lambda) = \frac{\exp \lambda - \exp s_\alpha(\lambda)}{1 - \exp \alpha}, \qquad \lambda \in N$$

(cf. $[\mathrm{D}]_1$). Let M_{s_α} be the operator defined by

$$M_{s_\alpha}: \mathbf{Z}[N] \to \mathbf{Z}[N]$$
$$M_{s_\alpha}(\exp \lambda) = (\exp \rho)\, L_\alpha(\exp(\lambda - \rho))$$

(ρ being half the sum of positive roots). For $\tau \in W$, fixing a reduced expression $\tau = s_{\alpha_1} \cdots s_{\alpha_r}$, let M_τ be the operator $M_\tau = M_{s_{\alpha_1}} \circ \cdots \circ M_{s_{\alpha_r}}$. It can be easily seen that M_τ is independent of the reduced expression chosen for τ (for example, this follows from Theorem (2) of $[\mathrm{D}]_1$) and that $M_\tau(\exp \lambda) = \exp \rho \cdot L_\tau(\exp(\lambda - \rho))$, $\lambda \in N$. Now we have the following

PROPOSITION 3.1. *With notations as above, we have for any $w \in W/W_P$,*

$$M_\tau(\exp(-\omega)) = \sum \exp[\tfrac{1}{2}(\lambda(-\omega) + \mu(-\omega))]$$

where the summation on the R.H.S. is over all admissible pairs (λ, μ) such that $\tau \geqslant \lambda$.

Proof. (By induction on dim $X(\tau)$.) If dim $X(\tau)=0$, then $\tau=\text{Id}$ and the result is trivially true. Let then dim $X(\tau)\geqslant 1$. Let $X(\phi)$ be a moving divisor in $X(\tau)$ moved by α, so that $\tau=s_\alpha\phi$. Let

$$I_\phi=\{\text{all admissible pairs } (\lambda, \mu)/\phi\geqslant\lambda\}.$$

For simplicity of notations, let us denote an admissible pair (λ, μ) by just δ and let $\delta(\pm\omega)$ denote $\frac{1}{2}(\lambda(\pm\omega)+\mu(\pm\omega))$. Let us partition I_ϕ into the following four subsets:

$$I^0=\{\delta\in I_\phi/\langle\delta(\omega), \alpha^*\rangle=0\}$$
$$I^-=\{\delta\in I_\phi/\langle\delta(\omega), \alpha^*\rangle<0\}$$
$$I^+=\{\delta\in I_\phi/\langle\delta(\omega), \alpha^*\rangle>0 \text{ and } \phi\geqslant s_\alpha\lambda \text{ where } \delta=(\lambda, \mu)\}$$
$$I^e=\{\delta\in I_\phi/\langle\delta(\omega), \alpha^*\rangle>0 \text{ and } \phi\ngeqslant s_\alpha\lambda \text{ where } \delta=(\lambda, \mu)\}.$$

Let us define

$$F^0=\sum_{\delta\in I^0}\exp\delta(-\omega);\qquad F^-=\sum_{\delta\in I^-}\exp\delta(-\omega)$$

$$F^+=\sum_{\delta\in I^+}\exp\delta(-\omega);\qquad F^e=\sum_{\delta\in I^e}\exp\delta(-\omega).$$

By the definition of the operator M_{s_α}, the following can be checked easily:

$$\begin{aligned} M_{s_\alpha}(\exp\lambda) &= \exp\lambda+\exp(\lambda+\alpha)+\cdots+\exp(\lambda+n\alpha) \quad \text{if } n=-\langle\lambda, \alpha^*\rangle\geqslant 0\\ &=0 \quad \text{if } -\langle\lambda, \alpha^*\rangle=-1\\ &=-[\exp(\lambda-\alpha)+\exp(\lambda-2\alpha)+\cdots+\exp(\lambda-q\alpha)],\\ &\qquad \text{if}\quad q=\langle\lambda, \alpha^*\rangle-1\geqslant 1, \text{ i.e., } -\langle\lambda, \alpha^*\rangle\leqslant -2\end{aligned}$$

for any $\lambda\in\text{Hom}(T, \mathbf{G_m})$. From this, we deduced easily the following: For any $\lambda\in\text{Hom}(T, \mathbf{G_m})$,

(i) $M_{s_\alpha}(\exp\lambda)=\exp\lambda$, if $\langle\lambda, \alpha^*\rangle=0$.

(ii) $M_{s_\alpha}(\exp\lambda+\exp(s_\alpha\lambda))=\exp\lambda+\exp(s_\alpha\lambda)$.

Using (i) we obtain $M_{s_\alpha}(F^0)=F^0$ and using (ii) we claim that

$$M_{s_\alpha}(F^++F^-)=F^++F^-. \tag{$*$}$$

To prove $(*)$ we first observe that the map $(\lambda, \mu)\mapsto(s_\alpha\lambda, s_\alpha\mu)$ define a bijection of I^+ onto I^- (in view of Corollaries 2.10 and 2.11; note that if $\langle\lambda(\omega)+\mu(\omega), \alpha^*\rangle<0$, then $\langle\lambda(\omega), \alpha^*\rangle\leqslant 0$ so that $s_\alpha\lambda\leqslant\lambda\leqslant\phi$ and hence if $(\lambda, \mu) \in I^-$, then $(s_\alpha \lambda, s_\alpha \mu) \in I^+$). Further $\langle s_\alpha \lambda(\omega) + s_\alpha \mu(\omega), \alpha^* \rangle = -\langle \lambda(\omega) + \mu(\omega), \alpha^* \rangle$. Claim (*) now follows using (ii) above.

We next claim that

$$M_{s_\alpha}(F^e) = \sum_{\delta \in I^e \cup I_\tau - I_\phi} \exp(\delta(-\omega)). \qquad (**)$$

To prove this, we proceed as follows. For $\delta \in I_\phi$, say, $\delta = (\lambda, \mu)$, let $n(\delta) = (\langle \lambda(\omega), \alpha^* \rangle, \langle \mu(\omega), \alpha^* \rangle)$. If $\delta \in I^e$, the possibilities for $n(\delta)$ are (2, 2), (2, 0), (1, 1) (note that $\phi \not\geq s_\alpha \lambda$ in particular implies that $s_\alpha \lambda > \lambda$ and hence $\langle \lambda(\omega), \alpha^* \rangle > 0$). Now we have

$$M_{s_\alpha}(\exp(\delta(-\omega))) = \sum_{j=0}^{r} \exp(\delta(-\omega) + j\alpha)$$

where $r = -\langle \delta(-\omega), \alpha^* \rangle$. We shall now evaluate $M_{s_\alpha}(\exp \delta(-\omega))$ explicitly (where $\delta = (\lambda, \mu) \in I^e$).

(1) Let $n(\delta) = (2, 2)$, so that $r = 2$. In view of Corollary 2.10, $\delta_1 = (s_\alpha \lambda, \mu)$ and $\delta_2 = (s_\alpha \lambda, s_\alpha \mu)$ are again admissible pairs and they are both in $I_\tau - I_\phi$ (since $\phi \not\geq s_\alpha \lambda$). Further $\delta_i(-\omega) = \delta(-\omega) + i\alpha$, $i = 1, 2$.

(2) Let $n(\delta) = (2, 0)$ or (1, 1), so that $r = 1$. In view of Corollary 2.10, $\eta = (s_\alpha \lambda, s_\alpha \mu)$ is again an admissible pair. Further $\eta(-\omega) = \delta(-\omega) + \alpha$.

On the other hand, by Lemma 2.16, every $\delta' = (\lambda', \mu') \in I_\tau - I_\phi$ is (uniquely) of the form $(\lambda', \mu') = (s_\alpha \lambda, \mu)$ or $(s_\alpha \lambda, s_\alpha \mu)$ for an unique $(\lambda, \mu) \in I^e$.

This completes the proof of (**). Now the proposition follows from (*), (**), and the induction hypothesis. (One also uses the fact that $M_{s_\alpha}(F^0) = F^0$).

PROPOSITION 3.2. (cf. $[\mathrm{D}]_1$). *Notations as before, we have*

$$M_{w_0}(\exp(-\omega)) = \mathrm{ch}\ V_{i(\omega)}$$

where $V_{i(\omega)}$ is the irreducible G-module (in characteristic zero) with highest weight $i(\omega)$, $i(= -w_0)$ being the Weyl involution.

Proof. Recall the operator $L_{s_\alpha}: \mathbf{Z}[N] \to \mathbf{Z}[N]$,

$$L_{s_\alpha}(\exp \lambda) = \frac{\exp \lambda - \exp s_\alpha(\lambda)}{1 - \exp \alpha}, \qquad \lambda \in N = \mathrm{Hom}(T, \mathbf{G}_\mathbf{m}).$$

Also recall (cf. $[\mathrm{B}]_2$) the operator $J = \sum_{w \in W} (-1)^{l(w)} w$. We have (cf. $[\mathrm{S}]_2$)

$$\mathrm{ch}\ V_{i(\omega)} = \frac{J(\exp(i(\omega) + \rho))}{J(\exp \rho)} \qquad (1)$$

(where $\rho=\frac{1}{2}$ sum of positive roots). We first claim (cf. [D]$_1$)

$$L_{w_0}=\exp(-\rho)(J(\exp(-\rho)))^{-1}J. \qquad (*)$$

To prove (*), we first observe that for a simple root α, we have

$$L_{s_\alpha}L_{s_\alpha}=L_{s_\alpha}.$$

From this it can be easily seen that

$$L_wL_{s_\alpha}=L_w \qquad \text{if } ws_\alpha<w$$

and

$$L_{s_\alpha}L_w=L_w \qquad \text{if } s_\alpha w<w.$$

In particular, we obtain

$$L_{s_\alpha}L_{w_0}=L_{w_0}$$

for all simple roots α. From this we obtain

$$s_\alpha L_{w_0}=\exp\alpha\cdot L_{w_0} \qquad (2)$$

for any simple root α. Now set

$$A=\exp\rho\ J(\exp(-\rho))\ L_{w_0}. \qquad (3)$$

In view of (2), we have

$$s_\alpha A=-A$$

(one also uses the fact (cf. [B]$_2$) that J is anti-invariant, i.e., $wJ=(-1)^{l(w)}J$, $w\in W$). Hence we obtain

$$wA=(-1)^{l(w)}A. \qquad (4)$$

On the other hand, proceeding as in Lemma 3 in [D]$_2$, we obtain

$$L_{w_0}=(-1)^q\prod_{\alpha>0}(1-\exp\alpha)^{-1}w_0+\sum_{w'\neq w_0}a(w')\,w' \qquad (5)$$

where $a(w')\in$ quotient field of $\mathbf{Z}[N]$ and $q=\#\{\text{positive roots}\}$. Using (3) and (5) we obtain that the coefficient of w_0 in A is $=(-1)^q\exp\rho\ J(\exp(-\rho))\prod_{\alpha>0}(1-\exp\alpha)^{-1}=(-1)^q$ (for $J(\exp\rho)=\prod_{\alpha>0}(\exp(\alpha/2)-\exp(-\alpha/2)) = \exp(-\rho)\prod_{\alpha>0}(\exp\alpha-1)$). Hence $\sum_{\alpha>0}(1-\exp\alpha)^{-1}=((-1)^q\exp(-\rho))/(J(\exp\rho))$ and $J(\exp(-\rho))=(-1)^qJ(\exp\rho)$ (since J is anti-invariant, $w_0J(\exp\rho)$ $(=J(\exp(-\rho)))$

$= (-1)^{l(w_0)} J(\exp \rho) = (-1)^q J(\exp \rho))$. Now this together with the fact that A is anti-invariant (cf. (4) above) implies that $A = J$ (cf. [B]$_2$). From this (∗) follows immediately. Now

$$\begin{aligned} M_{w_0}(\exp(-\omega)) &= \exp \rho \cdot L_{w_0}(\exp(-\omega - \rho)) \\ &= (J(\exp(-\rho)))^{-1} J(\exp(-\omega - \rho)) \\ &= \frac{J(\exp(-\omega - \rho))}{J(\exp(-\rho))} \\ &= \frac{J(\exp(-w_0(\omega) + \rho))}{J(\exp \rho)} \\ &= \frac{J(\exp(i(\omega) + \rho))}{J(\exp \rho)} = \text{ch } V_{i(\omega)} \end{aligned}$$

(cf. [S]$_2$). This completes the proof of Proposition 3.2.

Remark 3.3. In view of Propositions 3.1 and 3.2, we have

(a) $\text{ch } Vi(\omega) = \sum \exp[\frac{1}{2}(\lambda(-\omega) + \mu(-\omega))]$

where the summation on the R.H.S. is over all admissible pairs (λ, μ) on G/P. In particular, we obtain

(b) $\dim V_\omega = \dim V_{i(\omega)} = \#\{\text{admissible pairs on } G/P\}$,

(c) for any weight χ in V_ω and any root α,

$$|\langle \chi(\omega), \alpha^* \rangle| \leqslant 2$$

(note that $|\langle \tau(\omega), \alpha^* \rangle| \leqslant 2$, for $\tau \in W$ and any root α).

LEMMA 3.4. *Let $\phi, \tau \in W^P_{\min}$ (= the set of minimal representatives of W/W_P in W (cf. [L-M-S]$_2$)) and let $\phi = s_\alpha \tau$, $l(\tau) = l(\phi) + 1$, where s_α is the reflection associated to the simple root α. Then we have*

$$U_{-\alpha, \mathbf{Z}} V_{\mathbf{Z}}(\phi) = V_{\mathbf{Z}}(\tau) \tag{∗}$$

(in particular $V_{\mathbf{Z}}(\phi) \subset V_{\mathbf{Z}}(\tau)$).

Proof. We first observe that

$$X_\alpha e_\phi = 0; \qquad X_{-\alpha} e_\tau = 0. \tag{1}$$

For, if $X_\alpha e_\phi \neq 0$, then it is a weight vector (for the $T_{\mathbf{Z}}$-action) of weight $\phi(\omega) + \alpha$. Also since α moves $X(\phi)$, we have $\langle \phi(\omega), \alpha^* \rangle > 0$ (cf. Lemma 1.5). Hence $\langle \phi(\omega) + \alpha, \alpha^* \rangle \geqslant 3$. But this is not possible (see Remark 3.3(c)). Thus $X_\alpha e_\phi = 0$. Similarly, $X_{-\alpha} e_\tau \neq 0$ leads to a contradiction. This proves (1). Now (1) implies that e_ϕ (resp. e_τ) is a highest (resp. lowest) weight vector for the three-dimensional subalgebra of Lie(G) isomorphic to $sl(2)$ associated to α. Then by the well-known $sl(2)$-theory (cf. [S]$_2$), we obtain

$$\begin{aligned} \pm e_\tau = s_\alpha e_\phi = X_{-\alpha} e_\phi \quad & \text{if} \quad \langle \phi(\omega), \alpha^* \rangle = 1 \\ = \frac{X_{-\alpha}^2 e_\phi}{2!} \quad & \text{if} \quad \langle \phi(\omega), \alpha^* \rangle = 2. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} \pm e_\phi = s_\alpha e_\tau = X_\alpha e_\tau \quad & \text{if} \quad \langle \tau(\omega), \alpha^* \rangle = -1 \\ = \frac{X_\alpha^2 e_\tau}{2!} \quad & \text{if} \quad \langle \tau(\omega), \alpha^* \rangle = -2. \end{aligned} \tag{2'}$$

Now $V_{\mathbf{Z}}(\tau)$ and $V_{\mathbf{Z}}(\phi)$ are both $T_{\mathbf{Z}}-\mathbf{Z}$ modules and hence are stable under the operators $\binom{H_\beta}{n}$, $\beta \in \Delta^+$, the set of positive roots (see [St] for the notation $\binom{H_\beta}{n}$). Since $T_{\mathbf{Z}}$ normalizes the group scheme $G_{\alpha,\mathbf{Z}}$, we conclude immediately that the LHS of $(*)$ is also a $T_{\mathbf{Z}}$-$\mathbf{Z}$ module, so that the L.H.S. of $(*)$ is stable under the operators $\binom{H_\beta}{n}$, $\beta \in \Delta^+$. To prove Lemma 3.4, we make use of the following

SUBLEMMA. (i) *Let $\alpha \in \Delta^+$. Then we have*

$$\text{(a)} \quad \frac{X_\alpha^m}{m!}\frac{X_{-\alpha}^n}{n!} = \sum_{j=0}^{\min(m,n)} \frac{X_{-\alpha}^{n-j}}{(n-j)!}\binom{H_\alpha - m - n + 2j}{j}\frac{X_\alpha^{m-j}}{(m-j)!}$$

$$\text{(b)} \quad \frac{X_{-\alpha}^m}{m!}\frac{X_{\alpha}^n}{n!} = \sum_{j=0}^{\min(m,n)} \frac{X_{\alpha}^{n-j}}{(n-j)!}\binom{H_{-\alpha} - m - n + 2j}{j}\frac{X_{-\alpha}^{m-j}}{(m-j)!}.$$

(ii) *Let α be a simple root and β a positive root such that $\beta \neq \alpha$. Then the subalgebra of U generated by $X_{-\alpha}^m/m!$ and $X_\beta^n/n!$, $m \geqslant 0$, $n \geqslant 0$, can be written as an integral linear combination of terms of either of the following forms*:

$$\text{(a)} \quad \frac{X_{\beta_1}^{m_1}}{m_1!}\cdots\frac{X_{\beta_r}^{m_r}}{m_r!}\frac{X_{-\alpha}^n}{n!}, \qquad \beta_i \in \Delta^+, 1 \leqslant i \leqslant r$$

$$\text{(b)} \quad \frac{X_{-\alpha}^n}{n!}\frac{X_{\beta_1}^{m_1}}{m_1!}\cdots\frac{X_{\beta_r}^{m_r}}{m_r!}, \qquad \beta_i \in \Delta^+, 1 \leqslant i \leqslant r.$$

Proof of Sublemma. The assertions (a) and (b) of (i) are just those of Lemma 5 in [St]. The assertions (a) and (b) of (ii) can be deduced from Lemma 8 in [St] as follows.

Let R be the set of roots which can be expressed in the form $i\beta - j\alpha$, $i \geqslant 0$, $j \geqslant 0$. Since α is a simple root and $\beta \in \Delta^+$, we see easily that

$$\text{if } \gamma \in R,\ \gamma \neq -\alpha, \text{ then } \gamma \in \Delta^+$$

and

$$\text{if } \gamma \in R, \text{ then } -\gamma \notin R.$$

Then the assertions (a) and (b) of (ii) are obtained from Lemma 8 of [St] by taking the set S of Lemma 8 (loc. cit.) as R.

Returning to the proof of Lemma 3.4, we now claim the following

$$\text{R.H.S. of } (*) \text{ is stable under } U_{-\alpha,\mathbf{Z}}. \tag{3}$$

This claim, in particular, would imply that the R.H.S. of $(*) \supseteq$ L.H.S. of $(*)$, for, it is clear from (2) that $V_{\mathbf{Z}}(\phi) \subseteq V_{\mathbf{Z}}(\tau)$. The claim (3) is proved by the following inductive argument. Let

$$\frac{X_{\gamma_1}^{m_1}}{m_1!} \cdots \frac{X_{\gamma_r}^{m_r}}{m_r!} e_\tau = F, \qquad m_1 > 0, \ldots, m_r > 0 \tag{4}$$

be an expression representing an element of $V_{\mathbf{Z}}(\tau)$. We call $(m_1 + \cdots + m_r)$ the *degree of* F. We now show

$$\frac{X_{-\alpha}^n}{n!} F \in V_{\mathbf{Z}}(\tau) \tag{5}$$

by induction on the degree of F and this would prove (3). Suppose $\deg F = 0$, i.e., $F = e_\tau$. Then by (1), $(X_{-\alpha}^n/n!)\, e_\tau = 0$ and thus (5) follows in this case. Suppose then that $\deg F > 0$ so that we can suppose that $m_1 > 0$. Consider first the case $\gamma_1 \neq \alpha$. Then by (ii)(a) of Sublemma, $(X_{-\alpha}^n/n!) \cdot (X_{\gamma_1}^{m_1}/m_1!)$ can be expressed as an integral linear combination of sums of type

$$\frac{X_{\beta_1}^{n_1}}{n_1!} \cdots \frac{X_{\beta_r}^{n_r}}{n_r!} \frac{X_{-\alpha}^m}{m!} F_1, \qquad F_1 = \frac{X_{\gamma_2}^{m_2}}{m_2!} \cdots \frac{X_{\gamma_r}^{m_r}}{m_r!} e_\tau \tag{6}$$

so that $\deg F_1 = m_2 + \cdots + m_r < \deg F$. Hence by induction hypothesis $(X_{-\alpha}^m/m!)\, F_1 \in V_{\mathbf{Z}}(\tau)$ and since $\beta_i \in \Delta^+$, it follows that the elements in (6) are in $V_{\mathbf{Z}}(\tau)$. This proves (5) if $\gamma_1 \neq \alpha$. Suppose now that $\gamma_1 = \alpha$. Then by (i)(b) of Sublemma, $(X_{-\alpha}^n/n!)(X_{\alpha_1}^{m_1}/m_1!)$ can be expressed as an integral linear combination of sums of type

$$\frac{X_{\alpha_1}^{l_1}}{l_1!} \binom{H_\alpha - l_1 - l_2}{j} \frac{X_{-\alpha}^{l_2}}{l_2!}, \qquad l_1 \geqslant 0,\ l_2 \geqslant 0.$$

Hence (5) can be expressed as an integral linear combination of sums of type

$$\frac{X_\alpha^{l_1}}{l_1!}\binom{H_\alpha - l_1 - l_2}{j}\frac{X_{-\alpha}^{l_2}}{l_2!}F_1, \qquad F_1 = \frac{X_{\gamma_2}^{m_2}}{m_2!}\cdots\frac{X_{\gamma_r}^{m_r}}{m_r!}e_\tau \tag{7}$$

so that $\deg F_1 = m_2 + \cdots + m_r < \deg F$. By the induction hypothesis $(X_{-\alpha}^{l_2}/l_2!)\,F_1 \in V_{\mathbf{Z}}(\tau)$. One knows that $\binom{H_\alpha - l_1 - l_2}{j}$ can be expressed as an integral linear combination of sums of type $\binom{H_\alpha}{p}$ and as $V_{\mathbf{Z}}(\tau)$ is stable under expressions of type $\binom{H_\alpha}{p}$ it follows that the elements in (7) are in $V_{\mathbf{Z}}(\tau)$. This concludes the proof of (5).

Remark 3.5. In proving the assertion (3) we have in fact shown that

$$U_{-\alpha,\mathbf{Z}} \cdot V_{\mathbf{Z}}(\tau) \subseteq V_{\mathbf{Z}}(\tau) \qquad \text{if } \tau \geqslant s_\alpha \tau.$$

Now to conclude the proof of Lemma 3.4, it suffices to show that

$$\text{R.H.S. of } (*) \subseteq \text{L.H.S. of } (*). \tag{8}$$

Now by (2), we have

$$e_\tau \in \text{L.H.S. of } (*)$$

so that to prove (8), it suffices to show that

$$\text{L.H.S. of } (*) \text{ is } U_{\mathbf{Z}}^{+}\text{-stable} \tag{9}$$

and then the proof of Lemma 3.4 would be complete. To prove (9), we have to show that an expression of type

$$\frac{X_\beta^n}{n!}\frac{X_{-\alpha}^m}{m!}F, \qquad \beta \in \Delta^+,\ F \in V_{\mathbf{Z}}(\phi) \tag{10}$$

belongs to the L.H.S. of $(*)$. Now we express $(X_\beta^n/n!)(X_{-\alpha}^m/m!)$ as an integral linear combination of expressions of type as in (i)(a) of Sublemma, in case $\beta = \alpha$, and (ii)(b) of Sublemma otherwise. Then using the fact that $V_{\mathbf{Z}}(\phi)$ is stable under

$$\binom{H_\gamma}{p} \qquad \text{and} \qquad \frac{X_\gamma^q}{q!},\ . \qquad \gamma \in \Delta^+,\ p > 0,\ q > 0$$

we see immediately that an element in (10) can be expressed as an integral linear combination of sums of type

$$\frac{X_{-\alpha}^s}{s!}F', \qquad F' \in V_{\mathbf{Z}}(\phi)$$

which shows that the elements in (10) are in the L.H.S. of (*). This proves the assertion (8) and the proof of Lemma 3.4 is now complete.

LEMMA 3.6. *Let ϕ, τ be as in Lemma 3.4. Suppose the* **Z**-*module* $V_{\mathbf{Z}}(\phi)$ *has a basis* $\{Q(\lambda, \mu)\}$ *indexed by admissible pairs* (λ, μ) *on* $X(\phi)$ *having the following properties*:

(1) $Q(\lambda, \mu)$ *is a weight vector of weight*

$$\tfrac{1}{2}(\lambda(\omega) + \mu(\omega)).$$

(2) *If* $W(\theta)$ *denotes the* **Z**-*submodule of* $V_{\mathbf{Z}}(\phi)$ *spanned by all* $Q(\lambda, \mu)$ *such that* $\phi \geqslant \theta \geqslant \lambda$, *then*

$$W(\theta) = V_{\mathbf{Z}}(\theta).$$

Let now (λ_1, μ_1) *be such that* $\phi \geqslant \lambda_1$, $\langle \lambda_1(\omega) + \mu_1(\omega), \alpha^* \rangle > 0$, *and* $\phi \ngeqslant s_\alpha \lambda_1$. *If* $\langle \frac{1}{2}(\lambda_1(\omega) + \mu_1(\omega)), \alpha^* \rangle = 1$, *set*

(a) $Q(\lambda, \mu) = X_{-\alpha} Q(\lambda_1, \mu_1)$, $(\lambda, \mu) = (s_\alpha \lambda_1, s_\alpha \mu_1)$.

If $\langle \frac{1}{2}(\lambda_1(\omega) + \mu_1(\omega)), \alpha^* \rangle = 2$, *set*

(b) $Q(\lambda, \mu) = (X_{-\alpha}^2/2!)\, Q(\lambda_1, \mu_1)$, $(\lambda, \mu) = (s_\alpha \lambda_1, s_\alpha \mu_1)$,

(c) $Q(s_\alpha \lambda_1, \mu_1) = X_{-\alpha} Q(\lambda_1, \mu_1)$.

We call the $Q(\lambda, \mu)$ *defined in* (a), (b), (c) *as the new basis elements* (*relative to* ϕ, τ). *Consider the set of all* $Q(\lambda, \mu)$ *above, i.e., the new basis elements as well as those given by the hypothesis. Then we have the following.*

(A) $\{Q(\lambda, \mu)\}$ *is a basis of the* **Z**-*module* $V_{\mathbf{Z}}(\tau)$ *and is indexed by* (*distinct*) *admissible pairs on* $X(\tau)$.

(B) $Q(\lambda, \mu)$ *is a weight vector of weight*

$$\tfrac{1}{2}(\lambda(\omega) + \mu(\omega)).$$

(C) *If* $W(\theta)$ *denotes the* **Z**-*submodule of* $V_{\mathbf{Z}}(\tau)$ *spanned by all* $Q(\lambda, \mu)$ *such that* $\theta \geqslant \lambda$, $\phi \geqslant \theta$, *then*

$$W(\theta) = V_{\mathbf{Z}}(\theta)$$

(*we call these* $Q(\lambda, \mu)$ *the basis elements of* $W(\theta)$).

Proof. That $\{Q(\lambda, \mu)\}$ is indexed by distinct admissible pairs on $X(\tau)$ follows from Lemma 2.16. The assertion (B) follows from the way $\{Q(\lambda, \mu)\}$ has been defined.

