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Standard Monomial Theory for G_2

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INTRODUCTION

Let G be a semi-simple, simply connected Chevalley group over a field k, T a maximal (split) torus, B a Borel subgroup, $B \supset T$. Let P be a maximal parabolic subgroup of G, $P \supset B$. Let L be a positive line bundle on G/B. When G is a classical group, an explicit basis for $H^0(G/B, L)$ (more generally for $H^0(X, L)$, X a Schubert variety in G/B) has been constructed in [9, 10] by means of standard monomials, as a generalization of the classical Hodge–Young standard monomial theory (cf. [5, 6]). When G is not classical, similar results have been obtained in [9, 10], for G/Q and its Schubert varieties, where Q is a parabolic subgroup of classical type, i.e., $Q = \bigcap_{i=1}^{r} P_i$, where $P_i, 1 \leqslant i \leqslant r$, is a maximal parabolic subgroup with the property that if ω_i is the associated fundamental weight, then $|(\omega_i, \alpha^*)| \leqslant 2$ for all roots α.

The problem of developing a standard monomial theory for G/P (more generally for G/Q), where P (resp. Q) is not of classical type, is open. In this paper we state a conjecture–arrived at in collaboration with Seshadri—towards this problem (cf. Section 2); further, we develop a standard monomial theory for Schubert varieties in G/B, G being of type $\mathbf{G}_2$ (which also verifies the conjecture for the case where G is of type $\mathbf{G}_2$). The spirit of this paper is the same as in [9]; namely, we first develop the theory for G/P, where P is a maximal parabolic subgroup of G. This is done by constructing an explicit basis for $H^0(G/P, L)$, where L is the ample generator of $\mathrm{Pic}(G/P)$, and defining the notion that a monomial in the basis elements is standard on a Schubert variety $X(\tau)$ in G/P, and then proving (cf. Theorem 6.3) that standard monomials of degree m on $X(\tau)$ form a basis for $H^0(X(\tau), L^m)$. The proof of linear independence of standard monomials (cf. Section 5) is carried out in the same spirit as in [9];

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the proof of generation by standard monomials is done in a more elegant way using the results of [10] so that we are able to avoid the tedious counting arguments (cf. 9, Sect. 4]).

Using the results for G/P, we then define standard monomials of bidegree (m_1, m_2) on a Schubert variety in G/B in the same spirit as in [8, 9, 11] and prove (cf. Theorem 8.18) that given $L = L(\lambda)$, where $\lambda = m_1\omega_1 + m_2\omega_2$, $m_i \in \mathbf{Z}^+$, $i = 1, 2$, standard monomials of bidegree (m_1, m_2) on a Schubert variety $X(\tau)$ form a basis of $H^0(X(\tau), L)$; the method of the proof is as in [10].

For $\tau \in W$, let $R(\tau) = \bigoplus_{L \geqslant 0} H^0(X(\tau), L)$. Then using the explicit basis for $H^0(X(\tau), L)$ by means of standard monomials, we prove the ring $R(\tau)_{(0)}$ (and hence also $R(\tau)$) is Cohen–Macaulay (by exhibiting a canonical system of parameters). Then using a result of Chevalley (cf. [3]) that Schubert varieties are non-singular in co-dimension 1, we also obtain that the ring $R(\tau)$ is normal.

As a consequence of standard monomial theory we also obtain a proof of Demazure's conjecture (cf. [4]) and hence also obtain the result that

$$H^i(X(\tau), L) = 0, \qquad i \geqslant 1, \quad L \geqslant 0.$$

In [1], K. Backlawski and J. Towber have obtained results similar to ours. It should be remarked that they prove results *only* for the big cell of G/B, G being of type $\mathbf{G}_2$. Their arguments are more computational. (Compare Section 4 and the Appendix of [1] with Section 8 of this paper.)

The sections are arranged as follows.

Section 1 deals with some preliminary results.

In Section 2, we state the conjecture (mentioned above).

In Section 3, we collect some facts about the group G of type $\mathbf{G}_2$.