Let

$$I^e_\phi = \quad (\lambda, \mu) \left\{ \begin{array}{ll} 1. & (\lambda, \mu) \text{ is an admissible pair on } X(\phi) \\ 2. & \langle \lambda(\omega)+\mu(\omega), \alpha^* \rangle > 0 \\ 3. & \phi \not\geqslant s_\alpha \lambda \end{array} \right\}$$

and let $V_1 = V_1(\phi)$ denote the $\mathbf{Z}$-submodule of $V_{\mathbf{Z}}(\phi)$ spanned by $\{Q(\lambda, \mu), (\lambda, \mu) \in I^e_\phi\}$ and let $V_2 = V_2(\phi)$ denote the $\mathbf{Z}$-submodule of $V_{\mathbf{Z}}(\phi)$ spanned by $\{Q(\lambda, \mu), (\lambda, \mu) \notin I^e_\phi\}$. We have therefore $V_{\mathbf{Z}}(\phi) = V_1 \oplus V_2$. We claim that

$$V_{\mathbf{Z}}(\tau) = U_{-\alpha,\mathbf{Z}} V_1 + V_2. \tag{1}$$

On account of Lemma 3.4, to prove the claim (1), it suffices to show that

$$\begin{array}{l} (\lambda, \mu), \text{ an admissible pair on } X, \text{ and} \\ (\lambda, \mu) \notin I^e_\phi, \text{ then } U_{-\alpha,\mathbf{Z}} Q(\lambda, \mu) \subseteq U_{-\alpha,\mathbf{Z}} \cdot V_1 + V_2. \end{array} \tag{1'}$$

Suppose now that $\langle \frac{1}{2}(\lambda(\omega)+\mu(\omega)), \alpha^* \rangle < 0$. Then we claim that $X_{-\alpha} Q(\lambda, \mu) = 0$. For otherwise, the weight of $X_{-\alpha} Q(\lambda, \mu)$ is $\frac{1}{2}(\lambda(\omega)+\mu(\omega)) - \alpha$ and we have

$$\langle \tfrac{1}{2}(\lambda(\omega)+\mu(\omega)) - \alpha, \alpha^* \rangle \leqslant -3$$

which leads to a contradiction (cf. Remark 3.3(c)). Suppose now that (λ, μ) is an admissible pair on $X(\phi)$ such that $\phi \geqslant s_\alpha \lambda$. Then we have either $\lambda \geqslant s_\alpha \lambda$ or $\phi \geqslant \theta = s_\alpha \lambda \geqslant \lambda$. If $\lambda \geqslant s_\alpha \lambda$, then we have (cf. Remark 3.5)

$$U_{-\alpha,\mathbf{Z}} \cdot V_{\mathbf{Z}}(\lambda) \subseteq V_{\mathbf{Z}}(\lambda) \qquad \text{and} \qquad V_{\mathbf{Z}}(\lambda) \subseteq V_{\mathbf{Z}}(\phi).$$

If $\phi \geqslant \theta = s_\alpha \lambda \geqslant \lambda$, then again we have (cf. Remark 3.5)

$$U_{-\alpha,\mathbf{Z}} \cdot V_{\mathbf{Z}}(\theta) \subseteq V_{\mathbf{Z}}(\theta) \qquad \text{and} \qquad V_{\mathbf{Z}}(\theta) \subseteq V_{\mathbf{Z}}(\phi).$$

Thus in this case, we have indeed

$$U_{-\alpha,\mathbf{Z}} Q(\lambda, \mu) \subseteq V_1 + V_2$$

so that, in particular, (1′) follows in this case. Thus to complete the proof of (1′), it remains only to prove it for the case $\langle \frac{1}{2}(\lambda(\omega)+\mu(\omega)), \alpha^* \rangle = 0$. In this case, we set

$$u = X_\alpha Q(\lambda, \mu) \in V_{\mathbf{Z}}(\phi).$$

We observe first that if $u = 0$, then $X_{-\alpha} Q(\lambda, \mu) = 0$. For, at first $Q(\lambda, \mu)$ is a highest weight vector for the $sl(2)$ associated to α and since the weight of $Q(\lambda, \mu)$ is $\frac{1}{2}(\lambda(\omega)+\mu(\omega))$ where $\langle\frac{1}{2}(\lambda(\omega)+\mu(\omega)), \alpha^*\rangle=0$, by the standard $sl(2)$-theory (cf. $[\mathrm{S}]_2$), we see also that $X_{-\alpha}Q(\lambda, \mu)=0$. If $X_{-\alpha}Q(\lambda, \mu)=0$, then the assertion (1′) is immediate. Let then $u\neq 0$. We now claim that

$$\frac{X^2_{-\alpha}}{2!}u=X_{-\alpha}Q(\lambda, \mu). \tag{2}$$

Let us first show how (2) completes the proof of assertion (1′). For this, it suffices to show that

$$X_{-\alpha}Q(\lambda, \mu)\in U_{-\alpha,\mathbf{Z}}V_1+V_2 \tag{3}$$

and hence by (2), it suffices to show that

$$\frac{X^2_{-\alpha}}{2!}U\in U_{-\alpha,\mathbf{Z}}\cdot V_1+V_2. \tag{3′}$$

Since $u\in V_{\mathbf{Z}}(\phi)$ and $\langle$weight of $u, \alpha^*\rangle=2$, we can write

$$\begin{aligned}&u=\sum a_{\theta,\sigma}Q(\theta, \sigma), \qquad a_{\theta,\sigma}\in\mathbf{Z}\\&(\theta, \sigma)\text{ admissible pair on }X(\phi),\ \langle\text{weight of }(\theta, \sigma), \alpha^*\rangle=2.\end{aligned} \tag{4}$$

Now (θ, σ) being as in (4), (θ, σ) either $\in I^e_\phi$ or $\notin I^e_\phi$. In either case, by our discussion above

$$U_{-\alpha,\mathbf{Z}}Q(\theta, \sigma)\subseteq U_{-\alpha,\mathbf{Z}}\cdot V_1+V_2.$$

Now (3′) and hence (3) follows immediately and consequently (1′) also follows. Thus to complete the proof of (1′), it just remains to prove (2) and we proceed as follows:

$$X_{-\alpha}X_\alpha Q(\lambda, \mu)=X_\alpha X_{-\alpha}Q(\lambda, \mu)$$

$$\text{(since }H_\alpha Q(\lambda, \mu)=0,\text{ as }\langle\text{weight of }Q(\lambda, \mu), \alpha^*\rangle=0)$$

$$\begin{aligned}\frac{X^2_{-\alpha}}{2!}u&=\frac{X_{-\alpha}}{2}(X_{-\alpha}X_\alpha Q(\lambda, \mu))\\&=\left(\frac{X_{-\alpha}}{2}X_\alpha\right)X_{-\alpha}Q(\lambda, \mu)\\&=\tfrac{1}{2}X_\alpha X_{-\alpha}X_{-\alpha}Q(\lambda, \mu)+X_{-\alpha}Q(\lambda, \mu)\\&\qquad\left(\text{since }\frac{X_{-\alpha}X_\alpha}{2}=\frac{X_\alpha X_{-\alpha}}{2}+\frac{H_\alpha}{2}\text{ and }\frac{H_\alpha}{2}(X_{-\alpha}Q(\lambda, \mu))\right.\\&=X_{-\alpha}Q(\lambda, \mu),\\&\qquad\left.\text{since }\langle\text{weight of }\langle Q(\lambda, \mu), \alpha^*\rangle=0\right).\end{aligned}$$

Also,

$$X^2_{-\alpha}\,Q(\lambda,\mu)=0$$

for otherwise $\langle$weight of $X^2_{-\alpha}\,Q(\lambda,\mu),\alpha^*\rangle=-4$ and this leads to a contradiction (cf. Remark 3.3(c)). This proves assertion (2) above and thus the proof of assertions (1′) and (1) is complete.

We now claim that

$$X_\alpha Q(\lambda,\mu)=0, \qquad (\lambda,\mu)\in I^e_\phi. \tag{5}$$

This is immediate, since, if $X_\alpha Q(\lambda,\mu)\neq 0$, then $\langle$weight of $X_\alpha Q(\lambda,\mu),\alpha^*\rangle=2+\langle$weight of $Q(\lambda,\mu),\alpha^*\rangle\geqslant 3$, since $(\lambda,\mu)\in I^e_\phi$. Now from (5), by standard $sl(2)$-theory, it follows that $U_{-\alpha,\mathbf{Z}}\,Q(\lambda,\mu)$ is spanned as a $\mathbf{Z}$-module by $X_{-\alpha}\,Q(\lambda,\mu)$ or by $X_{-\alpha}\,Q(\lambda,\mu)$ and $(X^2_{-\alpha}/2!)\,Q(\lambda,\mu)$ according as $\langle$weight of $Q(\lambda,\mu),\alpha^*\rangle=1$ or 2 $((\lambda,\mu)\in I^e_\phi)$. Thus (1) together with this observation shows that $V_{\mathbf{Z}}(\tau)$ is spanned as a $\mathbf{Z}$-module by $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $X(\tau)\}$.

Linear Independence

We prove the linear independence of $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $X(\tau)\}$ by proving the linear independence of $\{Q(\lambda,\mu)\ (\lambda,\mu)$ an admissible pair on $G/P\}$ (P being the maximal parabolic subgroup associated to the fundamental weight ω). To be very precise, starting with $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $X(\tau)\}$, and fixing a chain $\tau_0=\tau<\tau_1<\tau_2<\cdots<\tau_n=w_0$, where $X(\tau_i)$ is a moving divisor in $X(\tau_{i+1})$, $0\leqslant i\leqslant n-1$, we follow the above construction and arrive at the set $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $G/P\}$, which gives a set of generators for the $\mathbf{Z}$-module $V_{\mathbf{Z}}$. Now $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $G/P\}$ generates the $\mathbf{Q}$-vector space $V=V_{\mathbf{Z}}\otimes\mathbf{Q}$ (note that by our notations, $V=V_\omega$, the irreducible G-module with highest weight ω). The linear independence of $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $G/P\}$ follows from the fact that $\dim_{\mathbf{Q}}V=\#\{$admissible pairs (λ,μ) on $G/P\}$ (cf. Remark 3.3(b)) (since $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $G/P\}$ generates V_ω, by our construction). Hence we conclude that it is in fact a $\mathbf{Q}$-basis for V_ω. This in particular implies that $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $G/P\}$ is linearly independent over $\mathbf{Q}$ (and hence also over $\mathbf{Z}$). In particular $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on $X(\tau)\}$ is $\mathbf{Z}$-linearly independent. This concludes the proof of the assertion (A) of Lemma 3.6.

To conclude the proof of Lemma 3.6, it remains to prove the assertion (C). Let then $\theta\in W/W_P$ be such that $\tau\geqslant\theta$. Consider first the case $\phi\geqslant\theta$. Then because of the hypothesis on $\{Q(\lambda,\mu),\ (\lambda,\mu)$ an admissible pair on

$X(\phi)\}$ as well as the new $Q(\lambda, \mu)$ which have been defined, we see immediately in this case that $W(\theta) = V_{\mathbf{Z}}(\theta)$. Suppose then that $\phi \not\geqslant \theta$. Then we have $\theta_1 \in W/W_P$, $s_\alpha \theta_1 = \theta$ such that $\phi \geqslant \theta_1$ (cf. Lemma 1.5) so that $W(\theta_1) = V_{\mathbf{Z}}(\theta_1)$. By Lemma 3.4, we have $U_{-\alpha,\mathbf{Z}} \cdot V_{\mathbf{Z}}(\theta_1) = V_{\mathbf{Z}}(\theta)$ and hence

$$U_{-\alpha,\mathbf{Z}}\, W(\theta_1) = V_{\mathbf{Z}}(\theta). \tag{6}$$

The relation (6) implies that

$$V_{\mathbf{Z}}(\theta) = U_{-\alpha,\mathbf{Z}} \cdot V_1(\theta_1) + W(\theta_1)$$

where $V_1(\theta_1)$ is the $\mathbf{Z}$-submodule of $V_{\mathbf{Z}}(\theta)$ generated by $Q(\lambda_1, \mu_1)$ such that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ (the proof being the same as that of assertion (A)). Suppose now that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ is such that $(\lambda_1, \mu_1) \in I^e_\phi$. Then by the way in which the new basis elements (relative to ϕ and τ) have been defined, we see that $U_{-\alpha,\mathbf{Z}}\, Q(\lambda_1, \mu_1) \subseteq W(\theta)$. Suppose on the other hand that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ is such that $(\lambda_1, \mu_1) \notin I^e_\phi$. This means that $\phi \geqslant \lambda$ where $\lambda = s_\alpha \lambda_1$ (note that $X(\lambda_1)$ is a moving divisor in $X(\lambda)$ moved by α). By hypothesis, we have $V_{\mathbf{Z}}(\lambda) = W(\lambda)$; further, since $\lambda \geqslant s_\alpha \lambda = \lambda_1$, we have, by Remark 3.5, that

$$U_{-\alpha,\mathbf{Z}} \cdot V_{\mathbf{Z}}(\lambda) \subseteq V_{\mathbf{Z}}(\lambda).$$

Further, $Q(\lambda_1, \mu_1) \in V_{\mathbf{Z}}(\lambda_1) = W(\lambda_1) \subseteq V_{\mathbf{Z}}(\lambda) = W(\lambda) \subseteq W(\theta)$. Hence $U_{-\alpha,\mathbf{Z}}\, Q(\lambda_1, \mu_1) \subseteq W(\theta)$. Thus we conclude that $U_{-\alpha,\mathbf{Z}} \cdot V_1(\theta_1) \subseteq W(\theta)$ and since $W(\theta_1) \subseteq W(\theta)$, it follows that $V_{\mathbf{Z}}(\theta) \subseteq W(\theta)$. Thus to show that $V_{\mathbf{Z}}(\theta) = W(\theta)$, it finally remains to be shown that $W(\theta) \subseteq V_{\mathbf{Z}}(\theta)$. Let then $Q(\lambda, \mu)$ be a basis element of $W(\theta)$. Suppose that $\phi \geqslant \theta$, then $Q(\lambda, \mu) \in V_{\mathbf{Z}}(\lambda)$, since by hypothesis $Q(\lambda, \mu) \in W(\lambda) = V_{\mathbf{Z}}(\lambda)$. But we have $V_{\mathbf{Z}}(\lambda) \subseteq V_{\mathbf{Z}}(\theta)$ since $\theta \geqslant \lambda$, so that $Q(\lambda, \mu) \in V_{\mathbf{Z}}(\theta)$. Suppose on the other hand that $\phi \not\geqslant \lambda$, then if $\lambda_1 = s_\alpha \lambda$, $X(\lambda_1)$ is a moving divisor in $X(\lambda)$ moved by α and $\phi \geqslant \lambda_1$ (cf. Lemma 1.5). Further by the construction of the new basis elements, we have

$$Q(\lambda, \mu) \in U_{-\alpha,\mathbf{Z}} \cdot Q(\lambda_1, \mu_1), \qquad (\lambda_1, \mu_1) \in I^e_\phi$$

where $Q(\lambda_1, \mu_1) \in W(\theta_1)$. The relation (6) implies then that $Q(\lambda, \mu) \in V_{\mathbf{Z}}(\theta)$. This proves that $W(\theta) \subseteq V_{\mathbf{Z}}(\theta)$ and thus we conclude that $V_{\mathbf{Z}}(\theta) = W(\theta)$ and the proof of Lemma 3.6 is now complete.

COROLLARY 3.7. *Let $\tau \in W/W_P$. Then there exists a basis $\{Q(\lambda, \mu)\}$ of the $\mathbf{Z}$-module $V_{\mathbf{Z}}(\tau)$ indexed by admissible pairs (λ, μ) on $X(\tau)$ having the following properties*:

(B) *$Q(\lambda, \mu)$ is a weight vector of weight*

$$\tfrac{1}{2}(\lambda(\omega)+\mu(\omega)).$$

(C) *If $W(\theta)$ denotes the $\mathbf{Z}$-submodules of $V_{\mathbf{Z}}(\tau)$ spanned by all $\{Q(\lambda, \mu)\}$ such that $\theta \geqslant \lambda$, $\tau \geqslant \theta$, then $W(\theta)=V_{\mathbf{Z}}(\theta)$. In particular $V_{\mathbf{Z}}(\theta)$ is a direct summand in $V_{\mathbf{Z}}(\tau)$.*

This corollary is an immediate consequence of Lemma 3.6.

Remark 3.8. We would like to remark that once a choice of the extremal weight vectors $Q(\mu)$ has been made, the construction of the vectors $Q(\lambda, \mu)$ (as given by Lemma 3.6) is in fact canonical; to make it very precise, we shall now prove the following: Let (λ, μ) be an admissible pair on $X(\tau)$. Then the vector $Q(\lambda, \mu)$ (as constructed in Lemma 3.6) is given by

$$Q(\lambda, \mu)=X_{-\alpha_1}\cdots X_{-\alpha_r} Q(\mu) \qquad (*)$$

where $\lambda=\lambda_0>\lambda_1>\cdots>\lambda_n=\mu$, $X(\lambda_i)$ is a divisor in $X(\lambda_{i-1})$, and $\lambda_{i-1}=s_\alpha \lambda_i$, $1 \leqslant i \leqslant n$. In particular, $Q(\lambda, \mu)$ is uniquely determined by the admissible pair (λ, μ) (and does not depend on the path from $X(\lambda)$ to $X(\mu)$).

To prove $(*)$, we proceed as follows. Having fixed a moving divisor $X(\phi)$ in $X(\tau)$ moved by α (as in Lemma 3.6), let $X(w)$ be another Schubert divisor in $X(\tau)$; further let $m(w, \tau)$ be 2 (cf. Definition 2.1). Let $w=s_{\alpha_1}\tau$, so that α_1 is simple (cf. Lemma 2.6). Let $X(\phi_1)$ be the divisor in $X(\phi)$ moved out to $X(w)$ by α. Now we claim that

(a) $\phi_1=s_{\alpha_1}\phi$

(b) s_α and s_{α_1} commute $\qquad (**)$

(c) multiplicity of $X(\phi_1)$ in $X(w)=$ multiplicty of $X(\phi)$ in $X(\tau)$.

Proof. Let $\phi_1=s_\beta \phi$ where β is some positive root (possibly non-simple). Now

$$\tau=s_\alpha \phi=s_\alpha s_\beta \phi_1$$

and

$$\tau=s_{\alpha_1} w=s_{\alpha_1} s_\alpha \phi_1 .$$

Hence $s_{\alpha_1}s_\alpha = s_\alpha s_\beta$ (taking τ, ϕ, ϕ_1, w, etc., to be minimal representatives), i.e.,

$$s_\beta = s_\alpha s_{\alpha_1} s_\alpha$$

which implies that

$$\beta = s_\alpha(\alpha_1).$$

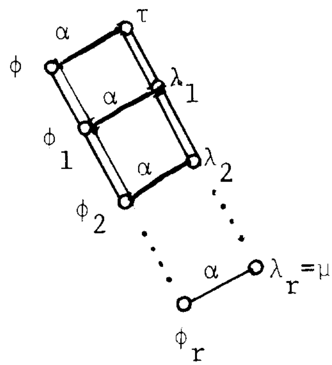
If s_α and s_{α_1} do not commute, let $s_\alpha(\alpha_1) = \alpha_1 + n\alpha$ where n is a *nonzero* positive integer. Now

$$\begin{aligned}\langle \phi(\omega), \alpha_1^* \rangle = \langle s_\alpha \tau(\omega), \alpha_1^* \rangle &= 2\frac{(s_\alpha \tau(\omega), \alpha_1)}{(\alpha_1, \alpha_1)}\\ &= \frac{2}{(\alpha_1, \alpha_1)}(\tau(\omega), s_\alpha(\alpha_1))\\ &= \frac{2}{(\alpha_1, \alpha_1)}(\tau(\omega), \alpha_1 + n\alpha)\\ &= \langle \tau(\omega), \alpha_1^* \rangle + \frac{(\alpha, \alpha)}{(\alpha_1, \alpha_1)} n \langle \tau(\omega), \alpha^* \rangle\\ &= -2 - n\frac{(\alpha, \alpha)}{(\alpha_1, \alpha_1)} a\end{aligned}$$

where $a = \langle \phi(\omega), \alpha^* \rangle$ (note that $a > 0$ (cf. Lemma 1.2) and that $\langle \tau(\omega), \alpha_1^* \rangle = -2$). Thus we obtain that $\langle \phi(\omega), \alpha_1^* \rangle < -2$ which is not possible (since ω is of classical type). From this it follows that s_α and s_{α_1} commute. This proves (b) of (**). Now (b) implies that $\beta = \alpha_1$ (since, from above, $\beta = s_\alpha(\alpha_1)$) and from this (a) follows. The assertion (c) of (**) follows by interchanging the rolls of ϕ and ω and using Lemma 2.5. This completes the proof of (**) above.

Now we return to the proof of (*). We prove (*) by induction on $\dim X(\tau)$ so that it is enough to prove (*) for $\lambda = \tau$. Given any chain $\lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_n = \mu$ (where $X(\lambda_i)$ is a Schubert divisor in $X(\lambda_{i-1})$, $1 \leqslant i \leqslant n$), we first observe that $X(\lambda_i)$ is in fact a double divisor in $X(\lambda_{i-1})$, $1 \leqslant i \leqslant n$ (cf. Remark 2.17). Hence, if $\lambda_i = s_{\alpha_i}\lambda_{i-1}$, then α_i is simple (cf. Lemma 2.6). We now distinguish the following two cases.

Case 1: $\langle \tau(\omega), \alpha^* \rangle = -1$. Let $X(\phi_1)$ be the divisor in $X(\phi)$ moved out to $X(\lambda_1)$ by α. Then by (c) of (**) (taking $w=\lambda_1$) we obtain that $\langle \lambda_1(\omega), \alpha^* \rangle = -1$. Thus continuing we obtain a path



$\phi_0=\phi>\phi_1>\phi_2>\cdots>\phi_r=\theta$ such that $\phi_i=s_\alpha\lambda_i$, $\phi_{i-1}=s_{\alpha_i}\phi_i$, and multiplicity of $X(\phi_i)$ in $X(\phi_{i-1})$ is 2, $1\leqslant i\leqslant r$. Further s_α commutes with s_{α_i}, $1\leqslant i\leqslant r$ (cf. (b) of (**)). Now, by our construction, $X_{-\alpha}Q(\phi,\theta)=Q(\tau,\mu)$ (where $\theta=\phi_r=s_\alpha\mu$). On the other hand $Q(\phi,\theta)=X_{-\alpha_1}\cdots X_{-\alpha_r}Q(\theta)$ (by the induction hypothesis). Now $X_{-\alpha_1}X_{-\alpha_2}\cdots X_{-\alpha_r}Q(\mu)=X_{-\alpha_1}\cdots X_{-\alpha_r}X_{-\alpha}Q(\theta)$ (since $\langle\theta(\omega),\alpha^*\rangle=1$, $X_{-\alpha}Q(\theta)=Q(\mu)$) $=X_{-\alpha}X_{-\alpha_1}\cdots X_{-\alpha_r}Q(\theta)=X_{-\alpha}Q(\phi,\theta)$ (since s_α commutes with s_{α_i}, we have $X_{-\alpha}X_{-\alpha_i}=X_{-\alpha_i}X_{-\alpha}$, $1\leqslant i\leqslant r$). Thus we obtain

$$X_{-\alpha_1}\cdots X_{-\alpha_r}Q(\mu)=X_{-\alpha}Q(\phi, s_\alpha\mu). \tag{1}$$

Now from (1) it follows that $Q(\tau,\mu)=X_{-\alpha_1}\cdots X_{-\alpha_r}Q(\mu)$ (since $X_{-\alpha}Q(\phi,\theta)=Q(\tau,\mu)$). We also obtain from (1) that for any two paths

$$\tau=\lambda>\lambda_1>\cdots>\lambda_r=\mu, \qquad \lambda_{i-1}=s_{\alpha_i}\lambda_i$$

and

$$\tau=\lambda>w_1>\cdots>w_r=\mu \qquad w_{i-1}=s_{\beta_i}w_i$$

we have

$$X_{-\alpha_1}\cdots X_{-\alpha_r}Q(\mu)=X_{-\beta_1}\cdots X_{-\beta_r}Q(\mu)=Q(\tau,\mu). \tag{2}$$

This completes the proof of (*) in this case.

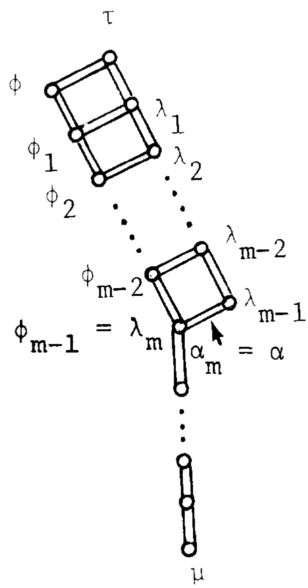
Case 2: $\langle\tau(\omega),\alpha^*\rangle=-2$. In this case, we have the following two possibilities. Either $\alpha\neq\alpha_i$, $1\leqslant i\leqslant r$, in which case, denoting by $X(\phi_i)$ the divisor in $X(\phi_{i-1})$ moved out to $X(\lambda_i)$ by α, we obtain a chain of double divisors (using (**)), $\phi > \phi_1 > \cdots > \phi_r = \theta$, or $\alpha = \alpha_i$, for some i, $1 \leqslant i \leqslant r$. In the former case, we have, by our construction,

$$Q(\tau, \mu) = \frac{X_{-\alpha}^2}{2!} Q(\phi, \theta)$$

(note that $\theta = s_\alpha \mu$). Now using the facts that s_α commutes with s_{α_i}, $1 \leqslant i \leqslant r$ (cf. (**) above) and $Q(\phi, \theta) = X_{-\alpha_1} \cdots X_{-\alpha_r} Q(\theta)$, we obtain

$$\begin{aligned} Q(\tau, \mu) &= \frac{X_{-\alpha}^2}{2!} X_{-\alpha_1} X_{-\alpha_2} \cdots X_{-\alpha_r} Q(\theta) \\ &= X_{-\alpha_1} X_{-\alpha_2} \cdots X_{-\alpha_r} \frac{X_{-\alpha}^2}{2!} Q(\theta) \\ &= X_{-\alpha_1} X_{-\alpha_2} \cdots X_{-\alpha_r} Q(\mu) \end{aligned}$$

(note that $Q(\mu) = (X_{-\alpha}^2/2!)\, Q(\theta)$, since $\langle \theta(\omega), \alpha^* \rangle = 2$ and $\mu = s_\alpha \theta$). In the latter case, let m be the smallest integer $\leqslant r$ such that $\alpha_m = \alpha$. Then again, we have (cf. (**) above) that s_α commutes with s_{α_i}, $1 \leqslant i \leqslant m-1$. Also, we have $\phi_{m-1} = \lambda_m$



Now

$$\begin{aligned} X_{-\alpha_1} \cdots X_{-\alpha_m} \cdots X_{-\alpha_r} Q(\mu) &= X_{-\alpha_m} X_{-\alpha_1} \cdots X_{-\alpha_{m-1}} X_{-\alpha_{m+1}} \cdots X_{-\alpha_r} Q(\mu) \\ &= X_{-\alpha} Q(\phi, \mu) \end{aligned}$$

(note that $\alpha_m = \alpha$ and s_α commutes with s_{α_i}, $1 \leqslant i \leqslant m-1$). On the other hand, by our construction $X_{-\alpha} Q(\phi, \mu) = Q(\tau, \mu)$. Thus $Q(\tau, \mu) = X_{-\alpha_1} \cdots X_{-\alpha_r} Q(\mu)$. This completes the proof of $(*)$.

Let us recall that

$$\mathbf{P}(V_{\mathbf{Z}}^*)(\mathbf{Z}) = \{\text{direct summands of } V_{\mathbf{Z}} \text{ of rank } 1\}.$$

If $v \in V_{\mathbf{Z}}$ is such that $\mathbf{Z}v$ is a direct summand in $V_{\mathbf{Z}}$, we denote by $\bar{v}$ the $\mathbf{Z}$-valued point of $\mathbf{P}(V_{\mathbf{Z}}^*)$ defined by $\mathbf{Z}v$. We observe that $\mathbf{Z}e$ is a direct summand in $V_{\mathbf{Z}}$ (this is, for example, a consequence of the assertion (C) of Corollary 3.7).

PROPOSITION 3.9. *Let $\bar{e}$ be the $\mathbf{Z}$-valued point of $\mathbf{P}(V_{\mathbf{Z}}^*)$ defined by $\mathbf{Z}e$. Then the isotropy subgroup scheme of $G_{\mathbf{Z}}$ at $\bar{e}$ is $P_{\mathbf{Z}}$. We obtain therefore a canonical closed immersion*

$$j\colon G_{\mathbf{Z}}/P_{\mathbf{Z}} \to \mathbf{P}(V_{\mathbf{Z}}^*).$$

The pull-back by j of the tautological line bundle on $\mathbf{P}(V_{\mathbf{Z}}^*)$ is the ample generator $L_{\mathbf{Z}}$ of Pic $G_{\mathbf{Z}}/P_{\mathbf{Z}}$. *In particular $L_{\mathbf{Z}}$ is very ample.*

Proof. Let H be the isotropy subgroup scheme of $G_{\mathbf{Z}}$ at $\bar{e}$. (Note that $P_{\mathbf{Z}} \subseteq H$.) In order to prove that $P_{\mathbf{Z}}$ is the isotropy subgroup scheme of $G_{\mathbf{Z}}$ at $\bar{e}$, it is clear that it suffices to show that

$$\operatorname{Lie}(P_{\mathbf{Z}} \otimes k) = \operatorname{Lie}(H \otimes k), \qquad \text{for every field } k. \tag{1}$$

Since

$$\operatorname{Lie}(G_{\mathbf{Z}} \otimes k) = (\operatorname{Lie} B_{\mathbf{Z}} \otimes \operatorname{Lie}(B_{\mathbf{Z}}^-)^u) \otimes k$$

where $(B_{\mathbf{Z}}^-)^u$ is the unipotent part of the Borel subgroup scheme $B_{\mathbf{Z}}^-$ *opposite to* $B_{\mathbf{Z}}$, to prove (1) it suffices to show

$$\operatorname{Lie}((H \cap (B_{\mathbf{Z}}^-)^u) \otimes k) \subseteq \operatorname{Lie}((P \cap (B_{\mathbf{Z}}^-)^u) \otimes k) \tag{2}$$

for every field k (since $B_{\mathbf{Z}} \subset P_{\mathbf{Z}} \subseteq H$). Now observing that $H \cap (B_{\mathbf{Z}}^-)^u$ is the isotropy subgroup scheme of the unipotent group scheme $(B_{\mathbf{Z}}^-)^u$ at the $\mathbf{Z}$-valued point of Spec $S(V_{\mathbf{Z}}^*)$ represented by $e \in V_{\mathbf{Z}}$, we see that if $e \otimes 1$ denotes the image of e under the canonical map $V_{\mathbf{Z}} \to V_{\mathbf{Z}} \otimes k$, then the subalgebra of $\operatorname{Lie}((B_{\mathbf{Z}}^-)^u \otimes k)$ which annihilates $e \otimes 1$ is precisely $\operatorname{Lie}((H \cap (B_{\mathbf{Z}}^-)^u) \otimes k)$. Thus, to prove (2), it suffices to show that

$$\text{Annihilator of } e \otimes 1 \text{ in } \operatorname{Lie}((B_{\mathbf{Z}}^-)^u \otimes k) = \operatorname{Lie}((B_{\mathbf{Z}}^-)^u \cap P_{\mathbf{Z}}) \otimes k). \tag{3}$$

Now

$$\text{Lie}(((B_{\mathbf{Z}}^{-})^{u} \cap P_{\mathbf{Z}}) \otimes k) = k\text{-span of } \{X_{-\alpha} \otimes 1\}, \qquad \alpha \in \Delta_P^+$$

where Δ_P^+ is the subset of Δ^+ (= the set of positive roots) spanned by the set of simple roots S_P associated to P. Hence, to prove (3), it suffices to prove

$$(X_{-\alpha} \otimes 1)(e \otimes 1) \neq 0 \text{ in } (V_{\mathbf{Z}} \otimes k) \text{ for every field } k,$$
$$\text{equivalently, } X_{-\alpha}e \text{ is } \textit{unimodular} \text{ in } V_{\mathbf{Z}} \text{ for } \alpha \in \Delta^+ - \Delta_P^+. \tag{4}$$

The proof of (4) follows from the following

LEMMA 3.10.(Deodhar). *Let V_λ be the irreducible G-module with highest weight λ and let e be a highest weight vector in V_λ; let $V_{\mathbf{Z}} = U_{\mathbf{Z}} \cdot e$. Let $\beta \in \Delta^+$. If $X_{-\beta}e \neq 0$, then $X_{-\beta}e$ is primitive in (the* **Z** *lattice) $V_{\mathbf{Z}}$ (here λ could be an arbitrary dominant weight).*

Proof. The proof of the Lemma is quite easy and may be obtained by using the properties of root systems and some commutation relations in Lie($G_{\mathbf{Z}}$) (for details of proof of Lemma 3.10 we refer the readers to Section 5 of [L-M-S]$_3$).

Let k be a field and let $\tau \in W/W_P$. Since we have

$$X_{\mathbf{Z}}(\tau)(k) \subset \mathbf{P}(V_{\mathbf{Z}}^*)(k) = V_k - (0)/k^*$$

where $V_{\mathbf{Z}} = U_{\mathbf{Z}}e$, e being a highest weight vector in the irreducible G-module (over **Q**) with highest weight ω and $V_k = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$, we can talk of the k-linear subspace of V_k generated by $X_{\mathbf{Z}}(\tau)(k)$. Also note that we have a canonical k-linear map

$$j_k\colon V_k(\tau) = V_{\mathbf{Z}}(\tau) \otimes k \to V_{\mathbf{Z}} \otimes k = V_k.$$

We have

PROPOSITION 3.11. *Let ω be a fundamental weight of classical type. Then*

(i) *j_k is an injection,*

(ii) *$V_k(\tau)$ is the subspace of V_k generated by $X_{\mathbf{Z}}(\tau) \otimes k$,*

(iii) char$(V_k(\tau))^* = M_\tau(\exp(-\omega))$

(M_τ being the operator as defined in the beginning of this section).

Proof. Part (i) follows form the fact that $V_{\mathbf{Z}}(\tau)$ is a direct summand in $V_{\mathbf{Z}}$ (cf. (C) of Corollary 3.7).

(ii) To prove (ii), we shall more generally prove the following. Let

$$\lambda=\sum_{i=1}^{r} a_i\omega_i, \qquad a_i>0$$

be a dominant weight. Let $Q=\bigcap_{i=1}^{r} P_i$ and $X(\tau)$ a Schubert variety in G/Q. Now since

$$X(\tau)\subset \mathbf{P}(V_\lambda^*)=(V_\lambda-(0))/k^*$$

(where $V_\lambda=V_{\lambda,\mathbf{Z}}\otimes k$) we can talk of the k-linear subspace of V_λ generated by $X(\tau)$. We have a canonical map

$$V_\lambda(\tau)=V(\tau)_{\lambda,\mathbf{Z}}\otimes k\to V_{\lambda,\mathbf{Z}}\otimes k=V_\lambda.$$

We denote by Im $V_\lambda(\tau)$ the image of the above map. Then we have (cf. [Se]$_2$)

LEMMA 3.12. *Notations being as above,*

$$\text{Im } V_\lambda(\tau)=k\text{-linear subspace of } V_\lambda \text{ generated}$$

by $X(\tau)$.

Proof. (by induction on dim $X(\tau)$). When dim $X(\tau)=0$, $V_{\lambda,\mathbf{Z}}(\tau)\approx\mathbf{Z}e$ (a direct summand of $V_{\lambda,\mathbf{Z}}$; here e is a highest weight vector in $V_{\lambda,\mathbf{Z}}\otimes\mathbf{Q}$ (which is unique up to scalars)) and $X_{\mathbf{Z}}(\tau)\approx\operatorname{Spec}\mathbf{Z}$ and the lemma is immediate. Now let dim $X(\tau)\geqslant 1$ and let $X(\phi)$ be a moving divisor in $X(\tau)$ moved by α. We see easily that it suffices to prove the lemma when the field k is algebraically closed. Let $q\in V_{\lambda,\mathbf{Z}}(\phi)$ and $\bar q$ the canonical image of q in $V_\lambda(\phi)$. Now $V_{\lambda,\mathbf{Z}}$ is a $G_{-\alpha,\mathbf{Z}}$ or equivalently $U_{-\alpha,\mathbf{Z}}$-module and V_λ is a $G_{-\alpha}$-module or equivalently $U_{-\alpha}$ module. The element $t\cdot\bar q$, when we identify t with an element of $G_{-\alpha}(k)\approx k$, is given by

$$t\cdot\bar q=\exp(tX_{-\alpha})\,\bar q=\left(1+tX_{-\alpha}+\cdots+t^n\frac{X_{-\alpha}^n}{n!}\right)\bar q \tag{$*$}$$

(where we choose n such that $(X_{-\alpha}^{n+1}/n+1)\cdot\bar q=0$). Since k is algebraically closed, we can find $t_1,\ldots,t_{n+1}\in k$ such that the Vandermonde determinant

$$\det\Delta=\det\begin{vmatrix} 1 & t_1 & \cdots & t_1^n\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ 1 & t_{n+1} & \cdots & t_{n+1}^n\end{vmatrix}\neq 0.$$

Set

$$\Delta \begin{bmatrix} \bar{q} \\ X_{-\alpha}\bar{q} \\ \vdots \\ X^n_{-\alpha}\bar{q} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \vdots \\ \vdots \\ \theta_n \end{bmatrix} \qquad (n+1 \times 1 \text{ matrix})$$

where $\theta_1 \in$ subspace of V_λ generated by $X(\tau)$ (by induction hypothesis Im $V_\lambda(\phi)$ is the subspace of V_λ generated by $X(\phi)$ and $X(\tau)$ is stable under $G_{-\alpha}$, etc.). Hence

$$\begin{bmatrix} \bar{q} \\ X_{-\alpha}\bar{q} \\ \vdots \\ X^n_{-\alpha}\bar{q} \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \theta_1 \\ \vdots \\ \vdots \\ \theta_{n+1} \end{bmatrix}.$$

From this we conclude (using Lemma 3.4) that

$$\text{Im } V_\lambda(\tau) \subset k \cdot \text{linear subspace of } V_\lambda \text{ generated by } X(\tau).$$

It remains to prove the inclusion in the other direction. It is not difficult to see that the image of the map

$$G_{-\alpha} \times X(\phi) \to X(\tau)$$

contains a non-empty open subset of $X(\tau)$ (since te image contains $X(\phi)$ and τ, etc.). Hence the k-linear subspace of V_λ spanned by $X(\tau)$ is the k-linear subspace of V_λ spanned by $G_{-\alpha} \cdot X(\phi)$. We have

$$G_{-\alpha} \cdot X(\phi) \subset G_{-\alpha}(V_{\lambda,\mathbf{Z}}(\phi) \otimes k).$$

It is clear (see $(*)$ above) that the R.H.S. is contained in

$$(U_{-\alpha,\mathbf{Z}} \cdot V_{\lambda,\mathbf{Z}}(\phi)) \otimes k = V_{\lambda,\mathbf{Z}}(\tau) \otimes k.$$

This completes the proof of Lemma 3.12. Now Lemma 3.12 together with (i) proves the assertion (ii) of Proposition 3.11. The assertion (ii), follows from Proposition 3.1. To make it very precise, the vector space $V_k(\tau)$ has a basis $\{q(\lambda, \mu), (\lambda, \mu)$ an admissible pair on $X(\tau)\}$ where $q(\lambda, \mu)$ is a weight vector of weight $\frac{1}{2}(\lambda(\omega) + \mu(\omega))$ (cf. Corollary 3.7; here $q(\lambda, \mu) = Q(\lambda, \mu) \otimes 1$). Hence

$$\text{ch } V_k(\tau) = \sum \exp[\tfrac{1}{2}(\lambda(\omega) + \mu(\omega))] \qquad (\dagger)$$

where the summation on the R.H.S. is over all admissible pairs (λ, μ) on

$X(\tau)$. But then by Proposition 3.1, the R.H.S of (†) is precisely $M_\tau(\exp(-\omega))$. Hence we obtain

$$\operatorname{ch} V_k(\tau)=M_\tau(\exp(-\omega)).$$

This completes the proof of assertion (iii) (and hence also of Proposition 3.11).

Remark 3.13. Now, we have a canonical isomorphism

$$H^0(\mathbf{P}(V_{\lambda,\mathbf{Z}}^*), L_{\lambda,\mathbf{Z}})\approx V_{\lambda,\mathbf{Z}}^*.$$

Lemma 3.12 says that the smallest projective subspace containing $X(\tau)$ is $(\operatorname{Im} V_\lambda(\tau))^*$. Hence the above canonical isomorphism induces a canonical k-linear map

$$j_k\colon (V_\lambda(\tau))^*\to H^0(X(\tau), L_\lambda)$$

and the image of this linear map can be canonically identified with $(\operatorname{Im} V_\lambda(\tau))^*$. In particular, taking $\lambda=\omega$, we obtain (in view of Proposition 3.11(i)) an injective k-linear map

$$j_k\colon (V(\tau))^*\hookrightarrow H^0(X(\tau), L)$$

(here $V(\tau)=V_{\omega,\mathbf{Z}}(\tau)\otimes k$, $L=L_{\omega,\mathbf{Z}}\otimes k$, etc.).

Remark 3.14. Taking $X_{\mathbf{Z}}(\tau)=G_{\mathbf{Z}}/P_{\mathbf{Z}}$, we obtain that the map j_k in Remark 3.11 is an isomorphism for all fields (using the well-known fact that $j_{\mathbf{Q}}$ is an isomorphism and the vanishing theorem (cf. [A], [Ha], or [K])). Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} V_k^* & \approx & H^0(G_k/P_k, L_k) \\ \big\downarrow & & \big\downarrow \\ (V_k(\tau))^* & \hookrightarrow & H^0(X_k(\tau), L_k) \end{array} \qquad \text{(restriction map).}$$

THEOREM 3.15. (First Basis Theorem). *There exists a basis $\{P(\lambda,\mu)\}$ of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$ indexed by admissible pairs (λ,μ) in W/W_P having the following properties*:

(i) *$P(\lambda,\mu)$ is a weight vector (for the $T_{\mathbf{Z}}$-action) and is of weight*

$$-\tfrac{1}{2}(\lambda(\omega)+\mu(\omega))$$

(ii) *the canonical rational morphism*

$$G_{\mathbf{Z}}/P_{\mathbf{Z}} \to \mathbf{P}(H^0(G_{\mathbf{Z}}, L_{\mathbf{Z}}))$$

is a closed imersion.

(iii) *Let* k *be any field. Set* $p(\lambda, \mu) = P(\lambda, \mu) \otimes 1$, $p(\lambda, \mu)$ *being the canonical image of* $P(\lambda, \mu)$ *in* $H^0(G_{\mathbf{Z}} \otimes k/P_{\mathbf{Z}} \otimes k, L_{\mathbf{Z}} \otimes k)$. *Then the restriction of* $p(\lambda, \mu)$ *to* $X_{\mathbf{Z}}(\tau) \otimes k$ *is not identically zero if and only if* $\tau \geqslant \lambda$.

Proof. The module $V_{\mathbf{Z}}$ has a basis $\{Q(\lambda, \mu)\}$, indexed by admissible pairs (λ, μ) in W/W_P as in Corollary 3.7. We now set $\{P(\lambda, \mu)\}$ to be the basis in $V_{\mathbf{Z}}^*$ dual to $\{Q(\lambda, \mu)\}$ (note that $V_{\mathbf{Z}}^* = H^0(G_{\mathbf{Z}}/B_{\mathbf{Z}}, L_{\mathbf{Z}})$—cf. Remark 3.14, for example). The assertion (i) is an immediate consequence of property (B) of Corollary 3.7. The assertion (ii) is just Proposition 3.9. To prove assertion (iii), first, it is clear that the kernel of the canonical homomorphism $V^*{}_k \to V_k(\tau)^*$ contains all $p(\lambda, \mu)$ such that $\tau \not\geqslant \lambda$. Hence the proof of assertion (iii) would be complete, once we show that if $\tau \geqslant \lambda$, then the restriction of $p(\lambda, \mu)$ to $X_k(\tau)$ is non-zero. But this follows in view of the injectivity of the map

$$V_k(\tau)^* \hookrightarrow H^0(X_k(\tau), L_k)$$

(cf. Remark 3.13). This completes the proof of Theorem 3.15.

Remark 3.16. (i) For any $\tau \in W/W_P$, the set $\{p(\lambda, \mu)/\tau \geqslant \lambda\}$ is a basis for $(V(\tau))^*$. In particular, the set $\{p(\lambda, \mu)/\tau \geqslant \lambda\}$ is a linearly independent set.

For, we have $V(\tau)^*$ is generated by $\{p(\lambda, \mu)/\tau \geqslant \lambda\}$ and further $\dim V(\tau)^* = \dim V(\tau) = \#\{\text{admissible pairs on } X(\tau)\}$ (cf. Corollary 3.7).

(ii) We observe that the basis $\{p(\lambda, \mu)\}$ is in fact canonical. For this, we first note that the elements $p(\tau) = p(\tau, \tau)$, being the extremal weight vectors, are uniquely determined up to ± 1 (in view of (iii) of Theorem 3.15). From this the uniqueness of $p(\lambda, \mu)$, (λ, μ) being a non-trivial admissible pair follows (cf. Remark 3.8).

(iii) It should be remarked, however, that the properties (i), (ii), (iii) of Theorem 3.15 do not characterize the set $\{p(\lambda, \mu)\}$ (cf. [L-M-S]$_3$, Remark 5.11]).

We shall now prove some lemmas which are for later use.

LEMMA 3.17. *Let* $\phi \in W/W_P$. *Then the* $\mathbf{Z}$*-submodule,* V, *of the* $B_{\mathbf{Z}}$ *module* $H^0(X_{\mathbf{Z}}(\phi), L_{\omega, \mathbf{Z}})$ *generated by all elements of the form* $P(\phi, \lambda)$ *is* $B_{\mathbf{Z}}$ *stable.*

Proof. We observe that any $F \in V$ vanishes on all the Schubert subschemes of codimension one or equivalently every Schubert subscheme

$X_{\mathbf{Z}}(\theta)$ such that $\theta<\phi$. Since obviously every Schubert scheme is $B_{\mathbf{Z}}$ stable, we deduced that if V' is the $B_{\mathbf{Z}}$-submodule of $H^0(X_{\mathbf{Z}}(\omega), L_{\omega,\mathbf{Z}})$, which is the $B_{\mathbf{Z}}$ span of V, every element $G\in V'$ vanishes on every $X_{\mathbf{Z}}(\theta)$ such that $\theta<\phi$. Suppose that $G\in V'$ is of the form

$$G=\sum_{i=1}^{t} a_i P(\lambda_i, \mu_i), \qquad a_i\in\mathbf{Z}, a_i\neq 0,$$

(λ_i, μ_i) are distinct admissible pairs and say $\phi>\lambda_1$ and λ_1 is a minimal element among $\{\lambda_i\}$.

We observe now that $G\,|\,X_{\mathbf{Z}}(\lambda_1)\neq 0$, which leads to a contradiction. This proves Lemma 3.17.

LEMMA 3.18. *Let $\phi\in W/W_P$ and α be* any *simple root. Then in the $B_{\mathbf{Z}}$-module (or equivalently the $U_{\mathbf{Z}}^+$-module, where $U_{\mathbf{Z}}^+$ denotes the enveloping algebra associated to* $\mathrm{Lie}(B_{\mathbf{Z}})$) *$H^0(X_{\mathbf{Z}}(\phi), L_{\omega,\mathbf{Z}})$ we have the following relations*:

(a) $X_\alpha P(\phi, \lambda)=a\,P(\phi, \mu)$, $a\in\mathbf{Z}$.

Furhter, if $a\neq 0$, we have necessarily the following:

$s_\alpha\lambda=\mu$ with $\mu>\lambda$ and $\langle\lambda(\omega), \alpha^*\rangle=2$, i.e., $X_{\mathbf{Z}}(\lambda)$ is a double divisor in $X_{\mathbf{Z}}(\mu)$ moved by α

(b) $X_\alpha P(\phi)=0$,

(c) $X_\alpha^2 P(\phi, \lambda)=0$.

In the above relations X_α denotes the usual element of $U_{\mathbf{Z}}^+$ associated to α.

Proof. By Lemma 3.17 it follows that

$$X_\alpha P(\phi, \lambda)=\sum_{i=1}^{s} a_i P(\phi, \mu_i), \qquad a_i\in\mathbf{Z}$$

where we can assume that μ_i are distinct. We see that

$$\text{weight of } P(\phi, \mu_i)\neq \text{ weight of } P(\phi, \mu_j), \quad \mu_i\neq\mu_j$$

using (1) of Theorem 3.15. We see also that $X_\alpha P(\phi, \lambda)$ is a weight vector. Hence we conclude that

$$X_\alpha P(\phi, \lambda)=a\,P(\phi, \mu). \tag{1}$$

Then if $a \neq 0$ we have

$$\begin{aligned} \text{weight of } X_\alpha P(\phi, \lambda) &= -\tfrac{1}{2}(\phi(\omega) + \lambda(\omega)) + \alpha \\ = \text{weight of } P(\phi, \mu) &= -\tfrac{1}{2}(\phi(\omega) + \mu(\omega)). \end{aligned}$$

This gives

$$\mu(\omega) = \lambda(\omega) - 2\alpha \qquad \text{if } a \neq 0 \text{ in (1).} \tag{2}$$

This implies that

$$\langle \mu(\omega), \alpha^* \rangle = \langle \lambda(\omega), \alpha^* \rangle - 4. \tag{3}$$

Since one knows that $|\langle \omega, \beta^* \rangle| \leqslant 2$ for any positive root (consequence of the property of ω being a classical fundamental weight, cf. Definition 2.2), we have in particular

$$|\langle \mu(\omega), \alpha^* \rangle| \leqslant 2. \tag{4}$$

Then one sees easily that (3) can hold only if

$$\langle \lambda(\omega), \alpha^* \rangle = 2. \tag{5}$$

In this case we get

$$s_\alpha \lambda(\omega) = \lambda(\omega) - \langle \lambda(\omega), \alpha^* \rangle \, \alpha = \lambda(\omega) - 2\alpha$$

which gives

$$\mu(\omega) = (s_\alpha \lambda)(\omega). \tag{6}$$

By the property of extremal weights, (6) gives that

$$\mu = s_\alpha \lambda.$$

The fact that $\mu > s_\alpha \lambda$ follows from the fact that $\langle \lambda(\omega), \alpha^* \rangle > 0$ (cf. Lemma 1.2). This completes the proof of the assertion (a) above.

The assertion (b) is an immediate consequence of (a); for

$$X_\alpha P(\phi) = X_\alpha P(\phi, \phi) = a\, P(\phi, \lambda), \qquad a \in \mathbf{Z}$$

and if $a \neq 0$, by (a) we get that $\lambda > \phi$, which contradicts the fact that $\phi > \lambda$ ((ϕ, λ) is an admissible pair). Hence $a = 0$ and the assertion (b) follows.

To prove (c), we observe that if

$$X_\alpha P(\phi, \lambda) = a\, P(\phi, \mu), \qquad a \neq 0$$

then we get that

$$\langle \mu(\omega), \alpha^* \rangle = -2.$$

Hence we have necessarily (using (5) above) that

$$X_\alpha P(\phi, \mu) = 0, \qquad \text{i.e.,} \qquad X_\alpha^2 P(\phi, \lambda) = 0.$$

This completes the proof of Lemma 3.18.

COROLLARY 3.19. *Let $\phi \in W/W_P$. Then the minimal $B_{\mathbf{Z}}$-submodule of the $B_{\mathbf{Z}}$ module $H^0(X_{\mathbf{Z}}(\phi), L_{\omega,\mathbf{Z}})$ containing $P(\phi, \lambda)$ consists of elements of the following form*:

$$\sum a_i P(\phi, \mu_i), \qquad a_i \in \mathbf{Z} \quad \text{and} \quad \mu_i \geqslant \lambda.$$

Proof. This is an immediate consequence of Lemma 3.18, for $P(\phi, \lambda)$ being a weight vector of the submodule $U_{\mathbf{Z}}^+ \cdot P(\phi, \lambda)$ of the $U_{\mathbf{Z}}^+$ module $H^0(X_{\mathbf{Z}}(\omega), L_{\omega,\mathbf{Z}})$ is the $\mathbf{Z}$ span generated by elements of the form

$$\frac{X_{\alpha_1}^{n_1}}{n!} \cdots \frac{X_{\alpha_s}^{n_s}}{n_s!} P(\phi, \lambda)$$

where the α_i occurring above are simple roots. Now Corollary 3.19 follows immediately from Lemma 3.18.

Remark 3.20. Note that the relations in Lemma 3.18 take place on the Schubert scheme $X_{\mathbf{Z}}(\phi)$, i.e., in the $B_{\mathbf{Z}}$-module $H^0(X_{\mathbf{Z}}(\phi), L_{\omega,\mathbf{Z}})$ and *not* in $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\omega,\mathbf{Z}})$. For example, it follows that $\mathbf{Z} \cdot P(\phi)$ is stable under $B_{\mathbf{Z}}$ and this cannot be true in $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\omega,\mathbf{Z}})$, for then $P(\phi)$ would be the highest weight vector. Note that $H^0(X_{\mathbf{Z}}(\phi), L_{\omega,\mathbf{Z}})$ is a quotient and not a submodule of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\omega,\mathbf{Z}})$.

4. YOUNG DIAGRAMS AND STANDARD MONOMIALS

Let $Q_{\mathbf{Z}}$ stand for a parabolic subgroup scheme of $G_{\mathbf{Z}}$ of *classical type* containing $B_{\mathbf{Z}}$ (cf. Definition 2.3). *Choose any arbitrary ordering*, say, $P_{i,\mathbf{Z}}$, $1 \leqslant i \leqslant r$, of the set $\{P_{i,\mathbf{Z}}\}$ of the maximal parabolic subgroup schemes containing $Q_{\mathbf{Z}}$. Let ω_i, $1 \leqslant i \leqslant r$, be the fundamental weight associated to $P_{i,\mathbf{Z}}$. We have

$$Q_{\mathbf{Z}} = \bigcap_{1 \leqslant i \leqslant r} P_{i,\mathbf{Z}}.$$

For any ring A, we denote by $L_{i,A}$ the line bundle $L_{\omega_i,A}$ on $G_A/P_{i,A}$. Note that for $A=\mathbf{Z}$ or a field k, $L_{i,A}$ is the ample generator of Pic $G_A/P_{i,A}$. Let $\mathbf{a}=(a_1,..., a_r)$, $a_i \in \mathbf{Z}$. Then we denote by $L_{\mathbf{a},A}$ the line bundle on G_A/Q_A (or G_A/B_A), defined by

$$L_{1,A}^{\otimes a_1} \otimes L_{2,A}^{\otimes a_2} \otimes \cdots \otimes L_{r,A}^{\otimes a_r}.$$

Note that $L_{\mathbf{a},A}$ is the line bundle, denoted by $L_{\lambda,A}$ (in Section 1), where λ is the weight $\lambda=\sum_{i=1}^r a_i\omega_i$.

In the sequel, unless otherwise stated $Q_{\mathbf{Z}}$ will always stand for a parabolic subgroup scheme of $G_{\mathbf{Z}}$ of classical type and $P_{i,\mathbf{Z}}$, $L_{i,A}$, etc., will be understood to be as above. Note that we have also chosen an ordering of the maximal parabolic subgroup schemes containing $Q_{\mathbf{Z}}$.

DEFINITION 4.1. Let $\mathbf{a}=(a_1,..., a_r)$, $a_i \in \mathbf{Z}^+$. Then a *Young diagram of type* $\mathbf{a}$ *or multi-degree* $\mathbf{a}$ in W/W_Q (or on $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$) is a pair (θ, δ), where

$\theta=(\theta_{ij})$, $\delta=(\delta_{ij})$, $1 \leqslant i \leqslant r$, and $(\theta_{ij}, \delta_{ij})$ is an admissible pair in W/W_i, $1 \leqslant j \leqslant a_i$.

If $a_t=0$ for some t, $1 \leqslant t \leqslant r$, we understand that the corresponding admissible pair $(\theta_{t,-}, \delta_{t,-})$ is empty, i.e., does not figure.

DEFINITION 4.2. A Young diagram (θ, δ) as above is said to be *standard* if there exists a pair (α, β) which we call a *defining pair* for (θ, δ) such that

(1) $\alpha=(\alpha_{ij})$, $\beta=(\beta_{ij})$, $\alpha_{ij}, \beta_{ij} \in W/W_Q$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant a_i$,

(2) each α_{ij} (resp. β_{ij}) is a lift in W/W_Q for the element θ_{ij} (resp. δ_{ij}) in W/W_i under the canonical morphism $W/W_Q \to W/W_i$, and

(3) $\alpha_{11} \geqslant \beta_{11} \geqslant \alpha_{12} \geqslant \beta_{12} \geqslant \cdots \geqslant \alpha_{1a_1} \geqslant \beta_{1a_1} > \alpha_{21} \geqslant \beta_{21} \geqslant \cdots \geqslant \alpha_{ra_r} > \beta_{ra_r}$.

We say that the Young diagram (θ, δ) is *standard with respect to* $\tau \in W/W_Q$ *if in addition to the conditions* (1), (2), and (3) we have also

(4) $\tau \geqslant \alpha_{11}$ in W/W_Q (often written as $\tau \geqslant \alpha$).

Note that the notion of a *standard Young diagram* depends upon the *ordering* (fixed above) of the maximal parabolic subgroup schemes containing $Q_{\mathbf{Z}}$.

DEFINITION 4.3. Let (θ, δ) be a Young diagram in W/W_Q of type $\mathbf{a}$. To this we can associate the following element of $H^0(G_{\mathbf{Z}}/Q_{\mathbf{Z}}, L_{\mathbf{a},\mathbf{Z}})$:

$$P(\theta_{11}, \delta_{11})\, P(\theta_{12}, \delta_{12}) \cdots P(\theta_{ra_r}, \delta_{ra_r}). \qquad (*)$$

This element is called a *Young monomial* and we denote this by $P(\theta, \delta)$. (To be precise, $P(\theta, \delta)$ is an element of $H^0(G_{\mathbf{Z}}/Q_{\mathbf{Z}}, L_{\mathbf{a},\mathbf{Z}})$, indexed by the Young diagram (θ, δ).) If (θ, δ) is standard with respect to $\tau \in W/W_Q$, we call $P(\theta, \delta)$ a *standard monomial on* $X_{\mathbf{Z}}(\tau)$. If k is a field, we denote by $P_k(\theta, \delta)$ or just $p(\theta, \delta)$ the canonical image of $P(\theta, \delta)$ in $H^0(G_k/Q_k, L_{\mathbf{a},k})$ and call it *standard on* $X_k(\tau)$ if (θ, δ) is standard with respect to τ.

LEMMA 4.4. (Deodhar). *Let $Q_{\mathbf{Z}}$ be a parabolic subgroup scheme of $G_{\mathbf{Z}}$ and let W^Q be the set of* minimal representatives *of W/W_Q in the Weyl group W. Then given $\sigma, \sigma' \in W^Q$ such that $\sigma' \leqslant \sigma$ and $w \in W/W_Q$, there exists a unique $w' = \eta(\sigma', \sigma, w)$ such that $\sigma' w' \leqslant \sigma w$ and w' is maximal for this property, i.e., for any $w'' \in W_Q$ such that $\sigma' w'' \leqslant \sigma w$, one has $w'' \leqslant w'$ in W_Q.*

Proof. The proof is by induction on $l(\sigma)$. If $l(\sigma) = 0$, then $\sigma = \sigma' = \mathrm{Id}$ (identity element) and it is clear that $w' = w$ is the required element. Now let $l(\sigma) > 0$. Choose a simple root α such that $s_\alpha \sigma < \sigma$. Set $s = s_\alpha$. Note that $s\sigma \in W^Q$ (this follows from the facts that $\tau \in W^Q \Leftrightarrow \tau(\beta) > 0$ for all $\beta \in \Delta_Q^+$ (the set of positive roots associated to Q) and that given a simple root α, s_α leaves $\Delta^+ - \{\alpha\}$ stable). Now we distinguish the following two cases.

Case 1: $s\sigma' < \sigma'$. In this case we have again $s\sigma' \in W^Q$ (for the same reasons as above). We have also $s\sigma' < s\sigma$ (cf. Lemma 1.5). Hence by induction hypothesis $\eta(s\sigma', s\sigma, w)$ is defined. Call it τ'. Now we claim that w' exists and is in fact τ'. For, we first note that for any τ in W_Q

$$\sigma'\tau \leqslant \sigma w \Leftrightarrow s\sigma'\tau \leqslant s\sigma w \qquad (*)$$

(since under the hypothesis $\sigma > s\sigma$ and $\sigma' > s\sigma'$, we have $\sigma w > s\sigma w$ and $\sigma'\tau > s\sigma'\tau$). Now if $\sigma' w'' \leqslant \sigma w$, then we obtain from $(*)$ that $s\sigma' w'' \leqslant s\sigma w$ and hence $w'' \leqslant \tau'$; also, from $(*)$ we obtain $\sigma'\tau' \leqslant \sigma w$ (since $s\sigma'\tau' \leqslant s\sigma w$). This proves the required claim.

Case 2: $s\sigma' > \sigma'$. In this case $\sigma' \leqslant s\sigma$. Henc by induction hypothesis, $\eta(\sigma', s\sigma, w)$ is defined, call it τ'. We claim that w' exists and is in fact τ'. For, we first note that for any τ in W_Q

$$\sigma w \geqslant \sigma'\tau \Leftrightarrow s\sigma w \geqslant \sigma'\tau \qquad (**)$$

(in view of the facts that $\sigma' < s\sigma'$ and $s\sigma < \sigma$, we have that if $\sigma'\tau \leqslant \sigma w$, then $\sigma'\tau \leqslant s\sigma w$; conversely, if $\sigma'\tau \leqslant s\sigma w$, then $\sigma'\tau < \sigma w$, necessarily (since $s\sigma w \leqslant \sigma w$)). Now, if $\sigma' w'' \leqslant \sigma w$, then from $(**)$ we obtain that $\sigma' w'' < s\sigma w$. Hence $w'' < \tau'$. Further $\sigma'\tau' < s\sigma w$ implies (in view of $(**)$) that $\sigma'\tau' < \sigma w$.
This proves the required claim.