In Section 4, we construct a basis for $H^0(X(\tau), L)$, where $X(\tau) \subset G/P_2$ and L is the ample generator of $\text{Pic}(G/P_2)$. Here and elsewhere P_2 denotes the parabolic corresponding to the short simple root α_2.

In Section 5, we prove the linear independence of standard monomials on $X(\tau)$.

In Section 6, we prove that the set of standard monomials of deg m on $X(\tau)$ give a basis for $H^0(X(\tau), L^m)$, where $X(\tau) \subset G/P_2$. (Note that a standard monomial theory for G/P_1 has already been developed in [9], since ω_1 is a fundamental weight of classical type.)

In Section 7, we prove the Cohen–Macaulayness and normality for the cones over Schubert varieties in G/P, where $P = P_i$, $i = 1, 2$.

In Section 8, we develop a standard monomial theory for Schubert varieties in G/B and also prove the Cohen–Macaulayness and normality of the multicones over Schubert varieties in G/B.

1. Preliminaries

Let G be a semi-simple, simply connected Chevalley group over a field k, T a maximal (split) torus, B a Borel subgroup, $B \supset T$, and W the Weyl group of G. Let (,) be a W-invariant scalar product on $\mathrm{Hom}(T, \mathbf{G}_m)$. Let Q be a parabolic subgroup of G, $Q \supset B$ and W_Q the Weyl group of Q. For $w \in W/W_Q$, let $X(w)$ $(= \overline{BwQ} \pmod{Q})$ with the canonical reduced scheme structure denote the Schubert variety in G/Q.

Partial Order on W/W_Q. Given $w_1, w_2 \in W/W_Q$, we have the canonical partial order, namely $w_1 \geqslant w_2$, if and only if $X(w_1) \subseteq X(w_2)$.

The Sets $W_Q^{\min}$, $W_Q^{\max}$. Given a parabolic subgroup Q, let $W_Q^{\min}$ (resp. $W_Q^{\max}$) denote the set of minimal (resp. maximal) representatives of W/W_Q. (Note that

$$W_Q^{\min} = \{w \in W/w(\alpha) > 0, \alpha \in S_Q\}$$
$$W_Q^{\max} = \{w \in W/w(\alpha) < 0, \alpha \in S_Q\},$$

where S_Q is the set of simple roots associated to Q.)

Multiplicity of a Schubert Divisor in a Schubert Variety. Given a Schubert divisor $X(\phi)$ in $X(\tau)$, let $\phi = s_\alpha \tau$, for some root α (cf. [4]). Now letting Q be a maximal parabolic subgroup and H the unique co-dimension one subvariety in G/P, Chevalley (cf.. [3]) has shown that if

$$[X(\tau)] \cdot [H] = \sum d_\phi [X(\phi)]$$

in the Chow ring of G/P (where the summation runs over all Schubert divisors in $X(\phi)$ in $X(\tau)$), then

$$d_\phi = |(\phi(\omega), \alpha^*)|,$$

where ω is the fundamental weight associated to P. We shall refer to d_ϕ as the *multiplicity of $X(\phi)$ in the hyperplane section of $X(\tau)$ or just multiplicity of $X(\phi)$ in $X(\tau)$) and denote it by $m(\phi, \tau)$.*

Definition 1.1. Let ϕ, τ be as above. If $m(\phi, \tau) > 1$, then we shall call $X(\phi)$ a *multiple divisor* in $X(\tau)$. In particular, if $m(\phi, \tau) = 2$ (resp. 3), we shall call $X(\phi)$ a double (resp. triple) divisor in $X(\tau)$.

Definition 1.2. Let ϕ, τ be as above. We call $X(\phi)$ a *moving divisor in* $X(\tau)$ (moved by α) if α (where $\phi = s_\alpha \tau$) is a simple root.

Next we recall the condition for $X(\varphi)$ to be a moving divisor in $X(\tau)$ (cf. [9, Lemma 1.2]).

PROPOSITION 1.3. *Let $\phi, \tau \in W_P^{\min}$ (where P is a maximal parabolic subgroup). Further let $\phi = s_\alpha \tau$, where α is simple. Then $X(\phi)$ is a moving divisor in $X(\tau)$ if and only if $(\phi(\omega), \alpha^*) > 0$.*

Finally we want to recall the following Proposition (cf. [12]).