COROLLARY 4.5. *Let (θ, δ) be a standard Young diagram on the Schubert subscheme $X_{\mathbf{Z}}(\phi)$ of $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$, $\phi \in W/W_Q$. Then we can find a* unique maximal defining pair (λ^+, μ^+) *for* (θ, δ) *on* $X_{\mathbf{Z}}(\phi)$, i.e., *if* (λ, μ) *is an defining pair for* (θ, δ) *on* $X_{\mathbf{Z}}(\phi)$, *we have*

$$\lambda^+ \geqslant \lambda \quad \textit{and} \quad \mu^+ \geqslant \mu, \quad \textit{i.e.,} \quad \textit{if} \quad \lambda = (\lambda_{ij}), \quad \mu = (\mu_{ij}), \quad \lambda^+ = (\mu_{ij}^+), \\ \mu^+ = (\mu_{ij}^+), \ \lambda_{ij}^+ \geqslant \lambda_{ij} \ \textit{and} \ \mu_{ij}^+ \geqslant \mu_{ij}$$

Proof. This is an immediate consequence of the above Lemma. Le (λ, μ) be a defining pair for (θ, δ) on $X_{\mathbf{Z}}(\phi)$ and

$$\lambda = (\lambda_{ij}), \qquad \mu = (\mu_{ij}), \qquad \theta = (\theta_{ij}), \qquad \delta = (\delta_{ij}).$$

We define $\lambda^+ = (\lambda_{ij}^+)$ and $\mu^+ = (\mu_{ij})$ inductively as follows: We have $\phi \geqslant \lambda_1$ and the image of λ_{11} under the canonical map $W/W_Q \to^{\eta} W/W_{P_1}$ is θ_1 Hence by Lemma 4.4, we can find a unique maximal element λ_{11}^+ such tha $\phi \geqslant \lambda_{11}^+$ and the image of λ_{11}^+ under η is θ_{11}. Clearly $\lambda_{11}^+ \geqslant \lambda_{11}$. Now choos μ_{11}^+ to be maximal within λ_{11}^+ such that the image of μ_{11}^+ (under η) is δ_{11}, et This completes the proof of Corollary 4.5.

Lemma 4.4 and Corollary 4.5 admit dual versions as follows.

LEMMA 4.4′ (Deodhar). *We follow the notations of Lemma* 4.4. *Then give* $\sigma, \sigma' \in W^Q$ *and* $w \in W_Q$ *with* $\sigma' \geqslant \sigma$, *there exists a unique* $w' = \mu(\sigma', \sigma, w)$ *i* W_Q *such that* $\sigma' w' \geqslant \sigma w$ *and* w' *is minimal for this property, i.e., for an* $w'' \in W_Q$, *if* $\sigma' w'' \geqslant \sigma w$, *then* $w'' \geqslant w'$ *in* W^Q.

COROLLARY 4.5′. *We follow the notations of Corollary* 4.5. *Then we ca find a* unique minimal defining pair (λ^-, μ^-) *for* (θ, δ) *on* $X_{\mathbf{Z}}(\phi)$, *i.e.,* (λ, μ) *is any other defining pair for* (θ, δ) *on* $X_{\mathbf{Z}}(\phi)$, *then*

$$\lambda^- \leqslant \lambda \qquad \textit{and} \qquad \mu^- \leqslant \mu.$$

Remark 4.6. Note that the minimal defining pair (λ^-, μ^-) c Corollary 4.5′ depends only on (θ, δ) and is independent of the choice c any Schubert scheme $X_{\mathbf{Z}}(\phi)$ on which (θ, δ) is standard. In particular, not that if (θ, δ) is a Young diagram on $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$, then it is standard on Schubert scheme $X_{\mathbf{Z}}(\Psi)$ if and only if $\Psi \geqslant \lambda_{11}^-$, where (λ^-, μ^-) is th minimal defining pair for (θ, δ) on $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$.

Remark 4.7. Given $w \in W/W_Q$ and $\bar{\lambda} \in W/W_{P_1}$, such that $\bar{w} \geqslant \bar{\lambda}$ (where $\bar{w}$ denotes the image of w under the canonical map $W/W_Q \to W/W_{P_1}$), let denote the unique maximal element in W/W_Q such that $w \geqslant \lambda$ and λ pro jects to $\bar{\lambda}$ (which exists by Lemma 4.4). We shall often refer to λ as the *uni que maximal lift* of $\bar{\lambda}$, *less than* w, and to $X_{\mathbf{Z}}(\lambda)$ as the *unique maximal lif* of $X_{\mathbf{Z}}(\lambda)$ in $X_{\mathbf{Z}}(w)$. It is clear that a standard Young diagram (θ, δ) o $X_{\mathbf{Z}}(w)$ remains standard on $X_{\mathbf{Z}}(\lambda)$.

The following technical result will be used only in Section 5, but we give it here as its proof uses the ideas of this section.

LEMMA 4.8. *Let $X_{\mathbf{Z}}(\phi)$ be a moving divisor in $X_{\mathbf{Z}}(\tau)$, moved by a simple root α, where $\phi, \tau \in W/W_Q$ ($Q_{\mathbf{Z}}$ being as usual). We denote by $\bar{\phi}$, etc., the image of ϕ, etc., under the canonical morphism $W/W_Q \to W/W_{P_1}$. Suppose that $\bar{\phi}$ $(=s_\alpha \bar{\tau}) < \bar{\tau}$. For any $\bar{\lambda} \in W/W_{P_1}$ such that $\bar{\lambda} < \bar{\phi}$, let λ be the* maximal lift of $\bar{\lambda}$ less than ϕ *(see Remark* 4.7*). Then we have the following*:

(1) *if $\bar{\lambda} < s_\alpha \bar{\lambda}$, λ (resp. $s_\alpha \lambda$) is also the maximal lift of $\bar{\lambda}$ (resp. $s_\alpha \bar{\lambda}$) less than τ,*

(2) *if $\bar{\lambda} = s_\alpha \bar{\lambda}$, the bigger of the two elements λ and $s_\alpha \lambda$ is the maximal lift of $\bar{\lambda}$ less than τ.*

Proof. Let μ be the maximal lift of $\bar{\lambda}$, less than τ. Then obviously $\mu \geqslant \lambda$. Let $s_\alpha \bar{\lambda} > \bar{\lambda}$. In this case, we claim that

(i) λ is the maximal lift of $\bar{\lambda}$, less than τ, i.e., $\mu = \lambda$, and

(ii) $s_\alpha \lambda$ is the maximal lift of $s_\alpha \bar{\lambda}$, less than τ.

We shall first prove (i). If $\mu \neq \lambda$, then we obtain that $\mu > \lambda$. Now $\mu > \lambda$ implies that $\mu \nleqslant \phi$ (since $\bar{\mu} = \bar{\lambda}$ and λ is the maximal lift of $\bar{\lambda}$, less than ϕ). Hence $\mu = s_\alpha \nu$ for some $\nu < \phi$ (by Lemma 1.5). Now $\nu = s_\alpha \mu$ implies that $\bar{\nu} = s_\alpha \bar{\mu} = s_\alpha \bar{\lambda}$. This, in turn, implies (under the hypothesis $s_\alpha \bar{\lambda} > \bar{\lambda}$) that $X_{\mathbf{Z}}(\nu)$ has a bigger projection in $G_{\mathbf{Z}}/P_{1,\mathbf{Z}}$ than that of $X_{\mathbf{Z}}(\mu)$, which is a contradiction (since $\mu > \nu$). Thus we obtain $\mu = \lambda$ as claimed in (i). (Note that as a particular case (with $\bar{\lambda} = \bar{\phi}$), we obtain that ϕ is the maximal lift of $\bar{\phi}$ less than τ.) Now to prove (ii), let $\theta < \tau$ be such that $\bar{\theta} = s_\alpha \bar{\lambda}$. Then $s_\alpha \theta$ projects onto $\bar{\lambda}$ (under the canonical map $W/W_Q \to W/W_{P_1}$). Hence by (i) above,

$$s_\alpha \theta \leqslant \lambda < s_\alpha \lambda.$$

Now $X_{\mathbf{Z}}(s_\alpha \lambda)$ is stable under the canonical action of the group scheme $G_{-\alpha,\mathbf{Z}}$ associated to α (see (iii) of Lemma 1.2) or equivalently stable under the action of the minimal parabolic subgroup scheme $P_{\alpha,\mathbf{Z}}$ associated to α and hence for any $w \leqslant s_\alpha \lambda$ (in particular for $w = s_\alpha \theta$), we have $s_\alpha w \leqslant s_\alpha \lambda$. Thus $\theta \leqslant s_\alpha \lambda$, which proves (ii) above.

Let now $s_\alpha \bar{\lambda} = \bar{\lambda}$. Then taking ν to be the bigger of the two elements λ and $s_\alpha \lambda$, we have $\bar{\nu} = \bar{\lambda}$. Hence $\nu \leqslant \mu$. On the other hand, if θ is the smaller of the two elements μ and $s_\alpha \mu$, we have $\theta \leqslant \phi$ (see Lemma 1.5). Further $\bar{\theta} = \bar{\lambda}$ and hence $\theta \leqslant \lambda$. Thus $\theta \leqslant \nu$ and hence $s_\alpha \theta \leqslant \nu$ (since $X_{\mathbf{Z}}(\nu)$ is stable under the action of the minimal parabolic subgroup scheme $P_{\alpha,\mathbf{Z}}$, etc.). This, in particular, implies that $\mu \leqslant \nu$. This, together with the fact that $\nu \leqslant \mu$ (which was proved above), implies that $\mu = \nu =$ bigger of the two elements λ and $s_\alpha \lambda$. The proof of Lemma 4.8 is now complete.

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5. Linear Independence of Standard Monomials

Let Q be a parabolic subgroup of classical type. Let P be a maximal parabolic subgroup containing Q, with associated fundamental weight ω. Let $\tau \in W/W_Q$ and let $X(w)$ be the projection of $X(\tau)$ under $G/Q \to G/P$. Suppose the following relations hold on $X_{\mathbf{Z}}(w)$.

(1) $P(w, \theta)^2 = \pm P(w) \cdot P(\theta)$.

(2) Conversely, if $F \in H^0(X_{\mathbf{Z}}(w), L_{\omega,\mathbf{Z}})$ is such that $F^2 = P(w)\, P(\theta)$, then either $F = 0$ on $X_{\mathbf{Z}}(w)$, i.e., $w \not\geqslant \theta$, or (w, θ) is an admissible pair with $F = \pm P(w, \theta)$. (*)

(3) $P(w, \theta_1)\, P(w, \theta_2) = \pm P(w) F$, where $F^2 = P(\theta_1)\, P(\theta_2)$.

In the sequel, we shall refer to these as "*special quadratic relations.*"

For the rest of this section, we shall assume that (*) *holds for any* $X(\tau)$ *that is considered in this section. Let* Y *be a union of Schubert subschemes of* $G_{\mathbf{Z}}/Q_{\mathbf{Z}}$, *say,*

$$Y = X_{\mathbf{Z}}(\phi_1) \cup \cdots \cup X_{\mathbf{Z}}(\phi_s), \qquad \phi_i \in W/W_Q, \qquad 1 \leqslant i \leqslant s.$$

Let (θ, δ) be a Young diagram in W/W_Q of type $\mathbf{a}$ (see Definition 4.2). We say (θ, δ) is *standard with respect to* $(\phi_1, ..., \phi_s)$ if (θ, δ) is standard with respect to at least one among the ϕ_i, $1 \leqslant i \leqslant s$. For any field k, if $p_k(\theta, \delta)$ is the Young monomial associated to (θ, δ), we call it *standard on the union of Schubert varieties* Y_k *if* (θ, δ) is standard with respect to $(\phi_1, ..., \phi_s)$

Theorem 5.1. *The set* $\{p_k(\theta, \delta)\}$, *where* (θ, δ) *runs over distinct standard Young diagrams with respect to* $(\phi_1, ..., \phi_s)$ *of type* $\mathbf{a}$, *is a linearly independent set* (*over* k) *in* $H^0(Y_k, L_{\mathbf{a},k})$, Y_k being the union of Schubert varieties $X_k(\phi_1) \cup \cdots \cup X_k(\phi_s)$ *in* G_k/Q_k (*for the notation* $L_{\mathbf{a},k}$ *see the beginning of Section* 4).

Proof. Let $\mathbf{a} = (a_1, ..., a_r)$. The proof is by a double induction argument on a_1 and r. The proof for the case $r = 1$ is contained in the proof of the general case. Suppose that $a_1 = 0$. If $r = 1$, again it is immediate. Suppose then that $r \geqslant 2$. Let $Q'_k = P_{2,k} \cap \cdots \cap P_{r,k}$ (recall $Q_{\mathbf{Z}} = P_{1,\mathbf{Z}} \cap \cdots \cap P_{r,\mathbf{Z}}$). We have a canonical morphism $\pi\colon G_k/Q_k \to G_k/Q'_k$ and let $X_k(w')$, $w' \in W/W_{Q'}$, be the image of $X_k(w)$ under π. Now a standard monomial of type $(0, a_2, ..., a_r)$ on $X_k(w)$ can be identified as the pull-back by π of a standard monomial of type $(a_2, ..., a_r)$ on $X_k(w')$ (to be precise, we identify the line bundle $L_{\mathbf{b},k}$ on $X_k(w)$, $\mathbf{b} = (0, a_2, ..., a_r)$, as the pull-back of the line bundle $L_{\mathbf{b}',k}$ on $X_k(w')$ and what we mean is the pull-back by π of a section of $L_{\mathbf{b}',k}$ on $X_k(w')$, identified as a section of $L_{\mathbf{b},k}$ on $X_k(w)$). Now linear independence is preserved under this pullback since $\pi\colon X_k(w) \to X_k(w')$ is a dominant morphism. By our induction hypothesis on r, standard monomials on $X_k(w')$ of type $(a_2,\dots, a_r)$ are linearly independent. Thus standard monomials on $X_k(w)$ of type $(0, a_2,\dots, a_r)$ are linearly independent.

Suppose now $a_1 \geqslant 1$. Set $\mathbf{a}' = (a_1 - 1, a_2,\dots, a_r)$. Let $P_k(\theta^{(l)}, \delta^{(l)})$, $1 \leqslant l \leqslant t$, be a *minimal set of linearly dependent* standard monomials on Y_k of type $\mathbf{a}$. We shall show that this leads to a contradiction, which would prove the theorem. Let $(\alpha^{(l)}, \beta^{(l)})$ be a defining pair for $(\theta^{(l)}, \delta^{(l)})$, $1 \leqslant l \leqslant t$. We write

$$\theta^{(l)} = (\theta_{ij}^{(l)}), \qquad \delta^{(l)} = (\delta_{ij}^{(l)}), \qquad \alpha^{(l)} = (\alpha_{ij}^{(l)}),$$
$$\beta^{(l)} = (\beta_{ij}^{(l)}), \qquad 1 \leqslant i \leqslant r, \quad 1 \leqslant j \leqslant a_i.$$

Recall that $(\theta_{ij}^{(l)}, \delta_{ij}^{(l)})$ is an admissible pair in $W/W_{P_i} = W/W_i$ ($P_{i,\mathbf{Z}}$ being the maximal parabolic subgroup schemes containing $Q_{\mathbf{Z}}$). We have the following relation on Y_k:

$\sum_{1 \leqslant l \leqslant t} c_l P_k(\theta^{(l)}, \delta^{(l)}) = 0, \qquad c_l \in k,\ c_l \neq 0$ for every l, $1 \leqslant l \leqslant t$ (because of the minimality property above) and $(\theta^{(l)}, \delta^{(l)})$ are distinct Young diagrams. (1)

Let λ be a minimal element of elements $\{\theta'_{11}\}$ and I the subset of $\{1,\dots, t\}$ defined by

$$I = \{l \mid \theta_{11}^{(l)} = \lambda,\ 1 \leqslant l \leqslant t\}.$$

Let Z_1 be the union of the Schubert varieties defined by

$$Z_1 = \bigcup_{l \in I} X_k(\alpha_{11}^{(l)}).$$

Since, by hypothesis, $P_k(\theta^{(l)}, \delta^{(l)})$ is standard on $Y_k = \bigcup X_k(\phi_i)$, $1 \leqslant i \leqslant s$, we see that every $\alpha_{11}^{(l)} \leqslant$ some ϕ_i. We deduce that $Z_1 \subseteq Y_k$. We note also $P_k(\theta^{(l)}, \delta^{(l)})$ is standard on $X_k(\alpha_{11}^{(l)})$. We can write

$$P_k(\theta^{(l)}, \delta^{(l)}) = P_k(\theta_{11}^{(l)}, \delta_{11}^{(l)})\, F_l$$

where F_l is a standard monomial not only on Y_k but in fact on $X_k(\beta_{11}^{(l)})$. Now restrict the relation (1) to Z_1. Then we obtain the following relation on Z_1:

$$\sum_{l \in I} c_l P_k(\lambda, \delta_{11}^{(l)})\, F_l = 0, \qquad c_l \neq 0\ \forall l \in I. \tag{2}$$

Let μ be a minimal element of the set $\{\delta_{11}^{(l)} \mid l \in I\}$. Without loss of generality we can suppose that $\mu = \delta_{11}^{(1)}$. Then multiplying the L.H.S. of (2) by $P_k(\lambda, \delta_{11}^{(1)})$ we obtain (using (*)) the following relation on Z_1:

$$P_k(\lambda) \sum_{l \in I} c_l (P_k(\mu)\, P_k(\delta_{11}^{(l)}))^{1/2} F_l = 0, \qquad c_l \neq 0 \ \forall l \in I. \tag{3}$$

We observe that as $P_k(\lambda)$ does not vanish identically on every $X_k(\alpha_{11}^{(l)})$ for $l \in I$, we can cancel $P_k(\lambda)$ in the relation (3) and obtain the following relation on Z_1:

$$\sum_{l \in I} c_l (P_k(\mu)\, P_k(\delta_{11}^{(l)}))^{1/2} F_l = 0, \qquad c_l \neq 0, \ \forall l \in I. \tag{4}$$

Let Z_2 be the union of the Schubert varieties defined by

$$Z_2 = \bigcup_{l \in J} X_k(\beta_{11}^{(l)}), \qquad \text{where} \quad J = \{ l \mid l \in I, \delta_{11}^{(l)} = \mu \}.$$

Now restrict the relation (4) to Z_2. Because of the minimality nature of $\mu = \delta_{11}^{(1)}$, we obtain the following relation on Z_2:

$$P_k(\mu) \left\{ \sum_{l \in J} c_l F_l \right\} = 0, \qquad c_l \neq 0 \ \forall l \in J. \tag{5}$$

We can cancel $P_k(\mu)$ in the above relation since $P_k(\mu)$ does not vanish identically on every $X_k(\beta_{11}^{(l)})$, $l \in J$. Thus we obtain the following relation on Z_2:

$$\sum_{l \in J} c_l F_l \neq 0, \qquad c_l \neq 0 \ \forall l \in J. \tag{6}$$

As we observed above F_l is standard on $X_k(\beta_{11}^{(l)})$ and so it is standard on Z_2. We note also that F_l are distinct standard monomials of type $\mathbf{a}'$ on Z_2. Hence we obtain a contradiction to the inductive hypothesis that the set of standard monomials of type $\mathbf{a}'$ on a Schubert variety in G_k/Q_k is a linearly independent set. As we observed above, this completes the proof of the theorem.

COROLLARY 5.2. *Let Y be the scheme-theoretic union of a finite number of Schubert schemes*

$$Y = X_{\mathbf{Z}}(w_1) \cup \cdots \cup X_{\mathbf{Z}}(w_s), \qquad w_i \in W/W_Q.$$

Note that Y is reduced since $X_{\mathbf{Z}}(w_i)$ is reduced. Let $S_{\mathbf{Z}}(Y, \mathbf{a})$ be the $\mathbf{Z}$-submodule of $H^0(Y, L_{\mathbf{a},\mathbf{Z}})$ generated by standard monomials of type $\mathbf{a}$. Then $S_{\mathbf{Z}}(Y, \mathbf{a})$ is a direct summand in $H^0(Y, L_{\mathbf{a},\mathbf{Z}})$.

Proof. It suffices to show that the canonical homomorphism

$$j_1: S_{\mathbf{Z}}(Y, \mathbf{a}) \otimes_{\mathbf{Z}} k \to H^0(Y, L_{\mathbf{a},\mathbf{Z}}) \otimes_{\mathbf{Z}} k$$

is injective for every field k. We have obviously a canonical k-linear map (with $Y_k = X_k(w_1) \cup \cdots \cup X_k(w_s)$):

$$j_2: H^0(Y, L_{\mathbf{a},\mathbf{Z}}) \otimes_{\mathbf{Z}} k \to H^0(Y_k, L_{\mathbf{a},k}).$$

By Theorem 5.1, it follows immediately that the k-linear map

$$j_2 \circ j_1: S_{\mathbf{Z}}(Y, \mathbf{a}) \otimes k \to H^0(Y_k, L_{\mathbf{a},k})$$

is injective for every field k. This implies that j_1 is injective for every field k, which proves Corollary 5.2.

DEFINITION 5.3. Let X be a union of Schubert varieties in G_k/Q_k (with the usual notations). We set

$S_k(X, \mathbf{a})$ (or simply $S(X, \mathbf{a})$) = the k-linear space of standard monomials on X of type $\mathbf{a}$ (1)

$$s(X, L_{\mathbf{a}}) = s(X, \mathbf{a}) = \dim S(X, \mathbf{a}). \quad (2)$$

(Note that in view of (*), we have linear independence of standard monomials.)

PROPOSITION 5.4. *Let Y_1, Y_2 be respectively unions of Schubert varieties in G_k/Q_k. Then we have*

$$s(Y_1 \cup Y_2, \mathbf{a}) = s(Y_1, a) + s(Y_2, \mathbf{a}) - s((Y_1 \cap Y_2)_{\text{red}}, \mathbf{a}).$$

(*Observe that the underlying reduced scheme of the scheme-theoretic intersection $(Y_1 \cap Y_2)$, i.e., $(Y_1 \cap Y_2)_{\text{red}}$, is a union of Schubert varieties on G_k/Q_k. Of course, the scheme-theoretic union in $Y_1 \cup Y_2$ is always reduced as one sees easily.*)

Proof. It is seen easily that we have only to show that if $P_k(\theta, \delta)$ is a standard monomial on Y_1 as well as Y_2, then it is standard on $(Y_1 \cap Y_2)_{\text{red}}$. By definition, $P_k(\theta, \delta)$ is standard on some irreducible component $X_k(\phi)$ of Y_1 as well as some irreducible component $X_2(\Psi)$ of Y_2. If $(\bar{\lambda}, \bar{\mu})$ denotes the *minimal defining pair* of (θ, δ), then we have (see Remark 4.6)

$$\phi \geqslant \lambda_{11}^{-}, \qquad \text{i.e., } X_k(\phi) \supseteq X_k(\lambda_{11}^{-})$$

and

$$\psi \geqslant \lambda_{11}^{-}, \qquad \text{i.e., } X_k(\psi) \supseteq X_k(\lambda_{11}^{-}).$$

Obviously this implies that there is an irreducible component $X_k(\mu)$ of $X_k(\phi) \cap X_k(\psi)$ such that $X_k(\lambda_{11}^{-}) \subseteq X_k(\mu)$. This means that $P_k(\theta, \delta)$ is standard on $X_k(\mu)$. Now one observes that $X_k(\mu) \subseteq (Y_1 \cap Y_2)_{\text{red}}$, which implies immediately that $P_k(\theta, \delta)$ is standard on $(Y_1 \cap Y_2)_{\text{red}}$.

DEFINITION 5.5. Let $w \in W/W_Q$. We set

$$H(w) = \text{the scheme-theoretic intersection } X_k(w) \cap \{P_k(\bar{w}) = 0\}$$

where $\bar{w}$ denotes as usual the canonical image of w in W/W_{P_1} (and we denote by the same $p_k(\bar{w})$ its pull-back by the canonical morphism $\pi\colon G_k/Q_k \to G_k/P_k$). It can be seen without much difficulty that $H(w)$ is a union of Schubert varieties (set-theoretically). This is a consequence of the fact that $H(w) = X_k(w) \cap \pi^{-1}(H(\bar{w}))$ (set-theoretically) and the following

LEMMA 5.6 (cf. [Se]$_1$). *Let P be* any *maximal parabolic subgroup of G (note that we do not assume P to be of classical type). Then the zero set of $p(\lambda)$ in $X(\lambda)$ is (set-theoretically) the union of all* codim 1 *subvarieties of $X(\lambda)$.*

Proof. Let X_0 be the zero set in G of $f \in H^0(G/P, L_\omega)$, where f is a generator of the unique B-fixed line in $H^0(G/P, L_\omega)$ (here we identify f with a regular function on G—recall that $H^0(G/P, L_\omega) = \{F\colon G \to k/F(gb) = F(g)\,\omega(b),\ g \in G,\ b \in B\}$). Let Y_0 denote the image of X_0 under $G/B \to G/P$. (Note that Y_0 is nothing but the unique codimension one Schubert divisor in G/P.) It can be easily seen that for any $h \in G$,

$$\text{zero-set of } h \circ f = hX_0.$$

In particular, we have, for $\lambda \in W/W_P$,

$$\text{zero-set of } p(\lambda) = \lambda w_0 X_0$$

(note that $p(\lambda)$ is the λw_0-translate of f—for example, by weight considerations). Now the following can be easily proved:

(a) $X(w) \subseteq \lambda w_0\, y_0$ if $\lambda \nleqslant w$,

(b) $X(w) \nsubseteq \lambda w_0\, y_0$ if $\lambda \leqslant w$,

(c) $B\lambda P \cap \lambda w_0 X_0 = \varnothing$.

(Parts (a) and (b) can be proved using the axioms of the Tits system; proof of (c) is fairly straightforward, for example, one can show easily that $B\lambda P \cap \lambda w_0 X_0 \neq \varnothing$ would imply that $w_0 \in X_0$, which is not true.) Thus we obtain that the zero set of $p(\lambda)$ in $X(\lambda)$ = union of all codimension one Schubert subvarieties of $X(\lambda)$. This completes the proof of Lemma 5.6.

Remark 5.7. Note that the proof of Lemma 5.6 implies that for $w \in W/W_P$

$$X(w) = \bigcap_{w \not\geq \lambda} \text{zero set of } p_\lambda \qquad \text{(set-theoretically)}.$$

For, if Z denotes the R.H.S., then $X(w) \subset Z$ (cf. (a) in the proof of Lemma 5.6). Now, for any $\phi \in W/W_P$, we have

$$B\phi P \cap \text{zero set of } p_\phi = \emptyset$$

(cf. (c) in the proof of Lemma 5.6).

Hence $Z \subset \bigcup_{w \geq \phi} B\phi P$. But then $\bigcup_{w \geq \phi} B\phi P$ is nothing but $X(w)$. Thus $Z = X(w)$, as asserted.

PROPOSITION 5.8. *Let $w \in W/W_Q$. Then we have*

$$s(X_k(w), \mathbf{a}) = s(X_k(w), \mathbf{a}') + s(H(w)_{\text{red}}, \mathbf{a}) + \sum_\lambda s(X_k(\lambda), \mathbf{a}') \qquad (\dagger)$$

where in the R.H.S. of (†) λ runs over all elements having the following properties:

(i) *λ is the maximal lift less than w of $\bar{\lambda}$, where as usual $\bar{\lambda}$ is the image of λ under the canonical map $W/W_Q \to W/W_{P_1}$.*

(ii) *$(\bar{w}, \bar{\lambda})$ is a non-trivial admissible pair in W/W_{P_1} (as usual $\mathbf{a} = (a_1,..., a_r)$ and $\mathbf{a}' = (a_1 - 1, a_2,..., a_r)$).*

Proof. We have the following:

$$S(X_k(w), \mathbf{a}) = S_1 \cup S_2 \cup S_3 \qquad \text{(disjoint union)}$$

where S_1 is the subset of $S(X_k(w), \mathbf{a})$ consisting of standard monomials beginning with $P_k(\bar{w})$, S_2 is the subset of $S(X_k(\bar{w}), \mathbf{a})$ of standard monomials beginning with $P_k(\bar{w}, \bar{\lambda})$, where $(\bar{w}, \bar{\lambda})$ is a non-trivial admissible pair in W/W_1, and S_3 is the subset of $S(X_k(\bar{w}), \mathbf{a})$ of standard monomials beginning with $P_k(\mu, v)$, where (μ, v) is an admissible pair in W/W_1 such that $\bar{w} > \mu$. It is clear that dim $S_1 = s(X_k(w), \mathbf{a}')$ and dim S_2 = the last term on the R.H.S. of (†) of Proposition 5.8 (see Remark 4.7). We have only to show that dim $S_3 = s(H(w)_{\text{red}}, \mathbf{a})$. Now if an element of S_3 is of the form $P_k(\mu, v)F$, since $\bar{w} > \mu$, if v is a maximal representative of μ less than w, we see that $X_k(v) \subseteq H(w)_{\text{red}}$ (by the description of $H(w)_{\text{red}}$ given above). We see now easily that we have in fact $S_3 = S(H(w)_{\text{red}}, \mathbf{a})$. The above proposition now follows.

6. Unions and Intersections of Schubert Varieties

We shall now prove some lemmas whose motivation is explained in Remark 6.4 below. *In this section, again we assume that the special quadratic relations in* $(*)$ *of Section* 5 *hold for the Schubert varieties considered here,* so that we have linear independence of standard monomials (cf. Theorem 5.1).

Lemma 6.1. *Let* Y_1, Y_2 *be respectively unions of Schubert varieties in* G/Q (*with the usual notations and dropping the subscript* k). *Let* ρ *be the* r-*tuple* $\rho=(1, 1,..., 1)$. *Note that* L_ρ *is an ample line bundle on* G/Q. *Suppose that* (*see Definition* 5.3)

$$h^0(Y_i, L_{\mathbf{m}\rho})=s(Y_i, m_\rho), \qquad m\gg 0, \quad i=1, 2$$

(*i.e., standard monomials on* Y_i *of type* $\mathbf{m}\rho$ *give a basis for* $H^0(Y_i, L_{m\rho})$, $i=1, 2$). *Then we have the following*:

(a) *the scheme-theoretic intersection* $Y_1 \cap Y_2$ *is reduced,*

(b) $h^0(Y_1 \cup Y_2, L_{m\rho})=s(Y_1 \cup Y_2, m\rho)$, *and* $h^0(Y_1 \cap Y_2, L_{m\rho})=s(Y_1 \cap Y_2, m\rho)$ *for* $m\gg 0$.

Proof. We have the following exact sequences (as sheaves of $\mathcal{O}_{G/Q}$-modules):

$$0 \to \mathcal{O}_{Y_1 \cup Y_2} \to \mathcal{O}_{y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0,$$

where we follow the usual convention of denoting the structure sheaf a scheme Z by $\mathcal{O}_Z$ and $Y_1 \cap Y_2$ (resp. $Y_1 \cup Y_2$) denotes the scheme-theoretic intersection (resp. union). If F is a coherent sheaf on G/Q, we denote by $F(m)$ the sheaf $F \otimes L_{m\rho}$ (here $L_{m\rho}$ denotes the sheaf associated to $m\rho$). Then tensoring the above exact sequence by $L_{m\rho}$, we obtain the exact sequence

$$0 \to \mathcal{O}_{Y_1 \cup Y_2}(m) \to \mathcal{O}_{Y_1}(m) \oplus \mathcal{O}_{Y_2}(m) \to \mathcal{O}_{Y_1 \cap Y_2}(m) \to 0.$$

Writing the cohomology exact sequence and using Serre's vanishing theorem (cf. $[\mathrm{S}]_1$) we get

$$\begin{aligned} h^0(Y_1 \cap Y_2, L_{m\rho}) &= h^0(Y_1, L_{m\rho})+h^0(Y_2, L_{m\rho}) \\ &\quad - h^0(Y_1 \cup Y_2, L_{m\rho}) \qquad \text{for} \quad m\gg 0. \end{aligned} \tag{1}$$

Now by Theorem 5.1, we have

$$h^0(Y_1 \cup Y_2, L_{m\rho}) \geqslant s(Y_1 \cup Y_2, m\rho). \tag{2}$$

Hence we obtain for $m \gg 0$

$$\text{R.H.S. of } (1) \leqslant s(Y_1, m\rho) + s(Y_2, m\rho) - s(Y_1 \cup Y_2, m\rho). \tag{3}$$

Now by Proposition 5.4,

$$\text{R.H.S. of } (3) = s((Y_1 \cap Y_2)_{\mathrm{red}}, \mathrm{m}\rho). \tag{4}$$

Thus we conclude that

$$h^0(Y_1 \cap Y_2, L_{m\rho}) \leqslant s((Y_1 \cap Y_2)_{\mathrm{red}}, m\rho), \qquad m \gg 0. \tag{5}$$

On the other hand, if $Y_1 \cap Y_2$ is *not* reduced, we see easily that

$$h^0(Y_1 \cap Y_2, L_{m\rho}) > h^0((Y_1 \cap Y_2)_{\mathrm{red}}, L_{m\rho}). \tag{6}$$

Further, by Theorem 5.1, we have

$$h^0((Y_1 \cap Y_2)_{\mathrm{red}}, L_{m\rho}) \geqslant s((Y_1 \cap Y_2)_{\mathrm{red}}, m\rho). \tag{7}$$

Thus (6) and (7) imply that

$$h^0(Y_1 \cap Y_2, L_{m\rho}) > s((Y_1 \cap Y_2)_{\mathrm{red}}, m\rho)$$

which contradicts (5). Hence it follows that $Y_1 \cap Y_2$ is reduced.

Now by Theorem 5.1, we have

$$h^0(Y_1 \cap Y_2, L_{m\rho}) \geqslant s((Y_1 \cap Y_2), m\rho), \qquad m \gg 0. \tag{8}$$

Hence from (8) and (5), we get

$$h^0(Y_1 \cap Y_2, L_{m\rho}) = s((Y_1 \cap Y_2), m\rho), \qquad m \gg 0 \tag{9}$$

which proves the second part of the assertion (b) of Lemma 6.1. Then from (1), we get (for $m \gg 0$)

$$\begin{aligned} h^0(Y_1 \cup Y_2, L_{m\rho}) &= h^0(Y_1, L_{m\rho}) + h^0(Y_2, L_{m\rho}) - h^0(Y_1 \cap Y_2, L_{m\rho}) \\ &= s(Y_1, m\rho) + s(Y_2, m\rho) - s(Y_1 \cap Y_2, m\rho) \\ &= s(Y_1 \cup Y_2, m\rho) \qquad \text{(by Proposition 5.4)}. \end{aligned}$$

This proves the first part of the assertion (b) of Lemma 6.1 and the proof of Lemma 6.1 is complete.

Notation 6.2. Let Schub(t) denote the set defined as follows: A member of Schub(t) is a subscheme of G/Q, which is the scheme-theoretic union of Schubert subvarieties of G/Q of dimension $\leqslant t$.

LEMMA 6.3. *Suppose that for every Schubert* variety $X \in \text{Schub}(d)$, *we have*

$$s(X, \mathbf{a}) = h^0(X, L_{\mathbf{a}}), \qquad \mathbf{a} = (a_1, ..., a_r),\ \mathbf{a} \geqslant 0,\ \text{i.e., } a_i \geqslant 0,\ 1 \leqslant i \leqslant r. \quad (*)$$

Then we have the following:

(i) *for any* Y_1, $Y_2 \in \text{Schub}(d)$, $Y_1 \cap Y_2$ *is reduced,*

(ii) *the assertion* $(*)$ *holds for* any $X \in \text{Schub}(d)$.

Proof. Suppose that $Y \in \text{Schub}(d)$. Then by induction on the number of components of Y, it follows, by an immediate extension of the argument of Lemma 6.1, that $h^0(Y, L_{m\rho}) = s(Y, m\rho)$ for $m \gg 0$. Then again by Lemma 6.1, the assertion (i) above follows. Thus, it remains only to prove the assertion (ii) above.

We observe that (ii) obviously holds for Schub(0). We now prove (ii) by induction on t, i.e., let $t \leqslant d$ and suppose that (ii) holds for every $X \in \text{Schub}(l)$, $l < t$; then we shall show that (ii) also holds for every $X \in \text{Schub}(t)$. We make another inductive argument; namely, if $C(X)$ denotes the number of irreducible components of $X \in \text{Schub}(t)$, we suppose that (ii) is true for $X \in \text{Schub}(t)$ such that $C(X) \leqslant (r-1)$. We now take $X \in \text{Schub}(t)$ such that $C(X) = r$, and will show that (ii) holds for X. This would prove the lemma. Let then

$$X = X_1 \cup \cdots \cup X_r$$

where X_i are the *distinct* irreducible components of X. We set

$$Y_1 = X_1 \cup \cdots \cup X_{r-1}, \qquad Y_2 = X_r.$$

By (1), $Y_1 \cap Y_2$ is reduced and we get the following exact sequence of $\mathcal{O}_{G/Q}$-modules (if F is a coherent sheaf on G/Q, we denote by $F(\mathbf{a})$ the sheaf $F \otimes L_{\mathbf{a}}$):

$$0 \to \mathcal{O}_X(\mathbf{a}) \to \mathcal{O}_{Y_1}(\mathbf{a}) \oplus \mathcal{O}_{Y_2}(\mathbf{a}) \to \mathcal{O}_{Y_1 \cap Y_2}(\mathbf{a}) \to 0. \qquad (1)$$

We see that $Y_1 \cap Y_2 \in \text{Schub}(t-1)$, since X_i are the distinct irreducible components of X. By our inductive hypothesis, (ii) holds for Y_1, Y_2 and $Y_1 \cap Y_2$. This implies that

$$h^0(Y_1 \cap Y_2, L_{\mathbf{a}}) = s(Y_1 \cap Y_2, \mathbf{a})$$

i.e., $H^0(Y_1 \cap Y_2, L_{\mathbf{a}})$ has a basis formed of standard monomials on $Y_1 \cap Y_2$. This implies that the canonical mapping

$$H^0(\mathcal{O}_{Y_1}(\mathbf{a}) \oplus \mathcal{O}_{Y_2}(\mathbf{a})) \to H^0(\mathcal{O}_{Y_1 \cap Y_2}(\mathbf{a}))$$

is *surjective*. Writing the cohomology sequence of (1) we get then

$$(2)\quad h^0(Y_1, L_{\mathbf{a}}) + h^0(Y_2, L_{\mathbf{a}}) = h^0(X, L_{\mathbf{a}}) + h^0(Y_1 \cap Y_2, L_{\mathbf{a}}).$$

We have on the other hand

$$h^0(Y_i, L_{\mathbf{a}}) = s(Y_i, \mathbf{a}), \qquad i = 1, 2$$
$$h^0(Y_1 \cap Y_2, L_{\mathbf{a}}) = s(Y_1 \cap Y_2, \mathbf{a}).$$

Hence, from (2) we get

$$h^0(X, L_{\mathbf{a}}) = s(Y_1, \mathbf{a}) + s(Y_2, \mathbf{a}) - s(Y_1 \cap Y_2, \mathbf{a})$$
$$= s(Y_1 \cup Y_2, \mathbf{a}) \qquad \text{(by Proposition 5.4)}.$$

Hence, we get that

$$h^0(X, L_{\mathbf{a}}) = s(X, \mathbf{a}) \qquad (\text{note } X = Y_1 \cup Y_2).$$

This proves Lemma 6.3.

Remark 6.4. We are interested in proving that

$H^0(X, L_{\mathbf{a}})$ has a basis given by standard monomials on X of type $\mathbf{a}$ $(*)$

for any Schubert *variety* X in G/Q. This will be done by induction on the dimension of X. Let us assume then that $(*)$ holds for any Schubert variety X in G/Q of dimension $\leqslant t$ and then we would like to prove that it holds for a Schubert variety of dimension $(t+1)$. In the course of carrying out this inductive proof we shall require the fact that $(*)$ holds for *any* X in Schub(t) and that scheme-theoretic intersections of members in Schub(t) behave well. The Lemmas 6.1 and 6.3 achieve this purpose.

Remark. 6.5. In Lemma 6.3, if in addition to $(*)$, we suppose also that for every Schubert *variety* X in Schub(d) we have

$$H^i(X, L_{\mathbf{a}}) = 0, \qquad i > 0$$

we see that the same proof gives, in addition, that

$$H^i(X, L_{\mathbf{a}}) = 0, \qquad i > 0 \quad \forall X \in \mathrm{Schub}(d).$$

LEMMA 6.6. *Suppose that for every* $Y \in \mathrm{Schub}(d)$, *one has*

$$h^0(Y, L_{\mathbf{a}}) = s(Y, \mathbf{a}), \qquad \mathbf{a} = (a_1, \dots, a_r), \quad a_i \geqslant 0.$$

Let $w \in W/W_Q$ and $X(w) \in \mathrm{Schub}(d)$. Let $H(w)$ be the closed subscheme o X(w) defined by $H(w) = X(w) \cap \{P_k(\bar{w}) = 0\}$ (cf. *Definition* 5.5). *Then w have*

$$h^0(X(w), L_{\mathbf{a}}) = h^0(X(w), L_{\mathbf{a}'}) + h^0(H(w)_{\mathrm{red}}, L_{\mathbf{a}}) + \sum_{\lambda} h^0(X(\lambda), L_{\mathbf{a}'}) \quad (*$$

where λ is the R.H.S. *of* $(*)$ *and runs over all elements λ such that* (i) *λ is th maximal lift of $\bar{\lambda}$ in w and* (ii) *$(\bar{w}, \bar{\lambda})$ is a non-trivial admissible pair i W/W_{P_1}.*

Proof. This is an immediate consequence of Proposition 5.8 (which i just the relation $(*)$ above with h^0 replaced by s) and Lemma 6.3 fron which it follows that

$$h^0(H(w)_{\mathrm{red}}, L_{\mathbf{a}}) = s(H(w)_{\mathrm{red}}, \mathbf{a}).$$

7. Filtration for the Ideal of $H(w)$

We keep the convention as stated in the beginning of Section 4.

One known that the ample line bundle L_i on G/P_i is in fact *very ampl* (cf. Proposition 3.7). Let us denote by

$$G/P_i \hookrightarrow \mathbf{P}^{m_i}$$

the projective embedding defined by $H^0(G/P_i, L_i)$. Since we have canonical immersion

$$G/Q \hookrightarrow \prod_{i=1}^{r} G/P_i$$

we get a canonical immersion of G/Q (or more generally a Schubert variet $X(w)$ in G/Q) in a multi-projective space as follows:

$$X(w) \hookrightarrow G/Q \hookrightarrow \prod_{i=1}^{r} G/P_i \hookrightarrow \prod_{i=1}^{r} \mathbf{P}^{m_i}.$$

Definition 7.1. Let $w \in W/W_Q$. Then we define the multi-graded rin $R(w)$ as follows:

$$R(w) = \bigoplus_{\mathbf{a}} H^0(X(w), L_1^{a_1} \otimes \cdots \otimes L_r^{a_r}), \qquad \mathbf{a} = (a_1, \ldots, a_r),\ a_i \geq 0.$$

We write

$$R(w)_{\mathbf{a}} = H^0(X(w), L_1^{a_1} \otimes \cdots \otimes L_r^{a_r})$$

$$R(w)_{\mathbf{a}} = 0 \qquad \textit{if } \mathbf{a} = (a_1, ..., a_r),\ a_i \in \mathbf{Z} \text{ with at least one } a_i,\ a_i < 0.$$

When $w_0 \in W/W_Q$ is the unique element of maximal length we write $R(w_0) = R$, i.e.,

$$R = \bigoplus_{\mathbf{a}} H^0(G/Q, L_1^{a_1} \otimes \cdots \otimes L_r^{a_r}).$$

Given $\mathbf{b} = (b_1, ..., b_r)$, $b_i \in \mathbf{Z}$, we define the multi-graded ring $R(w)(\mathbf{b})$ as follows:

$$R(w)(\mathbf{b})_{\mathbf{a}} = R(w)_{\mathbf{a}+\mathbf{b}}.$$

If F is any graded $R(w)$ module, we can associate to it canonically a sheaf of $\mathcal{O}_{X(w)}$-modules, which we denote by $\mathbf{F}$. When $F = R(w)(\mathbf{b})$, we write

$$\mathbf{F} = \mathcal{O}_{X(w)}(\mathbf{b}).$$

We see easily that $\mathcal{O}_{X(w)}(\mathbf{b})$ is the sheaf associated to the line bundle $L_1^{b_1} \otimes \cdots \otimes L_r^{b_r}$ on $X(w)$. (These assertions can be seen, for example, if we use the above immersion of $X(w)$ in a multiprojective space. We see that $\mathcal{O}_{X(w)}(\mathbf{b})$ is the restriction to $X(w)$ of the sheaf $\mathcal{O}_{\mathbf{P}^{m_1}}(b_1) \otimes \cdots \otimes \mathcal{O}_{\mathbf{P}^{m_r}}(b_r)$ on $\mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_r}$, where $\mathcal{O}_{\mathbf{P}^{m_i}}(b_i)$ denotes the usual sheaf in the sense of Serre (cf. [S]$_1$).)

DEFINITION 7.2. Let $w \in W/W_Q$. Recall that (cf. Definition 5.5) $H(w)$ is the scheme $X(w) \cap \{P(\bar{w}) = 0\}$ ($\bar{w}$ = image of w under the canonical map $W/W_Q \to W/W_{P_1}$). We set

$$I(H(w)) = \text{the (multi-graded) ideal } P(\bar{w})\, R(w) \text{ in } R(w).$$

Let M_w be the subset of W/W_{P_1} defined by

$$M_w = \{\bar{\lambda} \mid (\bar{w}, \bar{\lambda}) \text{ is an admissible pair in } W/W_{P_1}\}.$$

We take a total order in M_w (written ord $\bar{\lambda}$, $\bar{\lambda} \in M_w$) such that

$$(\operatorname{codim} \bar{\mu} \geqslant \operatorname{codim} \bar{\lambda}) \Rightarrow \operatorname{ord} \bar{\mu} \geqslant \operatorname{ord} \bar{\lambda} (\bar{\lambda}, \bar{\mu} \in M_w).$$

Then we write the elements of M_w as $\bar{\lambda}_0, \bar{\lambda}_1, ..., \bar{\lambda}_N$ such that ord $\bar{\lambda}_i = i$. Note that $\lambda_0 = w$. We now define the ideals (multigraded) in I_j, $0 \leqslant j \leqslant N$, in $R(w)$ as follows:

$$\begin{aligned}
I_{-1} &= 0 \\
I_0 &= I(H(w)) = P(\bar{w})\, R(w) \\
I_1 &= P(\bar{w})\, R(w) + P(\bar{w}, \bar{\lambda}_1)\, R(w) \\
&\vdots \\
I_j &= \sum_{0 \leqslant i \leqslant j} P(\bar{w}, \bar{\lambda}_i)\, R(w) \\
&\vdots \\
I_N &= \sum_{0 \leqslant i \leqslant N} P(\bar{w}, \bar{\lambda}_i)\, R(w).
\end{aligned}$$

(Note that I_j is B-stable, $0 \leqslant j \leqslant N$ (cf. Corollary 3.19).)

We have $I_0 \subset I_1 \cdots \subset I_N$ and define

$$\mathbf{I}_j = \text{ideal sheaf in } \mathcal{O}_{X(w)} \text{ associated to } I_j,\ 0 \leqslant j \leqslant N.$$

LEMMA 7.3. *Suppose that we have the following*:

(a) *For every* $X \in \text{Schub}(d)$ (cf. Notation 6.2)

$$h^0(X, \mathbf{a}) = s(X, \mathbf{a}), \qquad \mathbf{a} \geqslant 0.$$

(b) *Let Q' be any parabolic subgroup which contains Q properly. If Y is any Schubert variety in G/Q', we have*

$$h^0(Y, \mathbf{a}) = s(Y, \mathbf{a}), \qquad \mathbf{a} \geqslant 0.$$

(c) *For $X(w) \in \text{Schub}(d)$ the special quadratic relations in* $(*)$ *of Section 5 hold.*

Then, if I_j be the ideals of Definition 7.2, *we have the following*:

(1) *Let $F \in R(w)_{\mathbf{a}}$ such that $a_1 > 0$. Then we have*

$$F \in I_N \Leftrightarrow F \text{ vanishes on } H(w)_{\text{red}}.$$

(2) *We have a canonical homomorphism of multi-graded modules* (*of degree* $(0, 0, \ldots, 0)$):

$$f_j\colon I_j/I_{j-1} \to R(\lambda_j)(-1), \quad 0 \leqslant j \leqslant N \ (\textit{note } I_{-1} = (0) \textit{ and } \lambda_0 = w)$$

where λ_j is the maximal representative of $\bar{\lambda}_j$ less than w. Further, f_j induces isomorphisms:

$$(f_j)_{\mathbf{a}}\colon (I_j/I_{j-1})_{\mathbf{a}} \to (R(\lambda_j)(-1))_{\mathbf{a}} \qquad \textit{for} \quad a_1 \geqslant 2.$$

(3) *I_0 and I_j/I_{j-1} have canonical B-module structures (of course $R(\lambda_j)(-1)$ has a canonical B-module structure). Let χ_j be the character of T (i.e., an element of* $\operatorname{Hom}(T, \mathbf{G_m})$) *which defines the 1-dimensional T-module $\{kP_k(\bar{w}, \bar{\lambda}_j)\}$. We can consider this canonically as a B-module through the homomorphism $B \to T = B/B^u$ (B^u = unipotent part of B) and denote by the same χ_j the corresponding element of* $\operatorname{Hom}(B, \mathbf{G_m})$. *Then we have the following:*

$$b \cdot f_j(x) = \chi_j(b)^{-1} f_j(b_x)$$

i.e., f_j is a homomorphism of B-modules up to a twist by a character of B.

Proof. (1) Let $F \in R(w)_{\mathbf{a}}$ with $a_1 > 0$ and F vanish on $H(w)_{\mathrm{red}}$. In view of hypothesis (c), Theorem 5.1 holds for any $X \in \operatorname{Schub}(d)$ and hence in view of hypothesis (a), standard monomials on X of type $\mathbf{a}$ give a basis for $H^0(X, L_{\mathbf{a}})$. Hence we can write

$$F = \sum c_i p(\bar{\alpha}_i, \bar{\beta}_i) F_i, \qquad c_i \in k, \quad c_i \neq 0 \tag{$*$}$$

where $P(\bar{\alpha}_i, \bar{\beta}_i) F_i$ are distinct standard monomials on $X(w)$ and $(\bar{\alpha}_i, \bar{\beta}_i)$ are admissible pairs in W/W_{P_1}. We can suppose without loss of generality that $\bar{\alpha}_1$ is a minimal element among $\{\bar{\alpha}_i\}$. It suffices to show that $\bar{\alpha}_1 = \bar{w}$ (note that $\bar{w} \geqslant \bar{\alpha}_i$). Suppose that this is not the case, i.e., $\bar{w} > \bar{\alpha}_1$. Let α_1 be the maximal representative of α_1 less than w. We see that $X(\alpha_1) \subset H(w)_{\mathrm{red}}$ and hence F vanishes on $X(\alpha_1)$. Now restrict ($*$) to $X(\alpha_1)$. Now on the R.H.S. of ($*$) all the terms such that $\alpha_i \neq \alpha_1$ drop out and the remaining terms are standard on $X(\alpha_1)$ and are distinct. We therefore get a contradiction. This proves the assertion (1) of the above Lemma.

(2) Let F be an element of I_j/I_{j-1} represented by an element of I_j as follows:

$$F = \sum_{0 \leqslant i \leqslant j} P(\bar{w}, \bar{\lambda}_i) F_i.$$

Then we set

$$f_j(F) = F_j \mid X(\lambda_j).$$

We will now show that this defines a well-defined homomorphism of I_j/I_{j-1} into $R(\lambda_j)(-1)$. For this purpose we have only to show that if $F \in I_{j-1}$, then $F_j \mid X(\lambda_j)$ is zero. Suppose then that (on $X(w)$)

$$\sum_{0 \leqslant i \leqslant j} P(\bar{w}, \bar{\lambda}_i) F_i = \sum_{0 \leqslant i \leqslant j-1} P(\bar{w}, \lambda_i) G_i.$$

Multiplying throughout by $P(\bar{w}, \bar{\lambda}_j)$ and cancelling $P(\bar{w})$, this equation gives

$$\sum_{0 \leqslant i \leqslant j-1} (P(\bar{\lambda}_i)\, P(\bar{\lambda}_j))^{1/2} + P(\bar{\lambda}_j)\, F_j = \sum_{0 \leqslant i \leqslant j-1} (P(\bar{\lambda}_i)\, P(\bar{\lambda}_j))^{1/2} G_i.$$

Since $\bar{\lambda}_j \not\geqslant \bar{\lambda}_i$ for $1 \leqslant i \leqslant (j-1)$, the restriction of the above equation to $X(\lambda_j)$ gives that

$$P(\bar{\lambda}_j)\, F_j = 0 \qquad \text{on} \quad X(\lambda_j).$$

Since $P(\bar{\lambda}_j) \neq 0$ on $X(\lambda_j)$, we get that $F_j \mid X(\lambda_j) = 0$. This proves the existence of the required homomorphism f_j.

To prove the required isomorphism concerning $(f_j)_{\mathbf{a}}$, we see that it suffices to prove the following:

(i) $F \in R(w)_{\mathbf{a}},\ a_1 \geqslant 1$ and $\{F \mid X(\lambda_j) = 0\} \Rightarrow P(\bar{w}, \bar{\lambda}_j)\, F \in I_{j-1}$.

Write F as

(ii) $F = \sum_s c_s P(\bar{\alpha}_s, \bar{\beta}_s)\, F_s, \qquad c_s \in k,\ c_s \neq 0$ (on $X(w)$)

where $P(\bar{\alpha}_s, \bar{\beta}_s)\, F_s$ are distinct standard monomials on $X(w)$. We claim that

(iii) $\bar{\lambda}_j \not\geqslant \alpha_s$, $\forall s$, i.e., $\qquad P(\bar{\alpha}_s, \bar{\beta}_s) \mid X(\lambda_j) = 0, \qquad \forall s.$

Suppose this is not the case. Then we can suppose without loss of generality that $\bar{\lambda}_j \geqslant \bar{\alpha}_1$. Then restrict (ii) to $X(\lambda_j)$. Then all the terms on the R.H.S. of (ii) such that $\bar{\lambda}_j \not\geqslant \bar{\alpha}_s$ drop out, i.e., only the terms such that $\bar{\lambda}_j \geqslant \bar{\alpha}_s$ remain and there is at least one such term. Since $F \mid X(\lambda_j) = 0$, we get a contradiction. This proves the claim (iii) above. Thus we see that to prove (i) it suffices to prove the following:

(iv) $P(\bar{w}, \bar{\lambda}_j)\, P(\bar{\alpha}, \bar{\beta}) \in I_{j-1} \qquad$ if $\bar{\lambda}_j \not\geqslant \bar{\alpha}$.

Since $P(\bar{w}, \bar{\lambda}_j)\, P(\bar{\alpha}, \bar{\beta})$ vanishes on $H(\bar{w})_{\text{red}}$ (see Definition 5.5) by the same reasoning as in the proof of the assertion (i) above, we see that on $X(\bar{w})$ we have

(v) $P(\bar{w}, \bar{\lambda}_j)\, P(\bar{\alpha}, \bar{\beta}) = \sum_{i=1}^{t} c_i P(\bar{w}, \bar{\mu}_i)\, P(\bar{v}_i, \bar{\delta}_i), \qquad c_i \neq 0,$

where in the R.H.S. the summation runs over distinct standard monomials on $X(\bar{w}) \subset G/P_1$. It suffices to show that

(vi) $\bar{\mu}_i \geqslant \bar{\lambda}_j \qquad$ and $\qquad \bar{\mu}_i \neq \bar{\lambda}_j \qquad$ for every i, $1 \leqslant i \leqslant t$.

This assertion is a consequence of Lemma 7.1 of [L-M-S]$_3$; however, this is easily proved in the following way. Multiply (v) throughout by $P(\bar{w}, \bar{\mu}_1)$, then cancelling $P(\bar{w}_1)$ (using hypothesis (c)) and restricting to $X(\bar{\mu}_1)$, we obtain the following relation on $X(\bar{\mu}_1)$:

$$\text{(vii)} \quad (P(\bar{\mu}_1)\, P(\bar{\lambda}_j))^{1/2} P(\bar{\alpha}, \bar{\beta}) = c_1 P(\bar{\mu}_1)\, P(\bar{v}_1, \bar{\delta}_1) + \cdots$$

where the R.H.S. is a sum of distinct standard monomials on $X(\bar{\mu}_1)$. The R.H.S. is not zero on $X(\bar{\mu}_1)$ so that the L.H.S. is not zero on $X(\bar{\mu}_1)$, which gives

$$\text{(viii)} \quad \bar{\mu}_1 \geqslant \lambda_1 \qquad \text{and} \qquad \bar{\mu}_1 \geqslant \bar{\alpha}.$$

If $\bar{\mu}_1 = \bar{\lambda}_j$ then we deduce that $\bar{\lambda}_j \geqslant \bar{\alpha}$, which contradicts the hypothesis that $\bar{\lambda}_j \ngeqslant \bar{\alpha}$ and this completes the proof of (2) above.

(3) The fact that I_0 and I_j/I_{j-1} have canonical B-module structures is an immediate consequence of Corollary 3.19. The above isomorphism f_j is "formally" taking $x \in I_j/I_{j-1}$ and dividing it by θ, where θ is the image of $P(\bar{w}, \bar{\lambda}_j)$ in I_j/I_{j-1}. Now the one-dimensional space generated by θ is a B-module and the action of B is given by the character χ_j (by Corollary 3.19). Thus in a formal manner we can write

$$f_j = \frac{x}{\theta},$$

which gives

$$b \cdot f_j(x) = \chi_j(b)^{-1} \frac{(bx)}{\theta} = \chi_j(b)^{-1} f_j(bx).$$

PROPOSITION 7.4. *We assume the hypotheses of Lemma 7.3. Let $w \in W/W_Q$ such that $X(w) \in \mathrm{Schub}(d)$. Let $\mathbf{I}_j$ denote the sheaves of $\mathcal{O}_{X(w)}$-modules of Definition 7.2. Then we have the following*:

(1) *We have canonical isomorphisms (as $\mathcal{O}_{X(w)}$ modules)*

$$f_j\colon \mathbf{I}_j/\mathbf{I}_{j-1} \to \mathcal{O}_{X(\lambda_j)} \qquad (-1, 0, \ldots, 0), \quad 0 \leqslant j \leqslant N$$

λ_j being the maximal representative of $\bar{\lambda}$ less than w.

(2) *We have canonical B actions on I_j/I_{j-1}, $0 \leqslant j \leqslant N$. For $b \in B$, we have*

$$f_j(bx) = \chi_j(b)(b \cdot f_j(x))$$

where χ_j is the element of $\mathrm{Hom}(B, \mathbf{G_m}) \simeq \mathrm{Hom}(T, \mathbf{G_m})$, *associated to the* *weight of* $P_k(\bar{w}, \lambda_1)$ $(= -\frac{1}{2}(w(\omega_1) + \lambda_j(\omega_1)))$. *One can express this by saying that*

$$\mathcal{O}_{X(\lambda_j)}(-1, 0,..., 0) \simeq \chi_j^{-1} \otimes \mathbf{I}_j/\mathbf{I}_{j-1},$$

or

$$\mathbf{I}_j/\mathbf{I}_{j-1} \simeq \chi_j \otimes \mathcal{O}_{X(\lambda_j)} \qquad (-1, 0,..., 0)$$

where the isomorphisms are B-isomorphisms and χ_j *represents the* 1*-dimensional B-module canonically associated to the T-module* $\{kP_k(\bar{w}, \lambda_j)\}$.

(3) *The ideal sheaf* $\mathbf{I}(H(w)_{\text{red}})$ *in* $\mathcal{O}_{X(w)}$ *associated to the closed subscheme* $H(w)_{\text{red}}$ *of* $X(w)$ *is precisely* $\mathbf{I}_N$.