Let $w \in W$ and let $X(\theta)$ be the projection of $X(w)$ on G/P under $G/B \to G/P$, where P is a maximal parabolic subgroup. Let p_θ denote the extremal weight vector in $H^0(G/P, L)$ of weight $-\theta(\omega)$ (note that p_θ is unique up to scalars). It can be easily seen that

$$X(w) = (\text{zero set of } p_\theta \text{ in } G/B) = \bigcup_{w_i} X(w_i) \qquad (*)$$

(set theoretically), where $X(w_i)$ are the Schubert divisors in $X(w)$ such that the projection of $X(w_i)$ on G/P under $G/B \to G/P$ is $\subsetneq X(\theta)$.

PROPOSITION 1.4 (cf. [12]). *Let w, w_i be as above; further let $w_i = s_{\alpha_i} w$. If $|(w_i(\omega), \alpha^\vee)| = 1$, $\forall i$, then the intersection $X(w) \cap \{p_\theta = 0\}$ is reduced (in particular the set-theoretic equation $(*)$ is scheme-theoretic).*

2. A CONJECTURE

In this section we want to state a conjecture (arrived at in collaboration with Seshadri) on the indexing of a nice basis for $H^0(G/B, L(\chi))$, χ being a dominant weight. The indexing will be related to chains of Weyl group elements (to be very precise, the indexing will be by "admissible forms" on chains of Weyl group elements). Before we state the conjecture, we make some definitions.

DEFINITION 2.1. By a *chain* in W, we mean a sequence of Weyl group elements

$$\tau_0 > \tau_1 > \tau_2 > \cdots > \tau_r,$$

where $l(\tau_i) = l(\tau_{i-1}) - 1$, $1 \leqslant i \leqslant r$.

Remark 2.2. We shall regard any element of the Weyl group as a trivial chain (consisting of just one element).

DEFINITION 2.3. Assume given a chain $c=\{\tau_0, \tau_1, ..., \tau_r\}$ (as above). Let $m_i=|(\tau_i(\chi), \alpha_i^*)|$, where $\tau_{i-1}=s_{\alpha_i}\tau_i$, $1 \leqslant i \leqslant r$. We denote this by

Let $\lambda_0=m_1$; $\lambda_1=L\cdot C\cdot M(m_1, m_2)$, $\lambda_2=L\cdot C\cdot M(m_2, m_3)$,...,; $\lambda_r=m_r$. Let $\mu=L\cdot C\cdot M(\lambda_0, ..., \lambda_r)$ $(=L\cdot C\cdot M(m_1, ..., m_r))$ and let $\mu_i=\mu/\lambda_i$. We consider a form $\tau_0^{a_0}\tau_1^{a_1}\cdots\tau_r^{a_r}$ and call it an *admissible form* on c if

(i) $a_i \geqslant 0$, a_0 and a_r are non-zero,

(ii) $\sum_{i=0}^r (a_i/\lambda_i)=1$ (equivalently $\sum_{i=0}^r a_i\mu_i=\mu$),

(iii) $-\sum (a_i/\lambda_i)\,\tau_i(\chi)$ $(=(1/\mu)\sum_{i=1}^r a_i\mu_i\tau_i(\omega))$ is a weight in the T-module $H^0(G/B, L(\chi))$.

(iv) If i and j are such that $i<j$ and $a_k=0$ for some k, $i<k<j$, then $\tau_i^{a_i}\cdots\tau_j^{a_j}$ is an admissible form on the chain $c_{i,j}=\{\tau_i, \tau_{i+1}, ..., \tau_j\}$.

Remark 2.4. To an admissible form $\tau_0^{a_0}\cdots\tau_r^{a_r}$ on c, we associate a weight in the T-module $H^0(G/B, L(\chi))$, namely $-\sum_{i=0}^r (a_i/\lambda_i)\,\tau_i(\chi)$. The extremal weight $-\tau(\chi)$, $\tau \in W$, corresponds to the unique admissible form on the trivial chain $c=\{\tau\}$.

Remark 2.5. If $\chi=\omega$, a fundamental weight, then the condition (iii) above seems to be automatically satisfied.

Conjecture. $H^0(G/B, L(\chi))$ has a basis $\{p_\alpha\}$ indexed by admissible forms on chains c in W such that

(1) p_α is a weight vector of weight equal to that associated to the admissible form α (cf. Remark 2.4 above).

(2) If α is an admissible form on a chain $c=\{\tau_0, ..., \tau_r\}$, then $p_\alpha|_{X(w)} \neq 0$ if and only if $w \geqslant \tau_0$; further $\{p_\alpha/p_\alpha|_{X(w)} \neq 0\}$ is a basis for $H^0(X(w), L)$.

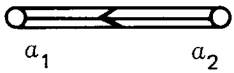
In this paper, among other things, we verify the conjecture for G of type $\mathbf{G}_2$ and $\chi=\omega_2$. The author has also verified that

$$\#\ \{\text{admissible forms}\}=\dim V_\chi,$$

where G is type $\mathbf{F}_4$ and $\chi=\omega_2$ or ω_3 and V_χ is the irreducible G-module (over $\mathbf{Q}$) with highest weight χ (note that for a fundamental weight of classical type, the admissible forms as defined here are precisely the admissible pairs as defined in [9]; in particular the conjecture holds in these cases (cf. [9]).

3. The Group $\mathbf{G}_2$

Let G be of type $\mathbf{G}_2$. Now G has the Dynkin diagram



(following the notations as in [2]). The Weyl roup of G has the following 12 elements:

$$\tau_0 = \phi_0 = \mathrm{Id}; \qquad \tau_6 = \phi_6 = s_1 s_2 s_1 s_2 s_1 s_2 = w_0$$

$$\tau_1 = s_2 \qquad \phi_1 = s_1$$

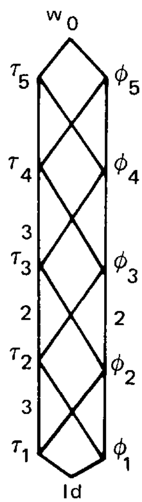
$$\tau_2 = s_1 s_2 \qquad \phi_2 = s_2 s_1$$

$$\tau_3 = s_2 s_1 s_2 \qquad \phi_3 = s_1 s_2 s_1$$

$$\tau_4 = s_1 s_2 s_1 s_2 \qquad \phi_4 = s_2 s_1 s_2 s_1$$

$$\tau_5 = s_2 s_1 s_2 s_1 s_2 \qquad \phi_5 = s_1 s_2 s_1 s_2 s_1$$

The configuration of the Schubert varieties is given by



Whenever $X(\phi)$ is a multiple divisor in $X(\tau)$ (cf. Definition 1.1) we have indicated the corresponding multiplicity.

Remark 3.1. It can be easily seen that the elements τ_i, $1 \leqslant i \leqslant 5$, belong to $W_{P_2}^{\min}$ and $W_{P_1}^{\max}$ and the elements ϕ_i, $1 \leqslant i \leqslant 5$, belong to $W_{P_1}^{\min}$ and $W_{P_2}^{\max}$.

Recall (cf. [7, 9]) that ω_1 is a fundamental weight of classical type (i.e., $|(\omega_1, \alpha^*)| \leqslant 2$, for all roots α) and that a standard monomial theory has been developed for Schubert varieties in G/P_1. In fact the Schubert varieties in G/P_1 are totally ordered (under inclusion); they are the $X(\phi_i)$'s, $0 \leqslant i \leqslant 5$. Further ω_1 is quasi-miniscule (cf. [7]). Thus each $X(\phi_i)$ occurs with multiplicity one in $X(\phi_{i+1})$ except when $i=2$ and $X(\phi_2)$ occurs with multiplicity two in $X(\phi_3)$. Recall (cf. [7], [9], or [11]) the following

THEOREM 3.2. *Let L be the ample generator of* Pic(G/P), *where $P = P_1$. Then there exists a basis $\{p_{\alpha,\beta}\}$ indexed by admissible pairs (cf. [7, 9, 11]) for $H^0(G/P, L)$ such that*