Proof. The homomorphisms $(f_j)_{\mathbf{a}}$ of Lemma 7.3 are isomorphisms for a_j sufficiently large. Hence by an easy generalization of the classical results of Serre (cf. $[\mathrm{S}]_1$) to the multi-projective case, it follows that f_j induce isomorphisms at the sheaf level and that $\mathbf{I}_N = \mathbf{I}(H(w)_{\text{red}})$. Thus Proposition 7.4 follows immediately from Lemma 7.3.

8. The Variety Z_ϕ

Let Q be any parabolic subgroup of G.

Let $X(\phi)$ be a Schubert divisor in a Schubert variety $X(\tau)$, moved by a simple root α with $\phi, \tau \in W/W_Q$. Set

$$B_\alpha = B \cap SL(2, \alpha)$$

where $SL(2, \alpha)$ is the "$SL(2)$" associated to the simple root α. One knows that for the canonical action of $SL(2, \alpha)$ on G/Q (induced by the canonical action of G on G/Q), $X(\tau)$ remains stable (cf. Lemma 1.2). Of course, any Schubert variety in G/Q remains stable under the action of B_α.

For any line bundle L on G/Q, we denote by the same L its restriction to a subscheme of G/Q (in particular a Schubert variety) when there is no confusion. We observe that $SL(2, \alpha)$ acts on L consistent with its action on $X(\tau)$. Similarly, B_α acts on L consistent with its action on $X(\phi)$.

Definition 8.1. Let ϕ, τ, α, and L be as above. We set:

(i) $Z_{\phi,\tau}$ = the fibre product $SL(2, \alpha) \times^{B_\alpha} X(\phi)$, i.e., $Z_{\phi,\tau}$ is the quotient variety modulo the equivalence relation in $SL(2, \alpha) \times X(\phi)$ defined by

$$(g, x) \sim (gb, b^{-1}x); \qquad g \in SL(2, \alpha), b \in B_\alpha, x \in X(\phi).$$

Sometimes we shall denote $Z_{\phi,\tau}$ by $Z_{\phi,\alpha}$ also or just by Z_ϕ, when there is no room for confusion. Note that if we define the action of B_α on $SL(2, \alpha) \times X(\phi)$ by

$$(g, x) \circ b = (gb, b^{-1}x)$$

then this action is a free action and Z_ϕ is the orbit space $(SL(2, \alpha) \times X(\phi))/B_\alpha$.

(ii) p is the canonical morphism

$$p: Z_\phi \to \mathbf{P}^1 = SL(2, \alpha)/B_\alpha.$$

Note that Z_ϕ is the fibre space with fibre $X(\phi)$, associated to the principal fibration $SL(2, \alpha) \to SL(2, \alpha)/B_\alpha$ with structure group B_α.

(iii) Ψ (or Ψ_ϕ if one wants to be precise) is the canonical morphism

$$\Psi: Z_\phi \to X(\tau) \subsetneq G/Q$$

defined by

$$(g, x) \mapsto g \cdot x, \qquad g \in SL(2, \alpha),\ x \in X(\phi)$$

(note that this map is well-defined since

$$\Psi(y) = \Psi(z), \qquad y \sim z \text{ (equivalence relation in (i) above)).}$$

(iv) $\tilde{L}$ is the line bundle on Z_ϕ, associated (in the sense of fibre spaces for the principal B_α fibration $SL(2, \alpha) \to \mathbf{P}^1$) to the B_α-line bundle $L \mid X(\phi)$, with its canonical B_α-action, i.e.,

$$\tilde{L} = SL(2, \alpha) \times^{B_\alpha} L.$$

More generally, let F be a coherent $\mathcal{O}_{X(\phi)}$-module with a B_α-action compatible with the action of B_α on $X(\phi)$. We set

$$\tilde{F} = SL(2, \alpha) \times^{B_\alpha} F.$$

Let q_1, q_2 be the canonical maps

$$q_1: SL(2, \alpha) \times X(\phi) \to X(\phi), \qquad q_2: SL(2, \alpha) \times X(\phi) \to Z_\phi,$$

then $q_1^*(F) \approx q_2^*(\tilde{F})$, i.e., $\tilde{F}$ is the sheaf on Z_ϕ to which the sheaf $q_1^*(F)$ on $SL(2, \alpha) \times X(\phi)$ (having a B_α-action) descends. An important example is when F is an ideal sheaf or the structure sheaf of a subscheme Y of $X(\phi)$ which is B_α-stable. Then $\tilde{Y}$ is a subscheme of Z_ϕ. It is easily seen as a consequence of the above remarks that if Y_1, Y_2 are subschemes of $X(\phi)$ which are B_α-stable, then $\widetilde{Y_1 \cap Y_2} = \tilde{Y}_1 \cap \tilde{Y}_2$ (scheme-theoretic intersections on

Z_ϕ). Further if Y is a *reduced* B_α-stable subscheme of $X(\phi)$, we see that $\tilde{Y}$ is also a reduced subscheme of Z_ϕ.

(v) $\bar{e}$ denotes the point of $\mathbf{P}^1$ corresponding to the coset eB_α ($e =$ identity of G).

(vi) For *any* $\lambda \in W/W_Q$ such that $\phi \geqslant \lambda$

$$Z_\lambda = SL(2, \alpha) \times^{B_\alpha} X(\lambda).$$

Note that we can identify $X(\phi)$ as the fibre of p over $\bar{e}$. Then Ψ induces an isomorphism of this fibre with $X(\phi) \subsetneq X(\tau)$. We see also that $Z_\lambda \subsetneq Z_\phi$ and we have the following commutative diagram:

$$\begin{array}{ccc} Z_\lambda & \hookrightarrow & Z_\phi \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathbb{P}^1 & \end{array}$$

where the same p denotes the canonical morphisms $Z_\lambda \to \mathbf{P}^1$ and $Z_\phi \to \mathbf{P}^1$ ($\mathbf{P}^1 = SL(2, \alpha)/B_\alpha$).

PROPOSITION 8.2. *With notations as above, we have*

$$\tilde{L} \simeq \psi^*(L).$$

Proof. Recall the following general fact on fibre spaces. Let H be a group, K a subgroup, and e the point of H/K representing K. Let W be a space on which H operates and $p: W \to H/K$ a map which is an H-morphism. If W_e denotes the fibre of W over e, we see that H operates on W_e. Then we see that the canonical map $H \times^K W_e$ defined by $(h, w) \mapsto h, w$ ($h \in H$, $w \in W_e$) is an isomorphism.

We now observe that ψ is an $SL(2, \alpha)$-morphism, i.e., ψ commutes with the canonical actions of $SL(2, \alpha)$ on Z_ϕ and $X(\tau)$.
Hence we get a canonical action of $SL(2, \alpha)$ on $\psi^*(L)$ compatible with the $SL(2, \alpha)$-action on Z_ϕ. On the other hand, ψ induces a B_α-isomorphism of $L \mid p^{-1}(e)$ with $L \mid X(\phi)$, which proves (by the above generalities on fibre spaces) that $\psi^*(L) \approx$ the object on Z_ϕ associated to the B_α-object $L \mid X(\phi)$, i.e., $\psi^*(L) \approx \tilde{L}$. This proves Proposition 8.2.

DEFINITION 8.3. Let M be a line bundle on Z_ϕ or more generally Z_λ (with notations as above, especially Definition 8.1). Then we set

$$M^{(m)} = M \otimes p^*(\mathcal{O}_{\mathbf{P}^1}(m)), \qquad m \in \mathbf{Z}$$

where $\mathcal{O}_{\mathbf{P}^1}(m)$ denotes the line bundle on $\mathbf{P}^1$ in the sense of Serre (cf. [S]$_1$). We are especially interested in the case $M = \Psi^*(L) \approx \tilde{L}$.

PROPOSITION 8.4. *With the above notations* (*especially Definition* 8.1), *we have the following*:

(i) *If $X(\lambda)$ is moved by α, i.e.,*

$$\lambda < s_\alpha \lambda = \mu \qquad \text{in} \quad W/W_Q$$

Ψ_λ maps Z_λ onto $X(\mu)$.

(ii) *If $X(\lambda)$ is not moved by α, i.e., $\lambda \geqslant s_\alpha \lambda$ in W/W_Q, then $X(\lambda)$ is stable under the canonical action of $SL(2, \alpha)$ on G/Q. Then Ψ_λ maps Z_λ onto $X(\lambda)$. In this case, the fibre space $p\colon Z_\lambda \to \mathbf{P}^1$* splits (*i.e., $\approx \mathbf{P}^1 \times X(\lambda)$*) *and $\Psi^*(L)$ can be identified with the pull-back of L (rather $L \mid X(\lambda)$) on the factor $X(\lambda)$ of Z_λ). Further for any L as above, one has*

(a) $H^i(Z_\lambda, (\Psi^* L)^{(-1)}) = 0$, *for all i,*

(b) $H^0(Z_\lambda, \Psi^*(L)) \approx H^0(X(\lambda), L)$.

Proof. The assertion (i) is immediate. In the assertion (ii) it is well known that $X(\lambda)$ is stable under the action of $SL(2, \alpha)$ on G/Q (cf. Lemma 1.3). It is also a well-known fact (and proved easily) that if one is given an $SL(2, \alpha)$ action on a variety Y (in particular B_α acts on Y) and we take the fibre space, associated to the B_α-principal fibre space $SL(2, \alpha) \to SL(2, \alpha)/B_\alpha$, then the corresponding fibre space splits. It follows also that $\Psi^*(L)$ comes from the factor $X(\lambda)$. Using this and the fact that

$$H^i(\mathcal{O}_{\mathbf{P}^1}(-1)) = 0, \qquad i \geqslant 0$$

the assertion (a) of (ii) follows by using the Künneth formula. The assertion (b) of (ii) is immediate (for example, again by the Künneth formula).

LEMMA 8.5. *Any fibre of $\psi\colon Z_\phi \to X(\tau)$ is either a point or* $\mathbf{P}^1$ (*set theoretically*).

Proof. This is a fairly known result. For the case of completeness, we sketch a proof in the case $Q = B$. Let $e(\theta)$ be the point of $X(\tau)$ corresponding to θ, $\theta \leqslant \tau$. Because of T-equivariance, it suffices to show that $\psi^{-1}(e(\theta))$ is either a point or $\mathbf{P}^1$ (set theoretically). Now it is easy to see that

(i) $SL(2, \alpha)\, e(\theta) \approx SL(2, \alpha)/B_\alpha \approx SL(2, \alpha)/B_{-\alpha} \approx \mathbf{P}^1$,

(ii) we have two possibilities, either

(a) $SL(2, \alpha)\, e(\theta) \subset X(\phi)$ or

(b) $SL(2, \alpha)\, e(\theta) \cap X(\phi) =$ one point.

From this it follows that

$$\begin{aligned} \psi^{-1}(e, \theta)) &= \mathbf{P}^1 && \text{if (ii)(a) holds} \\ &= \text{a point} && \text{if (ii)(b) holds.} \end{aligned}$$

From this the lemma follows.

PROPOSITION 8.6 (cf. [C]). *Schubert varieties are non-singular in codimension one.*

Proof. It suffices to show that given $w, w' \in W$ such that $w' < w$ and $l(w') = l(w) - 1$, $X(w)$ is smooth at the point $e(w')$ of $X(w)$ corresponding to w'. We prove this by decreasing induction on dim $X(w)$. If there exists a simple root α such that $w = s_\alpha w'$, then taking a lift n_α in $N(T)$ for $s_\alpha \in W = N(T)/T$, multiplication on the left by n_α defines an automorphism of $X(w)$ which maps $e(w)$ to $e(w')$. (Note that, since $l(s_\alpha w) < l(w)$, $X(w)$ is stable for multiplication on the left by elements of the minimal parabolic P_α (cf. Lemma 1.2).) Hence the results follows in this case (since $X(w)$ is smooth at $e(w)$). In the other case, let β be a simple root such that $w' < s_\beta w'$. Then w is $< s_\beta w$ (for $w > s_\beta w$ would imply $w > s_\beta w'$ which is not possible, since $l(w) = l(s_\beta w')$ and $w \neq s_\beta w'$). Let $\tau = s_\beta w$ and $Z_w = SL(2, \beta) \times^{B_\beta} X(w)$. By induction hypothesis $X(\tau)$ is smooth at $e(s_\beta w')$. Further, since $SL(2, \beta)\, e(w') \not\subseteq X(w)$, we have that $\psi^{-1}(e(s_\beta w'))$ is a point, namely, the point $(s_\beta, e(w'))$ (cf. Lemma 8.5, ψ or ψ_w being the map $\psi\colon Z_w \to X(\tau)$). Hence by Zariski's main theorem ψ induces an isomorphism in a neighborhood of $(s_\beta, e(w'))$; in particular, $(s_\beta, e(w'))$ is smooth on Z_ϕ. This implies that the fibre, say, Y, of $Z_w \to \mathbf{P}^1$ through $(s_\beta, e(w'))$ is smooth at $(s_\beta, e(w'))$. Now multiplication by s_β induces an isomorphism of $X(w)$ onto Y under which $e(w')$ is mapped into $(s_\beta, e(w'))$. Hence we obtain that $X(w)$ is smooth at $e(w')$. This completes the proof of Proposition 8.6.

The notations ϕ, τ, Z_ϕ, etc., being as above, let us denote $X(\bar{\tau})$ (resp.

$X(\bar{\phi})$) the projection of $X(\bar{\tau})$ (resp. $X(\phi)$) under $G/Q \to G/P_1$. Let us recall (cf. Definition 5.5) that

$$H(\phi)=X(\phi)\cap\{p(\bar{\phi})=0\}$$
$$H(\tau)=X(\tau)\cap\{p(\bar{\tau})=0\}$$
$$H(\bar{\phi})=X(\bar{\phi})\cap\{p(\bar{\phi})=0\}$$
$$H(\bar{\tau})=X(\bar{\tau})\cap\{p(\bar{\tau})=0\}.$$

With these notations, we have the following:

LEMMA 8.7. *Let $X(\theta)\subseteq H(\tau)_{\mathrm{red}}$, $\theta\in W/W_Q$. Then one of the following alternative holds*:

(i) *either $X(\theta)\subseteq X(\phi)$, or*

(ii) $\theta=s_\alpha\lambda$, $X(\lambda)\subseteq H(\phi)_{\mathrm{red}}$.

If (i) *does not hold, $X(\lambda)$ is a divisor in $X(\theta)$, moved by α.*

Proof. By Lemma 1.5, if (i) does not hold, we can find $\lambda\in W/W_Q$ such that $\theta=s_\alpha\lambda$ and $X(\lambda)\subseteq X(\phi)$. Observe that $X(\bar{\lambda})\subseteq X(\bar{\phi})$. To prove (ii), it suffices to show that $X(\bar{\lambda})$ is strictly contained in $X(\bar{\phi})$, for this will show that $X(\bar{\lambda})\subseteq H(\bar{\phi})_{\mathrm{red}}$ (one knows that (cf. Lemma 5.6) the union of all the proper Schubert subvarieties of $X(\bar{\phi})$ is $H(\bar{\phi})_{\mathrm{red}}$) and by definition $H(\phi)_{\mathrm{red}}$ is the inverse image (set theoretically) of $H(\bar{\phi})_{\mathrm{red}}$ under the canonical morphism $X(\phi)\to X(\bar{\phi})$. Suppose then that $X(\bar{\lambda})=X(\bar{\phi})$, i.e., $\bar{\lambda}=\bar{\phi}$ in W/W_{P_1}. Then we have $\bar{\theta}=s_\alpha\bar{\phi}$. By our choice of ϕ, $s_\alpha\bar{\phi}=\bar{\tau}$ so that we would get $\bar{\theta}=\bar{\tau}$. But by our choice $X(\bar{\theta})\subseteq H(\bar{\tau})_{\mathrm{red}}$, i.e., $\bar{\theta}<\bar{\tau}$ (and $\bar{\theta}\neq\bar{\tau}$). This leads to a contradiction. Thus $\bar{\lambda}<\bar{\phi}$ and this proves Lemma 8.7.

DEFINITION 8.8. If D is any closed subscheme of $X(\phi)$, we denote by $\mathbf{I}(D)$ the sheaf of ideals on $X(\phi)$, defined by D, and we employ this notation in a more general situation, for example, on Z_ϕ, $X(\tau)$, etc., Recall that if D is a closed subscheme of $X(\phi)$, which is B_α-stable, $\tilde{D}$ (resp. $\widetilde{\mathbf{I}(D)}$) denotes the subscheme on Z_ϕ (resp. ideal sheaf on Z_ϕ), defined by (cf. Definition 8.1)

$$\tilde{D}=SL(2,\alpha)\times^{B_\alpha}D,\qquad \widetilde{\mathbf{I}(D)}=SL(2,\alpha)\times^{B_\alpha}\mathbf{I}(D).$$

We now set $H(Z)$ to be the closed subscheme of Z_ϕ whose ideal sheaf $\mathbf{I}(H(Z))$ is given by $\mathbf{I}(H(Z))=\widetilde{\mathbf{I}(H(\phi))}\otimes M$ where M is the mth tensor power of $\mathbf{I}(X(\phi))$, $m=\langle\phi(\omega),\alpha^*\rangle$. (Here we identify $X(\phi)$ as the closed subscheme $p^{-1}(\bar{e})$ of Z_ϕ, i.e., the fibre of p over the point $\bar{e}$ of $\mathbf{P}^1$, $\bar{e}=$ coset eB_α, e being the identity element of $SL(2,\alpha)$.)

LEMMA *With notations as above, we have*

$$\psi^{-1}(H(\tau)) = H(Z) \qquad (\textit{set theoretically}).$$

Proof. We have

$$Z_\phi = P_\alpha \times^B X(\phi)$$

where P_α is the minimal parabolic subgroup associated to α. Further, $\psi: Z_\phi \to X(\tau)$ is P_α-equivariant. The subscheme $H(\tau)$ of $X(\tau)$ being the zero set of $p(\bar{\tau})$ in $X(\tau)$ is locally principal and hence $\psi^{-1}(H(\tau))$ is the underlying set of a locally principal closed subscheme of Z_ϕ. Hence we conclude that the irreducible components of $\psi^{-1}(H(\tau))$ are of codimension one in Z_ϕ. Now we have

$$H(\tau) = X(\phi) \cup S \qquad (\text{set theoretic})$$

where an irreducible component of S is of the form $X(\mu)$ where $X(\mu)$ is P_α-stable (an easy consequence of Lemma 1.5). In particular S is P_α-stable and hence $\psi^{-1}(S)$ is also P_α-stable. Since Z_ϕ is the fibre space associated to the principal fibration $P_\alpha \to P_\alpha/B \approx \mathbf{P}^1$, we deduce easily that $\psi^{-1}(S) = \tilde{T}$, where T is a closed B-stable subset of $X(\phi)$ of codimension 1. Hence $T \subseteq H(\phi)$ (note that for an irreducible component $X(\lambda)$ of T, $\bar{\lambda} \lneqq \bar{\phi}$; for $\bar{\lambda} = \bar{\phi}$ would imply $s_\alpha \bar{\lambda} = \bar{\tau}$, which in turn would imply $\psi(\widetilde{X(\lambda)}) \not\subseteq H(\tau)$, which would contradict the fact that $\psi(\tilde{T}) \subseteq H(\tau)$). Now, it is not difficult to see that $\psi^{-1}(X(\phi))$ is the union of $X(\phi)$ and all $\widetilde{X(\lambda)}$ where λ is such that $X(\lambda)$ is of codimension one in $X(\phi)$ and $X(\lambda)$ is stable under $SL(2, \alpha)$ (cf. Lemma 8.5). Hence $\psi^{-1}(X(\phi))$ is $\subseteq H(Z)$ and thus $\psi^{-1}(H(\tau)) \subseteq H(Z)$. On the other hand it is clear (using Lemma 8.7) that ψ maps $H(Z)$ onto $H(\tau)$. This completes the proof of Lemma 8.9.

LEMMA 8.10. *With notations as above, we have*:

(i) $\widetilde{H(\phi)}_{\mathrm{red}}$ *is a reduced subscheme of* Z_ϕ *so that* $H(Z)_{\mathrm{red}} = X(\phi) \cup \widetilde{H(\phi)}_{\mathrm{red}}$. *The irreducible components of* $\widetilde{H(\phi)}_{\mathrm{red}}$ *are distinct from* $X(\phi)$.

(ii) $\mathcal{O}_{Z_\phi}^{(-1)} \simeq \mathbf{I}(X(\phi))$ (*here* $X(\phi)$ *is considered as a subscheme of* Z_ϕ *as mentioned above*).

(iii) $\widetilde{I(H(\phi))}_{\mathrm{red}}^{(-1)} \simeq \mathbf{I}(H(Z)_{\mathrm{red}})$.

(*Note that* (ii) *implies that* $\mathbf{I}(H(Z)) = \widetilde{\mathbf{I}(H(\phi))}^{(-m)}$, $m = \langle \phi(\omega), \alpha^* \rangle$.)

Proof. The assertions (i) and (ii) are immediate. We have only to prove (iii). Let us write

$$J_1=I(\widetilde{H(\phi)}_{\mathrm{red}}),\qquad J_2=I(X(\phi)).$$

Now (iii) would follow if we show that $I(H(Z)_{\mathrm{red}})=J_1J_2$. This is a simple consequence of the fact that J_2 is *locally principal.* First of all, it is obvious that $J_1J_2\subseteq I(V)$. Hence we have only to show that $I(H(Z)_{\mathrm{red}})\subseteq J_1J_2$. Now if $f\in I(H(Z)_{\mathrm{red}})$, we can write *locally*

$$f=\theta g,\ \theta \text{ defines } X(\phi) \text{ locally,}$$

$$g \text{ a section of } \mathcal{O}_{Z_\phi} \text{ locally.}$$

By the fact that $X(\phi)$ is distinct from the irreducible components of $\widetilde{H(\phi)}_{\mathrm{red}}$, it follows that g vanishes on $\widetilde{H(\phi)}_{\mathrm{red}}$, i.e., $g\in J_1$. This proves that $f\in J_1J_2$ so that $I(H(Z)_{\mathrm{red}})\subseteq J_1J_2$. This proves (iii) and the proof of Lemma 8.10 is complete.

LEMMA 8.11. *Let T be a closed B_α-stable subscheme of $X(\phi)$. Then we have $\tilde{T}\cap X(\phi)=T$ (scheme theoretically), $X(\phi)$ being canonically identified as a subscheme of Z_ϕ.*

Proof. This is simple and can be seen as follows: We have

$$(SL(2,\alpha)\times T)\cap(B_\alpha\times X(\phi))=B_\alpha\times T\qquad\text{(scheme-theoretic).}\qquad(*)$$

Also,

$$SL(2,\alpha)\times^{B_\alpha}T=\tilde{T},\qquad B_\alpha\times^{B_\alpha}X(\phi)=X(\phi),\qquad B_\alpha\times^{B_\alpha}T=T.$$

The equality $(*)$ is preserved by taking quotients modulo B_α and thus we get the required assertion. We shall later need the following result.

LEMMA 8.12. *Notations being as above, suppose that $X(\tau)$ is normal. Then we have the following*:

(a) $\psi_*(\mathcal{O}_{Z_\phi})=\mathcal{O}_{X(\tau)}$,

(b) $R^i\psi_*(\mathcal{O}_{Z_\phi})=0,\ i>0$,

(c) *$H^i(X(\tau),M)\to H^i(Z_\phi,\psi^*(M))$ is an isomorphism for all vector bundles M on $X(\tau)$.*

Proof. We observe that (c) is an immediate consequence of (a) and (b). Further since $X(\tau)$ is normal, assertion (a) is also clear. Hence we have only to prove (b). To prove (b), we claim that it suffices to prove that

$$H^i(Z_\phi,\psi^*(L^n))=0,\qquad i>0,\quad n\gg 0\qquad(*)$$

where L is an ample line bundle on $X(\tau)$. To see this, we first observe that the Leray spectral sequence

$$\begin{aligned} &H^p(X(\tau), R^q\psi_*(\mathcal{O}_{Z_\phi})\otimes L^n) \\ &\quad = H^p(X(\tau), R^q\psi_*(\psi^*(L^n))) \Rightarrow H^{p+q}(Z_\phi, \psi^*(L^n)) \end{aligned}$$

degenerates for $n \gg 0$ (since $H^i(X(\tau), R^q\psi_*(\mathcal{O}_{Z_\phi})\otimes L^n) = n \gg 0$) (by $[S]_1$). Hence

$$H^0(X(\tau), R^q\psi_*(\mathcal{O}_{Z_\phi})\otimes L^n) \approx H^q(Z_\phi, \psi^*(L^n)), \qquad n \gg 0.$$

Hence if $(*)$ holds, then we would obtain

$$H^0(X(\tau), R^q\psi_*(\mathcal{O}_{Z_\phi})\otimes L^n) = 0, \qquad n \gg 0, \quad q > 0.$$

Now this implies that $R^q\psi_*(\mathcal{O}_{Z_\phi}) = 0$, $q > 0$. This proves the claim that it suffices to prove $(*)$ above. To prove $(*)$, we first observe that

$$H^0(X(\phi), L^n) \approx (V_{n\theta}(\phi))^*, \qquad n \gg 0$$

where θ is the (dominant) character associated to L. This is because we have (cf. $[S]_1$) that for $n \gg 0$ canonical map

$$(V_{n\theta}(\phi))^* \to H^0(X(\phi), L^n) \text{ is surjective} \tag{$**$}$$

and that

$$H^i(X(\phi), L^n) = 0.$$

This in particular implies that $\dim H^0(X(\phi), L^n)$ is the same in all characteristics for $n \gg 0$. Now denoting $\operatorname{Im}(V_{n\theta}(\phi))$, the image of $V_{n\theta,\mathbf{Z}}(\phi)\otimes_{\mathbf{Z}} k$ under $V_{n\theta,\mathbf{Z}}(\phi)\otimes k \to V_{n\theta,\mathbf{Z}}\otimes k$, we have (cf. Lemma 3.12) that the image of the map $(**)$ above can be identified with $(\operatorname{Im}(V_{n\theta}(\phi)))^*$. Hence we obtain $\dim \operatorname{Im}(V_{n\theta}(\phi))$ is the same in all characteristics; in particular, we obtain that $V_{n\theta,\mathbf{Z}}(\phi)$ is a direct summand in $V_{n\theta,\mathbf{Z}}$ and hence the map $(**)$ above is in fact an isomorphism. Now we consider the morphism $p\colon Z_\phi \to \mathbf{P}^1$, whose fibres are $\approx X(\phi)$. Hence for $n \gg 0$, we have

$$H^i(Z(\phi), \psi^*(L^n)) \approx H^i(\mathbf{P}^1, p_*(\psi^*(L^n)))$$

(since $H^i(X(\phi), L^n) = 0$, $n \gg 0$) (cf. $[S]_1$). Also, $p_*(\psi^*(L^n))$ gets identified with the vector bundle F on $\mathbf{P}^1$ associated to the B-module $H^0(X(\phi), L^n)$ and for $n \gg 0$, $H^0(X(\phi), L^n) \approx (V_{n\theta}(\phi))^*$ (as observed above). Denoting by E the bundle on $\mathbf{P}^1$ associated to $v^*_{n\theta}$, we have that E is trivial (since $V^*_{n\theta}$ is an $SL(2, \alpha)$-module). Considering the exact sequence of vector bundles on $\mathbf{P}^1$

$$0 \to K \to E \to F \to 0 \tag{$\dagger$}$$

(K being the kernel of $E\to F$). We have $H^i(\mathbf{P}^1, E)=0$, $H^{i+1}(\mathbf{P}^1, K)=0$, $i\geqslant 1$. Hence writing down the cohomology exact sequence of (†) we deduce that $H^i(\mathbf{P}^1, F)=0$ ($n\gg 0$) which proves that $H^i(Z_\phi, \psi^*(L^n))=0$, $n\gg 0$. This completes the proof of Lemma 8.12.

PROPOSITION 8.13. *Let ω be a fundamental weight of classical type (cf. Definition 2.2). Further, let $\tau\in W/Wp$ and $X(\phi)$ a moving divisor moved by α. Suppose*

(a) $H^i(X(\phi), L_\omega)=0$, $i\geqslant 1$,

(b) $H^0(X(\phi), L_\omega)=(V_\omega(\phi))^*$,

then

$$H^0(X(\tau), L_\omega)\approx H^0(Z_\phi, \psi^*(L_\omega))$$

(*for any field k*).

Proof. We proceed as in the proof of (∗) in Lemma 8.12, namely, we consider $p\colon Z_\phi\to\mathbf{P}^1$, whose fibres are $\approx X(\phi)$, then hypothesis (a) implies

$$H^i(Z_\phi, \psi^*(L_\omega))\approx H^i(\mathbf{P}^1, p_*(\psi^*(L_\omega)))\qquad \text{for all } i. \tag{1}$$

Then we use hypothesis (b) (and Corollary 3.7) and obtain the exact sequence

$$0\to K\to E\to F\to 0$$

where E (resp. F) is the vector bundle on $\mathbf{P}^1$ associated to the B_α-module $(V_\omega(\tau))^*$ (resp. $(V_\omega(\phi))^*$). Now using the fact that E is trivial (since $V_\omega(\tau)$ is an $SL(2,\alpha)$-module) we obtain

$$H^i(\mathbf{P}^1, F)=0,\qquad i\geqslant 1. \tag{2}$$

Now for any B-module W, we shall denote by $\mathscr{W}$ the vector bundle on $\mathbf{P}^1=SL(2,\alpha)/B_\alpha$ associated to the principal B_α-fibration $SL(2,\alpha)\to\mathbf{P}^1$ and we shall denote

$$\chi(\mathscr{W})=\operatorname{char} H^0(\mathbf{P}^1, \mathscr{W})-\operatorname{char} H^1(\mathbf{P}^1, \mathscr{W})$$

(where for any T-module M, char M denotes the character of M). Taking $W=(V_\omega(\phi))^*$, we have

$$\chi(W)=\operatorname{char} H^0(\mathbf{P}^1, (\mathscr{V}_\omega(\phi))^*) \tag{3}$$

(in view of (2)). Now we claim that

for any finite-dimensional B module (4)

$$\chi(\mathscr{W})=M_{s_\alpha}(\operatorname{char} W).$$

For this, we first observe that it is enough to prove the claim (4) in the case of a 1-dimensional B-module (since any finite-dimensional B-module W has a filtration by B-submodules such that its associated graded is a direct sum of 1-dimensional B-modules and χ is additive with respect to exact sequence, i.e., given an exact sequence $0 \to W_1 \to W \to W_2 \to 0$ of B-modules, $\chi(\mathscr{W}) = \chi(\mathscr{W}_1) + \chi(\mathscr{W}_2)$ etc.). Let then the 1-dimensional B-module W be given by $\mu \in \operatorname{Hom}(T, \mathbf{G_m})$; then we have

$$\begin{aligned} M_{s_\alpha}(\exp \mu) &= \exp \rho \cdot L_{s_\alpha}(\exp(\mu - \rho)) \qquad \text{(cf. Section 3 for the operator } L_{s_\alpha}) \\ &= \exp \rho \cdot \frac{\exp(\mu - \rho) - \exp(s_\alpha(\mu - \rho))}{1 - \exp \alpha} \\ &= \frac{\exp \mu - \exp(s_\alpha(\mu) + \alpha)}{1 - \exp \alpha} \\ &\qquad \text{(since } s_\alpha(\rho) = \rho - \alpha, \text{ for a simple root } \alpha) \\ &= \frac{\exp(\mu - \alpha/2) - \exp(s_\alpha(\mu - \alpha/2))}{\exp(-\alpha/2) - \exp \alpha/2}. \end{aligned}$$

Now the expression on the R.H.S. of the last equality can easily be seen to be char $H^0(\mathbf{P}^1, L_\mu) -$ char $H^1(\mathbf{P}^1, L_\mu)$, L_μ being the line bundle on $\mathbf{P}^1$, associated to μ. From this (and the remarks made already) claim (4) follows. Hence we obtain (using (1), (3), and (4)) that

$$\operatorname{char} H^0(Z_\phi, \psi^*(L_\omega)) = M_{s_\alpha}(\operatorname{char}(V_\omega(\phi))^*). \tag{5}$$

Now, in view of Proposition 3.9(iii) we have

$$\operatorname{char}(V_\omega(\phi))^* = M_\phi(\exp(-\omega)). \tag{6}$$

Hence from (5) and (6) we obtain

$$\operatorname{char} H^0(Z_\phi, \psi^*(L_\omega)) = M_\tau(\exp(-\omega)). \tag{7}$$

On the other hand (again by Proposition 3.11(iii)) we have

$$\operatorname{char}(V_\tau(\omega))^* = M_\tau(\exp(-\omega)). \tag{8}$$

Hence from (7) and (8), we obtain

$$H^0(Z_\phi, \psi^*(L_\omega)) \approx (V_\tau(\omega))^*. \tag{9}$$

Now the required result follows from (9) and the injections

$$j_k : (V_\omega(\tau))^* \hookrightarrow H^0(X(\tau), L_\omega)$$

(cf. Remark 3.13) and

$$H^0(X(\tau), L) \subsetneq H^0(Z_\phi, L)$$

(for any ample line bundle L). This completes the proof of Proposition 8.13.

COROLLARY 8.14. *Hypotheses being as in Proposition* 8.13, *we have*

$$H^0(X(\tau), L_\omega) \approx (V_\omega(\tau))^*.$$

Proof. This result follows from Proposition 8.13 and (9) in the proof of Proposition 8.13.

COROLLARY 8.15. *Hypothesis being as in Proposition* 8.13, *we have that* $\{p(\lambda, \mu)/\tau \geqslant \lambda\}$ *is a basis for* $H^0(X(\tau), L_\omega)$.

The proof follows from Remark 3.16(i) and Corollary 8.14.

9. THE MAIN THEOREM

DEFINITION 9.1. Let

$$\mathbf{I}_0 = \mathbf{I}(H(\phi)) \subset \mathbf{I}_1 \subset \cdots \subset \mathbf{I}_N \tag{1}$$

be the sequence of ideal sheaves on $X(\phi)$ as in Definition 7.2. Let $X(\phi) \in \text{Schub}(d)$ and further let us assume that the hypothesis in Lemma 7.3 holds. Recall that $\mathbf{I}_N = \mathbf{I}(H(\phi)_{\text{red}})$ (cf. Lemma 7.3 and Proposition 7.4). With the usual conventions (cf. Section 8) we denote

$$\tilde{I}_0 \subset \tilde{I}_1 \subset \cdots \subset \tilde{I}_N \tag{2}$$

the sequence of ideal sheaves on Z_ϕ associated to (1).

LEMMA 9.2. *With the above notations and assumptions we have the following*:

$$\tilde{\mathbf{I}}_j/\tilde{\mathbf{I}}_{j-1} \simeq \Psi^*(L_1^{-1})^{(m)} | Z_{\lambda_j}, \qquad 0 \leqslant j \leqslant N$$

where

$$m = \tfrac{1}{2}\langle \phi(\omega_1) + \lambda_j(\omega_1), \alpha^* \rangle.$$

(Recall that L_1 is the line bundle (or the ideal sheaf) associated to the fundamental weight ω_1.) We identify Z_{λ_j} with the closed subvariety $\widetilde{X(\lambda_j)}$

of Z_ϕ. Recall also that $(\bar{\phi}, \bar{\lambda}_j)$ is an admissible pair on $X(\bar{\phi})$, $\lambda_0 = \phi$, and that λ_j is the maximal representative of $\bar{\lambda}_j$ less than ϕ. Further I_{-1} and $\tilde{I}_{-1}$ are (0) and thus in particular, the lemma implies that $\mathbf{I}(H(Z))$ $(\approx \widetilde{\mathbf{I}(H(\phi))}^{(-m)}) \approx \psi^*(L_1^{-1})$.

Proof. We first observe that

$$\tilde{I}_j/\tilde{I}_{j-1} \simeq \widetilde{I_j/I_{j-1}}.$$

By Proposition 7.4, we have

$$I_j/I_{j-1} \simeq \chi_j \otimes (L_1^{-1} \mid X(\lambda_j))$$

where χ_j is the B-module associated to the T-module $kP_k(\bar{\phi}, \bar{\lambda}_j)$. Let $T_\alpha \approx \mathbf{G_m}$ be defined by $T_\alpha = T \cap SL(2, \alpha)$. Then the weight of $P_k(\bar{\phi}, \bar{\lambda}_j)$ with respect to T_α is

$$-\tfrac{1}{2}\langle \phi(\omega_1) + \lambda_j(\omega_1), \alpha^* \rangle.$$

Now if $\theta \in \mathrm{Hom}(B_\alpha, \mathbf{G_m}) \simeq \mathrm{Hom}(T_\alpha, \mathbf{G_m})$ is the element associated to the weight $-n$, then the line bundle on $\mathbf{P}^1 \simeq SL(2, \alpha)/B_\alpha$ associated to θ (in the sense of principal fibre spaces) is $\mathcal{O}_{\mathbf{P}^1}(n)$ (in the notation of [S]$_1$). Now the result follows by the way $\Psi^*(L_1^{-1})^{(m)}$ has been defined (cf. Definition 8.3).

COROLLARY 9.3. *We have the following (assumptions being as in Lemma* 9.2).

Case A: Suppose that $\langle \phi(\omega_1), \alpha^* \rangle = 2$. Then $\langle \lambda_j(\omega_1), \alpha^* \rangle = 2, -2$, or 0 and one has the following:

(i) $\widetilde{I_j/I_{j-1}} \simeq \Psi^*(L_1^{-1})^{(2)} \mid Z_{\lambda_j}$ if $\langle \lambda_j(\omega_1), \alpha^* \rangle = 2$.

In this case note that $X(\lambda_j)$ (resp. $X(\bar{\lambda}_j)$) is a Schubert divisor in $X(\mu_j)$ (resp. $X(\bar{\mu}_j)$) moved by α, $\mu_j = s_\alpha \lambda_j$.

(ii) $\tilde{I}_j/\tilde{I}_{j-1} \simeq \Psi^*(L_1^{-1}) \mid Z_{\lambda_j}$ if $\langle \lambda_j(\omega_1), \alpha^* \rangle = -2$.

In this case $X(\mu_j)$ (resp. $X(\bar{\mu}_j)$) is a Schubert divisor in $X(\lambda_j)$ (resp. $X(\bar{\lambda}_j)$) moved by α, $\mu_j = s_\alpha \lambda_j$.

(iii) $\tilde{I}_j/\tilde{I}_{j-1} \simeq \Psi^*(L_1^{-1})^{(1)} \mid Z_{\lambda_j}$ if $\langle \lambda_j(\omega_1), \alpha^* \rangle = 0$.

In this case $s_\alpha \bar{\lambda}_j = \bar{\lambda}_j$.

Case B: Suppose that $\langle \phi(\omega_1), \alpha^* \rangle = 1$. Then $\langle \lambda_j(\omega_1), \alpha^* \rangle = \pm 1$ and one has the following:

(i) $\tilde{I}_j/\tilde{I}_{j-1} \simeq \Psi^*(L_1^{-1})^{(1)} \mid Z_{\lambda_j}$ if $\langle \lambda_j(\omega_1), \alpha^* \rangle = 1$.

In this case $X(\lambda_j)$ is a Schubert divisor in $X(\mu_j)$ with $\mu_j = s_\alpha \lambda_j$.

(ii) $\tilde{I}_j/\tilde{I}_{j-1} \simeq \Psi^*(L_1^{-1})|Z_{\lambda_j}$ if $\langle \lambda_j(\omega_1), \alpha^* \rangle = -1$.

In this case $X(\mu_j)$ is a Schubert divisor in $X(\lambda_j)$, moved by α, $\mu_j = s_\alpha \lambda_j$.

Besides, the above computations exhaust all the possible values for $\langle \phi(\omega_1), \alpha^* \rangle$ and $\langle \lambda_j(\omega_1), \alpha^* \rangle$.

Proof. Since $X(\bar{\phi})$ is a divisor in $X(\bar{\tau})$ moved by α, we see that

$$\langle \bar{\phi}(\omega_1), \alpha^* \rangle = \langle \phi(\omega_1), \alpha^* \rangle > 0 \qquad \text{(cf. Lemma 1.2).}$$

Besides one knows that (by Definition 2.2 and Theorem 3.15)

$$\tfrac{1}{2} |\langle \phi(\omega_1) + \lambda_j(\omega_1), \alpha^* \rangle| \leqslant 2$$

and

$$\langle \phi(\omega_1) + \lambda_j(\omega_1), \alpha^* \rangle \in \mathbf{Z}.$$

Hence one sees easily that the above list in the Corollary exhausts all the possible values for $\langle \phi(\omega_1), \alpha^* \rangle$ and $\langle \lambda_j(\omega_1), \alpha^* \rangle$. The isomorphisms in the Corollary are then immediate consequences of Lemma 9.2.

LEMMA 9.4. *Let $X(\phi) \in \mathrm{Schub}(d)$ (assumptions being as in Lemma 9.2). Suppose that $\langle \phi(\omega_1), \alpha^* \rangle = 2$. Let us define the ideal sheaves K_i and M_0 on Z_ϕ as follows:*

$$M_0 = \tilde{\mathbf{I}}_0^{(-2)}, \quad K_0 = \tilde{\mathbf{I}}_0^{(-1)}, \qquad K_1 = \tilde{\mathbf{I}}_1^{(-1)}, \ldots, K_N = \tilde{\mathbf{I}}_N^{(-1)}$$

so that we have

$$\begin{array}{ccccccccccc} M_0 & \subset & K_0 & \subset & K_1 & \subset & K_2 & \subset \cdots \subset & K_N & \subset & \mathcal{O}_{Z_\phi} \\ \| & & \| & & \| & & \| & & \| & & \\ \tilde{I}_0^{(-2)} & \subset & \tilde{I}_0^{(-1)} & \subset & \tilde{I}_1^{(-1)} & \subset & \tilde{I}_2^{(-1)} & \subset \cdots \subset & \tilde{I}_N^{(-1)} & \subset & \mathcal{O}_{Z_\phi}. \end{array}$$

Then we have the following:

$$M_0 \simeq \Psi^*(L_1^{-1}) \qquad \text{(sheaves on } Z_\phi) \tag{1}$$

$$K_0/M_0 \simeq L_1^{-1}|X(\phi) \tag{2}$$

$$\mathcal{O}_{Z_\phi}/K_N \simeq \mathcal{O}_{H(Z)_{\mathrm{red}}}. \tag{3}$$

Further, if $1 \leqslant j \leqslant N$, either $X(\bar{\lambda}_j)$ is a Schubert divisor in $X(\bar{\mu}_j)$ moved by α with $\bar{\mu}_j = s_\alpha \bar{\lambda}_j$, or $s_\alpha \bar{\lambda}_j = \bar{\lambda}_j$, or $X(\bar{\mu}_j)$ is a Schubert divisor in $X(\bar{\lambda}_j)$ moved by α, and one has the following computations:

$$K_j/K_{j-1} \simeq \Psi^*(L_1^{-1})^{(1)}|Z_{\lambda_j} \tag{4}$$

if $X(\bar{\lambda}_j)$ is a Schubert divisor in $X(\bar{\mu}_j)$ moved by α, $\mu_j = s_\alpha \lambda_j$,

$$K_j/K_{j-1} \simeq \Psi^*(L_1^{-1}) \mid Z_{\lambda_j} \qquad \text{if} \quad s_\alpha \bar{\lambda}_j = \bar{\lambda}_j \tag{5}$$

$$K_j/K_{j-1} \simeq \Psi^*(L_1^{-1})^{(-1)} \mid Z_{\lambda_j} \tag{6}$$

if $X(\bar{\mu}_j)$ is a Schubert divisor in $X(\bar{\lambda}_j)$ moved by α, $\mu_j = s_\alpha \lambda_j$.

Proof. We see that the above assertions except (2) are immediate consequences of Corollary 9.3 and Lemma 8.10(iii). To prove (2), we observe that the pull-back by $p: Z_\phi \to \mathbf{P}^1$ of the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(1) \to \mathcal{O}_{\bar{e}} \to 0$$

($\mathcal{O}_{\bar{e}}$ = structure sheaf of the point $\bar{e} \in \mathbf{P}^1$) gives the exact sequence

$$0 \to \mathcal{O}_{Z_\phi} \to \mathcal{O}_{Z_\phi}^{(1)} \to \mathcal{O}_{X(\phi)} \to 0$$

as sheaves on Z_ϕ. Tensoring by $\Psi^*(L_1^{-1})$ we get

$$0 \to \Psi^*(L_1^{-1}) \to \Psi^*(L_1^{-1})^{(1)} \to L_1^{-1} \mid X(\phi) \to 0.$$

Now because of (1) above, we see that $K_1/K_0 \simeq L_1^{-1} \mid X(\phi)$. This proves (2) and the proof of Lemma 9.4 is complete.

LEMMA 9.5. *Assumptions being as in Lemma 9.2, let $X(\phi) \in \mathrm{Schub}(d)$ be such that $\langle \phi(\omega_1), \alpha^* \rangle = 1$. Let us define the ideal sheaves K_i on Z_ϕ as follows:*

$$K_0 = \tilde{\mathbf{I}}_0^{(-1)}, \qquad K_1 = \tilde{\mathbf{I}}_1^{(-1)}, \ldots, K_N = \tilde{\mathbf{I}}_N^{(-1)}$$

so that we have

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_N.$$

Then we have the following:

$$K_0 \approx \Psi^*(L_1^{-1}) \qquad (\textit{as sheaves on } Z_\phi) \tag{1}$$

$$\mathcal{O}_{Z_\phi}/K_N \approx \mathcal{O}_{H(Z)_{\mathrm{red}}}. \tag{2}$$

If $1 \leqslant j \leqslant N$, either $X(\bar{\lambda}_j)$ is a Schubert divisor in $X(\bar{\mu}_j)$ moved by α with $\bar{\mu}_j = s_\alpha \bar{\lambda}_j$ or $X(\bar{\mu}_j)$ is a Schubert divisor in $X(\bar{\lambda}_j)$ moved by α and we have

$$K_j/K_{j-1} \approx \Psi^*(L_1^{-1}) \mid Z_{\lambda_j} \tag{3}$$

if $X(\bar{\lambda}_j)$ is a Schubert divisor in $X(\bar{\mu}_j)$ moved by α, $\mu_j = s_\alpha \lambda_j$.

$$K_j/K_{j-1} \approx \Psi^*(L_1^{-1})^{(-1)} | Z_{\lambda_j} \tag{4}$$

if $X(\bar{\mu}_j)$ is a Schubert divisor in $X(\bar{\lambda}_j)$ moved by α, $\mu_j = s_\alpha \lambda_j$.

Proof. The proof of Lemma 9.5 is very similar to that of Lemma 9.4 (and in fact simpler) and of course we use the computations of case B of Corollary 9.3.

THEOREM 9.6. *Let $Q = \bigcap_{i=1}^r P_i$, $\tau \in W/W_Q$, and let $L = L_{\mathbf{a}}$ be a positive line bundle on G/Q. Then we have*

(a) $H^i(X(\tau), L) = 0$, $i \geqslant 1$,

(b) *$X(\tau)$ is normal,*

(c) $\dim H^0(X(\tau), L) = \#\{$*standard monomials on $X(\tau)$ of type* $\mathbf{a}\}$,

(d) *$V_\delta(\tau)^* = H^0(X(\tau), L)$, where δ is the character associated to L.*

Proof. We prove these results by induction on $\dim X(\tau)$ and on r. The proof of the results for $r = 1$ is contained in the proof of the general case. When $\dim X(\tau) = 0$, assertions are trivially true. Let then $\dim X(\tau) \geqslant 1$. Let us fix a moving divisor $X(\phi)$ in $X(\tau)$, moved by α, in such a way that $X(\phi')$ is a moving divisor in $X(\tau')$ moved by α (here, $X(\phi')$, $X(\tau')$ denote the projections of $X(\phi)$, $X(\tau)$, respectively, under $\pi: G/Q \to G/Q'$ where $Q' = \bigcap_{i=2}^r P_i$). We first claim

The special quadratic relations given by $(*)$ of Section 5 hold on $X(\tau)$. (1)

Proof (of claim (1)). (I) Let P be any maximal parabolic subgroup containing Q and let $X(\bar{\tau})$ be the projection of $X(\tau)$ under $G/Q \to G/P$. In view of Lemma 5.6, we have that the zero set of $p(\bar{\tau})$ in $X(\bar{\tau})$ is (set-theoretically) the union of all the codimension one subvarieties of $X(\bar{\tau})$. We now observe that $p(\bar{\tau})$ vanishes on a Schubert variety $X(\lambda)$ of codimension one in $X(\bar{\tau})$ up to order $\leqslant 2$; for, this order is precisely the multiplicity of $X(\lambda)$ in $[X(\bar{\tau})] \cdot [H]$ (cf. Section 2) which is $\leqslant 2$ (since ω is of classical type). Denoting $\bar{\tau}$ by δ (for simplicity of notations) we see that for an admissible pair $p(\delta, \theta)$ on $X(\delta)$, $p(\delta, \theta)$ vanishes on all the codimension one subvarieties $X(\lambda)$ of $X(\delta)$ (in view of (iii) of Theorem 3.15, so that $p(\delta, \theta)^2$ vanishes up to order $\geqslant 2$ on all Schubert subvarieties of codimension one in $X(\delta)$). Denoting $h = p(\delta, \theta)^2/p(\delta)$, we therefore obtain (using Proposition 8.6) that h is regular on an open subset of $X(\delta)$ whose complement has codimension $\geqslant 2$. Hence h becomes regular on the normalization $\widetilde{X(\delta)}$ of $X(\delta)$. Now fixing a moving divisor $X(\delta')$ in $X(\delta)$, we have $Z_{\delta'}(=Z_{\delta',\delta}$ (cf. Section 8)) is normal (in view of induction hypothesis $X(\delta')$ is normal) and hence we have the factorization

$$Z_{\delta'} \to \tilde{X}(\delta) \to X(\delta)$$

so that h becomes a regular section of $\Psi^*(L_\omega)$ on $Z_{\delta'}$. On the other hand, we have an isomorphism

$$(V_\omega(\delta))^* \simeq H^0(X(\delta), L_\omega) \overset{\Psi^*}{\simeq} H^0(Z_{\delta'}, \Psi^*(L_\omega))$$

(cf. Proposition 8.13 and Corollary 8.14—note that the hypotheses in Proposition 8.13 are satisfied, since $H^i(X(\delta'), L_\omega)=0$, $i \geqslant 1$, and $H^0(X(\delta'), L_\omega) \approx (V_\omega(\delta'))^*$, by the induction hypothesis). Hence $h \in (V_\omega(\delta))^*$ which is a quotient of V_ω^* (cf., Corollary 3.7). Now h is a weight vector of weight $-\theta(\omega)$, i.e., its weight is extremal and hence we conclude that $h=cp(\theta)$ (c is a non-zero scalar). Taking $k=\mathbf{Q}$, we get

$$p(\delta, \theta)^2 = cp(\delta) \cdot p(\theta), \qquad c \in \mathbf{Q}^*.$$

We now see easily that the above relation leads to the following relation:

$$aP(\delta, \theta)^2 = bP(\delta) \cdot P(\theta) \qquad \text{on } X_{\mathbf{Z}}(\delta),\ a, b \in \mathbf{Z} \text{ and are coprime.} \qquad \text{(A)}$$

Suppose that $a \neq \pm 1$. Then let p be a prime divisor of "a" and let $k=\mathbf{Z}/(p)$. Then (A) gives

$$p_k(\delta)\, p_k(\theta) = 0 \qquad \text{on } X_k(\delta).$$

This leads to a contradiction, since neither $p_k(\delta)$ nor $p_k(\theta)$ vanishes on $X_k(\delta)$ (in view of (iii) of Theorem 3.15; note that $\delta \geqslant \theta$, since (δ, θ) is an admissible pair). Thus we conclude that $a = \pm 1$. Similarly, if $b \neq \pm 1$, we get a contradiction to the fact that $p_k(\delta, \theta) \neq 0$ (for a suitable k). Thus we conclude that $a = \pm 1$, $b = \pm 1$, and (A) gives

$$P(\delta, \theta)^2 = \pm P(\delta)\, P(\theta) \qquad \text{on } X_{\mathbf{Z}}(\delta)$$

which proves (1) in (∗) of Section 5.

(II) Let $F \in H^0(X_{\mathbf{Z}}(\delta), L_{\omega,\mathbf{Z}})$ such that $F^2 = P(\delta)\, P(\theta)$. Then if $F \neq 0$ on $X_{\mathbf{Z}}(\delta)$, we can write

$$F = \sum a_{\lambda,\mu} P(\lambda, \mu), \qquad a_{\lambda,\mu} \in \mathbf{Z} \text{ and } \neq 0 \text{ on } X_{\mathbf{Z}}(\delta) \qquad \text{(B)}$$

(cf. Corollary 8.15—note that hypotheses in Corollary 8.15 are satisfied in view of induction hypothesis) where the R.H.S. of (B) runs over the distinct admissible pairs on $X_{\mathbf{Z}}(\delta)$. We first claim that for every (λ, μ) on the R.H.S.

of (B), we have $\lambda=\delta$. For, otherwise we have an admissible pair (λ_1, μ_1) on the R.H.S. of (B) such that $\delta>\lambda_1$. The restriction of F to $X_{\mathbf{Z}}(\lambda_1)$ is zero since $F^2=P(\delta)\,P(\lambda_1)$ and $P(\delta)\,|\,X_{\mathbf{Z}}(\lambda_1)=0$ (by (iii) of Theorem 3.15). But the restriction of the R.H.S. of (B) to $X_{\mathbf{Z}}(\lambda_1)$ is not zero, since the set of admissible pairs (λ, μ) which occur on the R.H.S. of (B) and such that $P(\lambda, \mu)\,|\,X(\lambda_1)\neq 0$ is not empty ((λ_1, μ_1) is in this set) and they are linearly independent (cf. Remark 3.16(i)). Thus we arrive at a contradiction. This proves the required claim that $\delta=\lambda$ for every (λ, μ) on the R.H.S. of (B). Now the weight of F $(=\sqrt{P(\delta)\,P(\theta)})$ is $-\frac{1}{2}(\delta(\omega)+\theta(\omega))$. On the other hand, the weight of any $P(\lambda, \mu)$ on the R.H.S. of (B) is $-\frac{1}{2}(\delta(\omega)+\mu(\omega))$ (since $\lambda=\delta$). We thus conclude that $\mu(\omega)=\theta(\omega)$ for every μ on the R.H.S. of (B); further, $\mu(\omega)$ and $\theta(\omega)$ being *extremal* weights, we deduce that $\mu=\theta$ for every such μ. Then (B) gives that $F=\sqrt{P(\delta)\,P(\theta)}=aP(\delta, \theta)$, $a\in\mathbf{Z}$. Then as we did above, we deduce that $a=\pm 1$.

(III) We see that $P(\delta, \theta_1)\,P(\delta, \theta_2)$ vanishes up to order $\geqslant 2$ on all the Schubert subvarieties of codimension one of $X_{\mathbf{Z}}(\delta)$. Hence as in (1) above, we see that (on $X_k(\delta)$)

$$P_k(\delta, \theta_1)\,P_k(\delta, \theta_2)=P_k(\delta)f, \qquad f\in H^0(X_k(\delta), L_{\omega,k})$$

which gives (taking $k=\mathbf{Q}$)

$$aP(\delta, \theta_1)\,P(\delta, \theta_2)=bP(\delta)F, \quad a, b\in\mathbf{Z}-\{0\}, F\in H^0(X_{\mathbf{Z}}(\delta), L_{\omega,\mathbf{Z}}).$$

Again as in (1) above, we deduce that (on $X_{\mathbf{Z}}(\delta)$)

$$P(\delta, \theta_1)\,P(\delta, \theta_2)=\pm P(\delta)F, \qquad F\in H^0(X_{\mathbf{Z}}(\delta), L_{\omega,\mathbf{Z}}).$$

On the other hand from (1) we deduce that (on $X_{\mathbf{Z}}(\delta)$)

$$P(\delta, \theta_1)^2 P(\delta, \theta_2)^2=P(\delta)^2 P(\theta_1)\,P(\theta_2)=P(\delta)^2 F^2.$$

Hence $F^2=P(\theta_1)\,P(\theta_2)$ (on $X_{\mathbf{Z}}(\delta)$). This proves (III) and the proof of claim (1) is complete.

Remark 9.7. In view of claim (1), we obtain that standard monomials on $X(\tau)$ of type $\mathbf{a}$ are linearly independent (cf. Theorem 5.1).

Next we claim (denoting Z_ϕ by just Z) that

(A) $H^i(H(Z)_{\text{red}}, \Psi^*(L))\approx H^i(H(\tau)_{\text{red}}, L), \qquad i\geqslant 0$

(B) Standard monomials on $X(\tau)$ of type $\mathbf{a}$ give a basis for $H^0(Z, \Psi^*(L))$

(C) $H^i(Z, \Psi^*(L))=0, \qquad i\geqslant 1$ (2)

(where, recall that $H(Z)_{\text{red}}=X(\phi)\cup\widetilde{H(\phi)}_{\text{red}}$ (cf. Section 8)).

Now we have $H^i(H(\tau)_{\text{red}}, L)=0$, $i\geqslant 1$ (in view of induction hypothesis (see also Remark 6.5)). Hence to prove (2)(A) we need to show

$$H^0(H(Z)_{\text{red}}, \Psi^*(L))\approx H^0(H(\tau)_{\text{red}}, L)$$

and (3)

$$H^i(H(Z)_{\text{red}}, \Psi^*(L))=0, \qquad i\geqslant 1.$$

Now $H(Z)_{\text{red}}=X(\phi)\cup \widetilde{H(\phi)}_{\text{red}}$ (scheme-theoretic) and

$$H(\phi)_{\text{red}}=\bigcup_{\lambda} \widetilde{X(\lambda)} \qquad \text{(scheme-theoretic)}$$

where $X(\lambda)$'s are the irreducible components of $H(\phi)_{\text{red}}$. Further if $\mu=$ bigger of $\{\lambda, s_\alpha\lambda\}$, then

$$H^i(\widetilde{X(\lambda)}, \Psi^*(L))\approx H^i(X(\mu), L), \qquad \text{for all } i$$

(if $s_\alpha\lambda>\lambda$, this follows from Lemma 8.12—note that $X(s_\alpha\lambda)$ is normal by induction hypothesis; and if $\lambda\geqslant s_\alpha\lambda$, this follows from the fact that $\widetilde{X(\lambda)}$ splits (and is $\approx\mathbf{P}^1\times X(\lambda)$)). In particular, we obtain

$$\begin{aligned} &H^i(\widetilde{X(\lambda)}, \Psi^*(L))=0, \qquad i\geqslant 1\\ &H^0(\widetilde{X(\lambda)}, \Psi^*(L))\approx H^0(X(\mu), L)\end{aligned} \tag{4}$$

(in view of the induction hypothesis $H^i(X(\mu), L)=0$, $i\geqslant 1$). Now proceeding as in Section 6 we claim that if $T=\bigcup_\lambda X(\lambda)$, where $X(\lambda)$ runs over some of the Schubert subvarieties $\subsetneqq X(\phi)$ and if (as above) $\mu=$ bigger $\{\lambda, s_\alpha\lambda\}$, then

$$H^0\left(\bigcup_\mu X(\mu), L\right)\to H^0(\tilde{T}, \Psi^*(L)) \text{ is an isomorphism}$$

and (5)

$$H^i(\tilde{T}, \Psi^*(L))=0, \qquad i>0.$$

To prove (5), we first note that if $T=X(\lambda)$, then $\tilde{T}=Z_\lambda$ and (5) follows from Lemma 8.12. If $T=\bigcup_\lambda X(\lambda)$, then we proceed as in Section 6, where we prove similar assertions for unions and intersections of Schubert varieties. For this purpose, we have only to observe that the functor $T\mapsto\tilde{T}$ preserves scheme-theoretic unions and intersections; for example, the exact sequence

$$0\to\mathcal{O}_{X(\lambda_1)\cup X(\lambda_2)}\to\mathcal{O}_{X(\lambda_1)}\oplus\mathcal{O}_{X(\lambda_2)}\to\mathcal{O}_{X(\lambda_1)\cap X(\lambda_2)}\to 0$$

gives the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X(\lambda_1)} \cup \widetilde{X(\lambda_2)}} \to \mathcal{O}_{\widetilde{X(\lambda_1)}} \oplus \mathcal{O}_{\widetilde{X(\lambda_2)}} \to \mathcal{O}_{\widetilde{X(\lambda_1)} \cap \widetilde{X(\lambda_2)}} \to 0.$$

Tensoring with $\Psi^*(L)$ and writing down the cohomology exact sequence we obtain

$$H^i(\widetilde{X(\lambda_1)} \cup \widetilde{X(\lambda_2)}, \Psi^*(L)) = 0, \qquad i \geqslant 1$$

and

$$H^0(\widetilde{X(\lambda_1)} \cup \widetilde{X(\lambda_2)}, \Psi^*(L)) \approx H^0(X(\mu_1) \cup X(\mu_2), L)$$

etc., and thus proceeding, we obtain (5). As particular case of (5) we have

$$H^0\left(\bigcup_{\mu} X(\mu), L\right) \simeq H^0(\widetilde{H(\phi)}_{\mathrm{red}}, \Psi^*(L))$$

and (6)

$$H^i(\widetilde{H(\phi)}_{\mathrm{red}}, \Psi^*(L)) = 0, \qquad i \geqslant 1.$$

Finally, we consider

$$0 \to \mathcal{O}_{H(Z)_{\mathrm{red}}} \to \mathcal{O}_{X(\phi)} \oplus \mathcal{O}_{\widetilde{H(\phi)}_{\mathrm{red}}} \to \mathcal{O}_{H(\phi)_{\mathrm{red}}} \to 0$$

(note that $\widetilde{H(\phi)}_{\mathrm{red}} \cap X(\phi) = H(\phi)_{\mathrm{red}}$ (scheme-theoretic) (cf. Lemma 8.11)). Tensoring the above exact sequence with $\Psi^*(L)$ and using the induction hypothesis we obtain that

$$H^0(X(\phi), L) \oplus H^0(\widetilde{H(\phi)}_{\mathrm{red}}, \Psi^*(L)) \to H^0(H(\phi)_{\mathrm{red}}, L)$$

is surjective (in view of the induction hypothesis, (c) of the theorem implies the surjectivity of $H^0(X(\phi), L) \to H^0(H(\phi)_{\mathrm{red}}, L)$, etc.). Hence we obtain

$$H^1(H(Z)_{\mathrm{red}}, \Psi^*(L)) = 0.$$

The vanishing of $H^i(H(Z)_{\mathrm{red}}, \Psi^*(L)) = 0$, for $i \geqslant 2$ follows in view of (6) (and the induction hypothesis). This completes the proof of claim (2)(A) for $i \geqslant 1$.