(1) *$p_{\alpha,\beta}$ is a weight vector of weight $-\frac{1}{2}(\alpha(\omega_1) + \beta(\omega_1))$.*

(2) *For a Schubert variety $X(w)$ in G/P_1, $p_{\alpha,\beta}|_{X(w)} \neq 0$ if and only if $w \geqslant \alpha$. Further $\{p_{\alpha,\beta}/w \geqslant \alpha\}$ is a basis for $H^0(X(w), L)$.*

(3) *Distinct standard monomials of* deg m *standard on $X(w)$ (i.e., monomials $p_{\alpha_1,\beta_1} p_{\alpha_2,\beta_2} \cdots p_{\alpha_m,\beta_m}$, $w \geqslant \alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots \geqslant \alpha_m \geqslant \beta_m$) form a k-basis of $H^0(X(w), L^m)$.*

(It should be remarked (cf. [7]) that all admissible pairs (α, β) are trivial admissible pairs; i.e., $\alpha = \beta$ except the unique non-trivial admissible pair (ϕ_3, ϕ_2).)

4. A BASIS FOR $H^0(X(\tau), L)$

In the following sections (Sections 4 through 6) we shall develop the standard monomial theory for Schubert varieties in $G/P_{\hat{\alpha}_2}$. Let us denote $P_{\hat{\alpha}_2}$ by P. Now the Schubert varieties in G/P are given by $X(\tau_i)$, $0 \leqslant i \leqslant 5$. They are again totally ordered (under inclusion).

Let $k = \mathbf{Q}$ and let V be the irreducible G-module with highest weight ω_2. Note that V is nothing but the adjoint representation of G. Fix a highest weight vector e in V and for $\tau \in W$, let $e_\tau = \tau e$. We may take e to be the element $X_{3\alpha_1 + 2\alpha_2}$ of the Chevalley basis (cf. [13]) of $\mathfrak{g}$, the Lie algebra of G; then we have

$$e_{\tau_0} = X_{3\alpha_1+2\alpha_2}; \qquad e_{\tau_5} = X_{-(3\alpha_1+2\alpha_2)}$$

$$e_{\tau_1} = X_{3\alpha_1+\alpha_2}; \qquad e_{\tau_4} = X_{-(3\alpha_1+\alpha_2)}$$

$$e_{\tau_2} = X_{\alpha_2}; \qquad e_{\tau_3} = X_{-\alpha_2}.$$

For $0 \leqslant i \leqslant 5$, let $V_{\mathbf{Z}}(\tau_i) = V_i = U_{\mathbf{Z}} e_{\tau_i}$, where $U_{\mathbf{Z}}$ is the canonical $\mathbf{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$. If $\tau_i = s_\alpha \tau_{i+1}$, we have (cf. [9, Lemma 5.2, Remark 5.4]) $V_{i+1} = U_{-\alpha,\mathbf{Z}} V_i$, where $U_{-\alpha,\mathbf{Z}}$ is the $\mathbf{Z}$-submodule of $U_{\mathbf{Z}}$ spanned by $X^n_{-\alpha}/n!$, $n \geqslant 0$. Using the commuting relations (in $\mathfrak{g}$) (cf. [13]

$$[X_\alpha, X_\beta] = \pm(r+1)\, X_{\alpha+\beta},$$

where r is the largest integer such that $\alpha - r\beta$ is a root, it is not difficult to see that

$$V_0 = \mathbf{Z}\text{-span of } X_{3\alpha_1+2\alpha_2}$$
$$V_1 = \mathbf{Z}\text{-span of } V_0 \text{ and } X_{3\alpha_1+\alpha_2}$$
$$V_2 = \mathbf{Z}\text{-span of } V_1 \text{ and } \{X_{2\alpha_1+\alpha_2}, X_{\alpha_1+\alpha_2}, X_{\alpha_2}\}$$
$$V_3 = \mathbf{Z}\text{-span of } V_2 \text{ and } \{H_{\alpha_2}, X_{\alpha_1}, X_{-\alpha_2}\}$$
$$V_4 = \mathbf{Z}\text{-span of } V_3 \text{ and } \{H_{\alpha_1}, X_{-(\alpha_1+\alpha_2)}, X_{-(2\alpha_1+\alpha_2)}, X_{-(3\alpha_1+\alpha_2)}, X_{-\alpha_1}\}$$
$$V_5 = \mathbf{Z}\text{-span of } V_4 \text{ and } X_{-(3\alpha_1+2\alpha_2)} = \mathfrak{g}_{\mathbf{Z}},$$