We shall now prove claim (2)(A) for $i = 0$, namely, that the map

$$\Psi^*: H^0(H(\tau)_{\mathrm{red}}, L)) \to H^0(H(Z)_{\mathrm{red}}, \Psi^*(L))$$

is an isomorphism. For this we first claim that

$$\Psi_*(\mathcal{O}_{H(Z)_{\mathrm{red}}}) = \mathcal{O}_{H(\tau)_{\mathrm{red}}}. \tag{7}$$

The claim (7), of course, would prove that Ψ^* is an isomorphism. Let U be an affine open subset of $H(\tau)_{\text{red}}$. Then we have only to show that a regular function f on $\Psi^{-1}(U)$ ($\subset H(Z)_{\text{red}}$ (cf. Lemma 8.9)) comes from U. Now, in view of Lemmas 8.5 and 8.9, we have that for $x \in H(\tau)_{\text{red}}$, $\Psi^{-1}(x)$ is either a point or a $\mathbf{P}^1$. Hence we conclude that f is constant on the fibres of $\Psi^{-1}(U) \to U$ (note that this does not necessarily imply that f goes down to a regular function on U). Let T be an irreducible component of $H(Z)_{\text{red}}$ which is the pull-back (under Ψ) of an irreducible component S of $H(\tau)_{\text{red}}$. By the induction hypothesis, S is normal and hence $\Psi_*(\mathcal{O}_T) = \mathcal{O}_S$ so that the restriction of f to $T \cap \Psi^{-1}(U)$ descends to a function on $S \cap U$. From this we conclude that if S_i are the irreducible components of $H(\tau)_{\text{red}}$, then f descends to a regular function f_i on $U \cap S_i$ such that $f_i - f_j$ vanishes on $(U \cap S_i \cap S_j)_{\text{red}}$. On the other hand, we known that the scheme-theoretic intersection $S_i \cap S_j$ is reduced (cf. Lemma 6.3 and the induction hypothesis). Hence f_i patch up to define a regular function on U and we see that f goes down to this function. This completes the proof of (2)(A) for $i = 0$. Thus proof of (2)(A) is now complete. Next, to prove (2)(B) and (C), we first prove them for $a_1 = 0$. When $a_1 = 0$, the commutative diagram

$$\begin{array}{ccc} SL(2, \alpha) \times X(\phi) & \longrightarrow & X(\tau) \\ \downarrow & & \downarrow \\ SL(2, \alpha) \times X(\phi') & \longrightarrow & X(\tau') \end{array}$$

(where, recall, $X(\tau')$ (resp. $X(\phi')$) is the projection of $X(\tau)$ (resp. $X(\phi)$) under $G/Q \to G/Q'$) yields the commutative diagram

$$\begin{array}{ccc} Z_\phi & \xrightarrow{\qquad} & Z_{\phi'} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathbf{P}^1 & \end{array}$$

Now $H^0(X(\phi), L) \approx H^0(X(\phi'), L)$ (when $a_1 = 0$) (since $X(\phi')$ is normal, under $\pi\colon X(\phi) \to X(\phi')$, $\pi_*(\mathcal{O}_{X(\phi)}) = \mathcal{O}_{X(\phi')}$) and hence we obtain

$$p_*(\Psi^*(L)) \approx p'_*(\Psi'^*(L)).$$

This then implies that

$$H^i(Z_\phi, \Psi^*(L)) \approx H^i(Z_{\phi'}, \Psi'^*(L))$$

and finally when $r = 1$ and $a_1 = 0$, we have $H^i(Z_\phi, \mathcal{O}_{Z_\phi}) = 0$, $i \geqslant 1$ (since $p_*(\mathcal{O}_{Z_\phi}) = \mathcal{O}_{\mathbf{P}^1}$, etc.) (note that proof of (2)(B) for the case $r = 1$, $a_1 = 0$ is trivial). Hence we may assume $a_1 \geqslant 1$. Now, we consider either one of the following two sets of exact sequences (according as $\langle\phi(\omega_1), \alpha^*\rangle = 2$ or 1) (let us denote Z_ϕ by just Z):

$$\begin{aligned} &0 \to M_0 \to \mathcal{O}_Z \to \mathcal{O}_Z/M_0 \to 0 \\ &0 \to K_0/M_0 \to \mathcal{O}_Z/M_0 \to \mathcal{O}_Z/K_0 \to 0 \\ &0 \to K_{i+1}/K_i \to \mathcal{O}_Z/K_i \to \mathcal{O}_Z/K_{i+1} \to 0, \qquad 0 \leqslant i \leqslant N-1 \end{aligned} \tag{I}$$

or

$$\begin{aligned} &0 \to K_0 \to \mathcal{O}_Z \to \mathcal{O}_Z/K_0 \to 0 \\ &0 \to K_{i+1}/K_i \to \mathcal{O}_Z/K_i \to \mathcal{O}_Z/K_{i+1} \to 0, \qquad 0 \leqslant i \leqslant N-1 \end{aligned} \tag{II}$$

(cf. Lemmas 9.4 and 9.5 for notations, M_0, K_i, etc.). Now we distinguish the following two cases.

Case 1: $\langle\phi(\omega_1), \alpha^*\rangle = 2$. In view of Lemma 9.4, we have (denoting $L' = L \otimes L_1^{-1}$)

$$\Psi^*(L) \otimes M_0 \approx \psi^*(L') \qquad (\text{on } Z_\phi)$$

$$\Psi^*(L) \otimes K_0/M_0 \approx L'|_{X(\phi)}$$

$$\mathcal{O}_{Z_\phi}/K_N \approx \mathcal{O}_{H(Z)_{\mathrm{red}}}.$$

Further,

$$\Psi^*(L) \otimes K_j/K_{j-1} \approx \Psi^*(L')^{(1)} | Z(\lambda_j)$$

if $X(\bar{\lambda}_j)$ is a Schubert divisor in $X(\bar{\mu}_j)$ moved by α with $\mu_j = s_\alpha \lambda_j$ and

$$\Psi^*(L) \otimes K_j/K_{j-1} \approx \Psi^*(L') | Z(\lambda_j)$$

if $s_\alpha \bar{\lambda}_j = \bar{\lambda}_j$ and

$$\Psi^*(L) \otimes K_j/K_{j-1} \approx \Psi^*(L')^{(-1)} Z(\lambda_j)$$

if $X(\bar{\mu}_j)$ is a Schubert divisor in $X(\bar{\lambda}_j)$ moved by α with $\mu_j = s_\alpha \lambda_j$. Also $H^i(H(Z)_{\mathrm{red}}, \Psi^*(L)) \approx H^i(H(\tau)_{\mathrm{red}}, L)$ for all i (cf. (2)(A) above). Hence, tensoring the exact sequence in (I) above by $\Psi^*(L)$ and writing down the cohomology long exact sequence, we obtain (recall (cf. Lemma 9.2) that $\mathbf{I}(H(Z)) = \Psi^*(L_1^{-1})$)

$$h^0(H(Z), \Psi^*(L)) = h^0(H(\tau)_{\mathrm{red}}, L) + h^0(X(\phi), L') + \sigma_1 + \sigma_2$$

where

$$\sigma_1 = \sum_\lambda h^0(X(\mu), L') + h^0(X(\lambda), L')$$

where the summation is over all distinct $\bar{\lambda}$ such that $(\bar{\phi}, \bar{\lambda})$ is a non-trivial admissible pair and $s_\alpha \bar{\lambda} > \bar{\lambda}$ and λ is the maximal representative of $\bar{\lambda}$ less than ϕ and $\mu = s_\alpha \lambda$ and

$$\sigma_2 = \sum_{\bar{\lambda}} h^0(X(\mu), L')$$

where the summation runs over all distinct $\bar{\lambda}$ such that $(\bar{\phi}, \bar{\lambda})$ is an admissible pair with $s_\alpha \bar{\lambda} = \bar{\lambda}$ and λ is the maximal representative of $\bar{\lambda}$ less than ϕ and μ is the larger of the two elements λ and $s_\alpha \lambda$ (here for any $w \in W/W_Q$, $X(\bar{w})$ denotes the projection of $X(w)$ under $G/Q \to G/P_1$). Also, when $\tau < s_\alpha \lambda > \lambda$, by considering the exact sequence of sheaves (on Z_λ)

$$0 \to \mathcal{O}_{Z_\lambda} \to \mathcal{O}_{Z_\lambda}^{(1)} \to \mathcal{O}_{X(\lambda)} \to 0$$

we obtain (in view of inductive hypothesis and Lemma 8.12) $h^0(Z_\lambda, \Psi^*(L')^{(1)}) = h^0(X(\lambda), L') + h^0(X(s_\alpha \lambda), L'))$. Now in view of Lemmas 2.8, 2.14, and 4.8, we obtain

$$h^0(H(Z), \Psi^*(L)) = h^0(H(\tau)_{\text{red}}, L) + \sum_{\lambda} h^0(X(\lambda), L') \tag{8}$$

where the summation on the R.H.S. is over all λ such that $(\bar{\tau}, \bar{\lambda})$ is a non-trivial admissible pair and λ is the maximal lift less than τ of $\bar{\lambda}$. Now, in view of (8) and Proposition 5.8, we obtain that $H^0(H(Z), \Psi^*(L))$ can be identified with the span of standard monomials on $X(\tau)$ of type $\mathbf{a}$ which do not involve $p(\bar{\tau})$. In particular this implies the surjectivity of $H^0(Z, \Psi^*(L)) \to H^0(H(Z), \Psi^*(L))$ and hence we obtain

$$h^0(Z, \Psi^*(L)) = h^0(Z, \Psi^*(L')) + h^0(H(Z), \Psi^*(L)).$$

Now by induction on a_1, we have $h^0(Z, \Psi^*(L')) = s(X(\tau), \mathbf{a}')$. This together with Proposition 5.8 implies

$$h^0(Z, \Psi^*(L)) = s(X(\tau), \mathbf{a}).$$

Now, this, in view of linear independence of standard monomials on $X(\tau)$ (cf. Remark 9.7), proves (2)(B) in case 1. The proof of (2)(C) follows again by considering the long exact cohomology sequence obtained by tensoring (I) by $\Psi^*(L)$ (and using the induction hypothesis).

This completes the proof of (2)(B), (C) in case 1.

Case 2. $\langle \phi(\omega_1), \alpha^* \rangle = 1$. The proof in this case is very similar (in fact simpler) to that of case 1. We tensor the exact sequences in (II) by $\Psi^*(L)$ and write down the cohomology long exact sequence and conclude as above (using (2)(A) and Lemmas 9.5, 2.9, 2.15, and 4.8, and

Proposition 5.8) that $H^0(H(Z), \Psi^*(L))$ can be identified with the span of standard monomials on $X(\tau)$ of type $\mathbf{a}$ not involving $p(\bar{\tau})$. Then we proceed as in case 1 and conclude

$$h^0(Z, \Psi^*(L)) = s(X(\tau), \mathbf{a}).$$

Now, this together with linear independence of standard monomials on $X(\tau)$ proves (2)(B) and the proof of (2)(C) is again as in case 1. This completes the proof of (2)(B) and (C). Now linear independence of standard monomials on $X(\tau)$ (cf. Remark 9.7) implies that

$$\dim H^0(X(\tau), L) \geqslant s(X(\tau), \mathbf{a}).$$

On the other hand from (2)(B) above, we obtain

$$\dim H^0(X(\tau), L) \leqslant s(X(\tau), \mathbf{a}).$$

Hence we obtain $\dim H^0(X(\tau), L) = s(X(\tau), \mathbf{a})$ and thus standard monomials on $X(\tau)$ of type $\mathbf{a}$ form a basis for $H^0(X(\tau), L)$ which proves (c) of Theorem 9.6.

Now in view of (c) and claim (2)(B), we obtain

$$H^0(X(\tau), L) \simeq H^0(\tilde{X}, f^*(L)) \tag{9}$$

where $\tilde{X}$ is the normalization of $X(\tau)$, for the map $\Psi\colon Z_\phi \to X(\tau)$ factors as

$$\begin{array}{ccc} Z_\phi & \xrightarrow{\quad\Psi\quad} & X(\tau) \\ & {}_{\theta}\searrow \quad \nearrow_{f} & \\ & \tilde{X} & \end{array}$$

(since Z_ϕ is normal, as $X(\phi)$ is normal by the induction hypothesis, etc.), and $H^0(Z_\phi, \Psi^*(L)) \approx H^0(X(\tau), L)$. Now (9) implies the normality of $X(\tau)$ which proves (b) of Theorem 9.6. The assertion (a) follows from (b), Lemma 8.12, and claim (2)(C). Finally, to complete the proof of Theorem 9.6, it remains to prove (d). Now in view of (c) (and Remark 9.7) we obtain that the canonical map

$$H^0(G/Q, L) \to H^0(X(\tau), L)$$

is surjective. Now we have a canonical isomorphism

$$H^0(G/Q, L) \approx (V_{\delta,\mathbf{Z}} \otimes k)^*$$

(cf. Remark 3.14) and hence we obtain that

$$(V_{\delta,\mathbf{Z}}\otimes k)^* \to H^0(X(\tau), L) \qquad (*)$$

is surjective. Now in view of (c), we have that dim $H^0(X(\tau), L)$ is the same in all characteristics; hence we obtain that dim $\operatorname{Im}(V_{\delta,\mathbf{Z}}\otimes k)^*$ is the same in all characteristics. Now if we denote Im $V_\delta(\tau)$ to be the image of the map

$$j_k\colon V_{\delta,\mathbf{Z}}(\tau)\otimes k \to V_{\delta,\mathbf{Z}}\otimes k$$

then $\operatorname{Im}(V_{\delta,\mathbf{Z}}\otimes k)^*$ can be identified with $(\operatorname{Im} V_\delta(\tau))^*$ (cf. Lemma 3.12). Thus we obtain that dim Im $V_\delta(\tau)$ is the same in all characteristics and hence in particular we obtain

$$\begin{aligned} &\text{(i)}\quad j_k \text{ is injective for all fields } k \text{ and}\\ &\text{(ii)}\quad V_\delta(\tau)^* \approx H^0(X(\tau), L). \end{aligned} \qquad (10)$$

Now (d) follows and the proof of Theorem 9.6 is complete.

COROLLARY 9.8. *Notations being as in Theorem* 9.6, *we have*:

(a) *$H^0(X(\tau), L)$ has a basis given by standard monomials on $X(\tau)$ of type* **a**.

(b) *$V_{\delta,\mathbf{Z}}(\tau)$ is a direct summand of $V_{\delta,\mathbf{Z}}$ (which is Demazure's conjecture (cf.* $[\mathrm{D}]_1$; *for notations $V_{\delta,\mathbf{Z}}(\tau)$, refer to Section* 3)).

(c) *The canonical map $H^0(G/Q, L) \to H^0(X(\tau), L)$ is surjective.*

(d) *$X_k(\tau) = X_{\mathbf{Z}}(\tau)\otimes k$.*

(e) *Char $H^0(X(\tau), L) = M_\tau(\exp(-\omega))$ (M_τ being the operator defined in Section* 3).

(f) *Let α be a simple root such that $s_\alpha\tau < \tau$; let $\phi = s_\alpha\tau$. Then the kernel K of the surjective map*

$$(\mathscr{V}_\delta(\tau))^* \to (\mathscr{V}_\delta(\phi))^* \to 0 \qquad (on\ \mathbf{P}^1)$$

is isomorphic to $\mathscr{F}(-1)$ where $\mathscr{F}$ is a trivial *vector bundle on $\mathbf{P}^1$, where $\mathscr{V}_\delta(\tau)$ (resp. $\mathscr{V}_\delta(\phi)$) denotes the vector bundle on $\mathbf{P}^1$ associated to $V_\delta(\tau)$ (resp. $V_\delta(\phi)$).*

Proof. Assertion (a) follows from Remark 9.7 and (c) of Theorem 9.6. Assertion (b) follows from (9)(i) in the proof of Theorem 9.6. Assertion (c) follows from (a).

To prove (d), we observe that since $X_{\mathbf{Z}}(\tau) \to \operatorname{Spec}\mathbf{Z}$ is $\mathbf{Z}$-flat, if L is ample on $X_{\mathbf{Z}}(\tau)$, then for $n \gg 0$, we have isomorphisms

$$H^0(X_{\mathbf{Z}}(\tau), L^n_{\mathbf{Z}})\otimes k \xrightarrow{\sim} H^0(X_{\mathbf{Z}}\otimes k, L^n_{\mathbf{Z}}\otimes k)$$

(by the semi-continuity theorems (cf. [M])). Also we have the commutative diagram

$$\begin{array}{ccc} H^0(X_{\mathbf{Z}}(\tau), L^n_{\mathbf{Z}}) \otimes k & \xrightarrow{\ i\ } & H^0(X_{\mathbf{Z}}(\tau) \otimes k,\ L^n_{\mathbf{Z}} \otimes k) \\ & \searrow^{u} & \downarrow j \\ & & H^0(X_k(\tau),\ L^n) \end{array}$$

Since $X_k(\tau) = (X_{\mathbf{Z}}(\tau) \otimes k)_{\text{red}}$, we see that j is surjective for $n \gg 0$ and that if j is an isomorphism for $n \gg 0$, then $X_{\mathbf{Z}}(\tau) \otimes k \approx X_k(\tau)$. On the other hand, in view of (a) we obtain that U is an isomorphism for $n \geqslant 0$. Hence we obtain that j is an isomorphism for $n \gg 0$, which proves that $X_k(\tau) = X_{\mathbf{Z}}(\tau) \otimes k$. To prove (e) and (f) we consider the exact sequence of vector bundles on $\mathbf{P}^1$

$$0 \to K \to (\mathscr{V}_\delta(\tau))^* \to (\mathscr{V}_\delta(\phi))^* \to 0$$

(ϕ being as in (f)). Now $V_\delta(\tau)$ being an $SL(2, \alpha)$ module, $(\mathscr{V}_\delta(\tau))^*$ is trivial. Hence

$$H^i(\mathbf{P}^1, (\mathscr{V}_\delta(\tau))^*) = 0, \qquad i \geqslant 1$$

and (1)

$$H^0(\mathbf{P}^1, (\mathscr{V}_\delta(\tau))^*) = (V_\delta(\tau))^*.$$

On the other hand, considering p: $Z_\phi \to \mathbf{P}^1$ we have

$$H^i(Z_\phi, \Psi^*(L)) \approx H^i(\mathbf{P}^1, p_*(\Psi^*(L))) \qquad \text{for all } i \tag{2}$$

(since the fibres of p are isomorphic to $X(\phi)$ and $H^i(X(\phi), L) = 0$, $i \geqslant 1$). Now $p_*(\Psi^*(L))$ gets identified with the vector bundle associated to $H^0(X(\phi), L)$. Hence $p_*(\Psi^*(L)) \approx \mathscr{V}_\delta(\phi)^*$ (cf. (d) of Theorem 9.6). Also, by Lemma 8.12 (and (b) of Theorem 9.6) we have

$$H^0(Z_\phi, \Psi^*(L)) \approx H^0(X(\tau), L) \qquad (\approx (V_\delta(\tau))^* \text{ by (d) of Theorem 9.6}).$$

Hence we conclude (using (2) above) that

$$H^0(\mathbf{P}^1, (\mathscr{V}_\delta(\phi))^*) \approx V_\delta(\tau)^*. \tag{3}$$

This together with (1) above proves that

$$H^i(\mathbf{P}^1, K) \approx 0, \qquad \text{for all } i.$$

From this assertion (f) follows. To prove (e), as in the proof of Proposition 8.13, if we denote

$$\chi(\mathscr{W}) = \text{char } H^0(\mathbf{P}^1, \mathscr{W}) - \text{char } H^1(\mathbf{P}^1, \mathscr{W})$$

where W is a B-module and $\mathscr{W}$ the associated vector bundle on $\mathbf{P}^1$, then taking $W = (V_\delta(\phi))^*$, we have

$$\chi(\mathscr{W}) = \text{char } H^0(\mathbf{P}^1, (V_\delta(\phi))^*) \tag{4}$$

(since $H^1(\mathbf{P}^1, (\mathscr{V}_\delta(\phi))^*) = 0$). On the other hand

$$\chi(\mathscr{W}) = M_{s_\alpha}(\text{char } W), \qquad \text{for any } B\text{-module } W$$

(cf. (4) in the proof of Proposition 8.13). Hence we obtain (using (3) and (4)) that

$$\text{char}(V_\delta(\tau))^* = M_{s_\alpha}(\text{char}(V_\delta(\phi))^*).$$

Now the assertion (e) follows using (d) of Theorem 9.6 and induction on $\dim X(\tau)$ (the result being trivially true when $\dim X(\tau) = 0$).

This completes the proof of Corollary 9.8.

10. Behaviour of Schubert Varieties under Unions and Intersections

In view of Theorem 9.6 and Lemma 6.3 we obtain that intersection of a family of Schubert varieties is reduced (note that a union of Schubert varieties is obviously reduced). In particular this enables us to compute the ideal sheaves of Schubert varieties as given by Theorem 10.3 below.

Let R be the multi-graded ring defined by

$$R_{\mathbf{a}} = H^0(G/Q, L_{\mathbf{a}})$$

where $L_{\mathbf{a}} = \bigoplus_{i=1}^r L_i^{a_i}$, $a_i \geq 0$, $Q = \bigcap_{i=1}^r P_i$ (Q being a parabolic subgroup of classical type). If J is a multi-graded ideal of R, then J determines a closed subscheme of G/Q, which we shall denote by $V(J)$. Conversely, any closed subscheme X of G/Q determines a multi-graded ideal $I(X)$ of R (namely, the ideal in R generated by all multi-homogeneous f in R vanishing on X). Let now J_i be a graded ideal of the homogeneous co-ordinate ring $R_i = \bigotimes_{m \geq 0} H^0(G/P_i, L_i^m)$ (L_i being the ample generator of $\text{Pic}(G/P_i)$) and let $\tilde{J}_i$ be the multi-graded ideal of R generalized by J_i. Now, if π_i: $G/B \to G/P_i$ is the canonical projection, then it can be easily seen that

$$V(\tilde{J}_i) = \pi_i^{-1}(V(J_i)) \qquad (\dagger)$$

in the scheme-theoretic sense.

LEMMA 10.1. *Let $w \in W$. For $1 \leqslant i \leqslant n$, let $X(\phi_i)$ be the pull-back under π_i: $G/B \to G/P_i$ of $\pi_i(X(w))$. Let $\theta \in W$ be such that $\theta \leqslant \phi_i$, $1 \leqslant i \leqslant l$ (where $l = \operatorname{rank} G$). Then $\theta \leqslant w$.*

Proof (By induction on $l(w)$). If $l(w) = 0$, then $w = \mathrm{Id}$ and ϕ_i is nothing but the element of maximal length in W_{P_i}. Now if $\theta \in W$ be such that $\theta \leqslant \phi_i$, $1 \leqslant i \leqslant l$, then θ is in fact the identity element (since $\bigcap_{i=1}^{l} P_i = B$). Thus $\theta = w$.

Now let $l(w) \geqslant 1$. Fix a simple root α such that $s_\alpha w < w$. Set $s_\alpha = s$. For $1 \leqslant i \leqslant n$, let $\sigma_i \in W_{P_i}$ be such that $\phi_i = w\sigma_i$ with $l(w\sigma_i) = l(w) + l(\sigma_i)$. We have $l(sw\sigma_i) = l(sw) + l(\sigma_i)$ (since $l(sw) + l(\sigma_i) \geqslant l(sw\sigma_i)$) and $l(w) + l(\sigma_i) = l(w\sigma_i) = l(s \cdot sw\sigma_i) \leqslant l(sw\sigma_i) + 1$, i.e., $l(sw\sigma_i) \geqslant l(sw) + l(\sigma_i)$ (since $l(w) = l(sw) + 1$). Now $l(sw\sigma_i) = l(w\sigma_i) - 1$ implies that $X(sw\sigma_i)$ is the pull-back of $\pi_i(X(sw))$ under π_i: $G/B \to G/P_i$. Now we distinguish the following two cases.

Case 1. For $1 \leqslant i \leqslant l$, $\theta \leqslant sw\sigma_i$. Then by induction hypothesis we obtain that $\theta \leqslant sw$ and hence $\theta < w$.

Case 2. For some i, $1 \leqslant i \leqslant l$, $\theta \nleqslant sw\sigma_i$. Then we obtain (by Lemma 1.5) that for such an i. $s\theta < sw\sigma_i$. In particular, in this case, we obtain that $\theta > s\theta$. Hence for any j, if $\theta \leqslant sw\sigma_j$, then $s\theta$ is also $< sw\sigma_j$ and for any j if $\theta \nleqslant sw\sigma_j$, then again $s\theta$ is $< sw\sigma_j$. Thus $s\theta < sw\sigma_j$, for all j, $1 \leqslant j \leqslant l$. Hence by induction hypothesis, $s\theta < sw$. Now this implies that $\theta < w$ (since $sw < w$). This completes the proof of Lemma 10.1. As an immediate consequence of Lemma 10.1, we have

COROLLARY 10.2. *Notations being as in Lemma* 10.1 *we have, set theoretically,*

$$X(w) = \bigcap_{i=1}^{l} X(\phi_i).$$

THEOREM 10.3. *For a Schubert variety $X = X(w)$ in G/Q, we have that the ideal sheaf of X in G/B is generated by the set of all $p(\tau_i, \phi_i)$, $1 \leqslant i \leqslant r$, such that (τ_i, ϕ_i) is an admissible pair with $w_i \ngeqslant \tau_i$, w_i being the projection of w under $W \to W/Wp_i$.*

Proof. By Corollary 10.2, we have

$$X = \bigcap_{i=1}^{l} \pi_i^{-1}(X(W_i)) \qquad \text{(set theoretically)}. \qquad (*)$$

Now if J_i denotes the ideal in R_i, generated by

$$\left\{ p(\tau_i, \phi_i) \;\middle/\; \begin{array}{l} (1)\ (\tau_i, \phi_i) \text{ is an admissible pair} \\ (2)\ w_i \not\geqslant \tau_i \end{array} \right\}$$

then we have (cf. Theorem 3.15 and (†) above)

(1) $V(J_i) = X(w_i)$

(2) $V(\tilde{J}_i) = \pi_i^{-1}(X(w_i))$

where $\tilde{J}_i$ denotes the multi-graded ideal of R generated by J_i. Now as mentioned above, as a consequence of Theorem 9.6, we obtain that the relation $(*)$ above is in fact scheme-theoretic from which the required result follows (using Theorem 3.15). Guided by Theorem 10.3, we make the following

Conjecture. G a semi-simple algebraic group and P a maximal parabolic subgroup:

(1) A Schubert variety $X(w)$ in G/P is defined scheme theoretically by the vanishing of certain weight vectors $f_i^w \in H^0(G/P, L)$, L being the ample generator of $\mathrm{Pic}(G/P)$ (note that set theoretically $X(w)$ is defined by the vanishing of all p_λ, $w \not\geqslant \lambda$) (cf. Remark 5.7).

(2) For a Schubert variety $X(w)$ in G/B, the ideal sheaf of $X(w)$ in G/B is generated by the set of all $f_i^{w_j}$, $1 \leqslant j \leqslant l$ (l being the rank of G), $X(w_j)$ being the projection of $X(w)$ under $G/B \to G/P_j$, $1 \leqslant j \leqslant l$.

(3) Unions and intersections of Schubert varieties are reduced.

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