where $\mathfrak{g}_{\mathbf{Z}}$ is the Chevalley $\mathbf{Z}$-form of $\mathfrak{g}$. Now the etremal weight vectors, e_{τ_i}, $0 \leqslant i \leqslant 5$, are the X_α, α being a long root. Let us denote the non-extremal weight vectors $\{H_{\alpha_1}, H_{\alpha_2}, X_\beta, \beta \text{ a short root}\}$ as follows.

$$X_{2\alpha_1+\alpha_2} = E(\tau_2, \tau_1); \qquad X_{-(2\alpha_1+\alpha_2)} = F(\tau_4, \tau_3)$$
$$X_{\alpha_1+\alpha_2} = F(\tau_2, \tau_1); \qquad X_{-(\alpha_1+\alpha_2)} = E(\tau_4, \tau_3)$$
$$X_{\alpha_1} = M(\tau_3, \tau_1); \qquad X_{-\alpha_1} = M(\tau_4, \tau_2)$$
$$H_{\alpha_1} = M(\tau_4, \tau_1); \qquad H_{\alpha_2} = M(\tau_3, \tau_2).$$

Now the weight of $E(\tau_2, \tau_1)$ (resp. $F(\tau_2, \tau_1)$) is $\frac{1}{3}(\tau_2(\omega) + 2\tau_1(\omega))$ (resp. $\frac{1}{3}(2\tau_2(\omega) + \tau_1(\omega))$ (where $\omega = \omega_2$); the weight of $E(\tau_4, \tau_3)$ (resp. $F(\tau_4, \tau_3)$) is $\frac{1}{3}(\tau_4(\omega) + 2\tau_3(\omega))$ (resp. $\frac{1}{3}(2\tau_4(\omega) + \tau_3(\omega))$; and

$$\text{weight of } M(\tau_3, \tau_1) = \tfrac{1}{6}(3\tau_3(\omega) + \tau_2(\omega) + 2\tau_1(\omega))$$
$$\text{weight of } M(\tau_3, \tau_2) = \tfrac{1}{2}(\tau_3(\omega) + \tau_2(\omega))$$
$$\text{weight of } M(\tau_4, \tau_1) = \tfrac{1}{6}(2\tau_4(\omega) + \tau_3(\omega) + \tau_2(\omega) + 2\tau_1(\omega))$$
$$\text{weight of } M(\tau_4, \tau_2) = \tfrac{1}{6}(2\tau_4(\omega) + \tau_3(\omega) + 3\tau_2(\omega)).$$

DEFINITION 4.1. Let $S(\tau, \phi)$ denote any one of the above 14 basis elements (note that if $S(\tau, \phi)$ is an extremal weight vector, then $\tau = \phi$). Define $\{N(\tau, \phi)\}$ to be the basis of $\mathfrak{g}^*_{\mathbf{Z}}$ (the $\mathbf{Z}$-dual of $\mathfrak{g}_{\mathbf{Z}}$) dual to $\{S(\tau, \phi)\}$.

Notation. The elements $N(\tau, \phi)$ being defined as in Definition 4.1, let us denote the elements corresponding to $M(\ ,\ )$ by $P(\ ,\ )$, those corresponding to $E(\ ,\ )$ by $Q(\ ,\ )$ and those corresponding to $F(\ ,\ )$ by $R(\ ,\ )$. Also the extremal weight vectors $N(\tau, \tau)$ in $\mathfrak{g}_{\mathbf{Z}}^*$ will be denoted by $P(\tau, \tau)$ or just $P(\tau)$. Recall (cf. [10, 14]) that we have a canonical closed immersion

$$j_{\mathbf{Z}}: G_{\mathbf{Z}}/P_{\mathbf{Z}} \to \mathbf{P}(V_{\mathbf{Z}}^*)$$

(here $V_{\mathbf{Z}}$ is $\mathfrak{g}_{\mathbf{Z}}$), which induces a canonical isomorphism

$$j_{\mathbf{Z}}: V_{\mathbf{Z}}^* \simeq H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}}).$$

Further, since in the present case, $V_{\mathbf{Z}}(\tau)$ is a direct summand in $V_{\mathbf{Z}}$, we also obtain a canonical injective homomorphism

$$j_{\mathbf{Z}}(\tau): V_{\mathbf{Z}}^*(\tau) \hookrightarrow H^0(X_{\mathbf{Z}}(\tau), L_{\mathbf{Z}}),$$

for all $\tau \in W/W_P$ (cf. [14]). Now over $\mathbf{Q}$ this last map is an isomorphism (cf. [4] or [9, Lemma 5.1]). Hence we obtain that $j_{\mathbf{Z}}(\tau)$ is an isomorphism and thus for every field k the map

$$j_{\mathbf{Z}}(\tau) \otimes 1: V_{\mathbf{Z}}^*(\tau) \otimes_{\mathbf{Z}} k \to H^0(X_{\mathbf{Z}}(\tau) \otimes k, L_{\mathbf{Z}} \otimes k)$$

is an injection.

In view of the above considerations we have the following

PROPOSITION 4.2. *For any field k, let*

$$n(\tau, \phi) = N(\tau, \phi) \otimes 1,$$

the image of $N(\tau, \phi)$ in $H^0(G_{\mathbf{Z}} \otimes k/P_{\mathbf{Z}} \otimes k, L_{\mathbf{Z}} \otimes k)$. Then the restriction of $n(\tau, \phi)$ to $X(w)$ $(= X_{\mathbf{Z}}(w) \otimes k)$ is not identically zero if and only if $w \geqslant \tau$.

Proof. Now $V(w)^* = V_{\mathbf{Z}}^*(w) \otimes_{\mathbf{Z}} \mathbf{Q} \approx H^0(X(w), L)$ (cf. [4; 9, Lemma 5.1]) and the canonical linear map is surjective (cf. [4]). This means that the kernel of the canonical surjective linear map

$$V^* = H^0(G/P, L) \to H^0(X(w), L) = V(w)^*$$

is surjective (cf. [4]). This means that the kernel of the canonical surjective linear map

$$V_{\mathbf{Z}}^* \otimes \mathbf{Q} \to H^0(X_{\mathbf{Z}}(w), L_{\mathbf{Z}}) \otimes \mathbf{Q} \qquad (= H^0(X(w), L))$$

coincides with the kernel of the surjective linear map

$$V_{\mathbf{Z}}^* \otimes \mathbf{Q} \to V_{\mathbf{Z}}^*(w) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Since the kernels of the canonical maps

$$V_{\mathbf{Z}}^{*} \to H^0(X_{\mathbf{Z}}(w), L_{\mathbf{Z}}), \qquad V_{\mathbf{Z}}^{*} \to V_{\mathbf{Z}}^{*}(w)$$

are direct summands, it follows that

$$\ker(V_{\mathbf{Z}}^{*} \to H^0(X_{\mathbf{Z}}(w), L_{\mathbf{Z}})) = \ker(V_{\mathbf{Z}}^{*} \to V_{\mathbf{Z}}^{*}(w)).$$

Now the kernel on the RHS is spanned by $\{N(\tau, \phi), \tau \not\leqslant w\}$ (by the way $N(\tau, \phi)$ has been constructed). Hence we obtain that $n(\tau, \phi)\,|_{X(w)} = 0$ if $w \not\geqslant \tau$, which is one part of the assertion of Proposition 4.2; the other part, namely $n(\tau, \phi)\,|_{X(w)} \neq 0$, if $w \geqslant \tau$, follows in view of the injection

$$V_{\mathbf{Z}}^{*}(w) \otimes k \to H^0(X_{\mathbf{Z}}(w) \otimes k, L_{\mathbf{Z}} \otimes k)$$

for every field k (as remarked above).

5. Standard Monomials and Their Linear Independence

The notations in this section are as in Section 4.

Definition 5.1. We call a monomial $n(\theta_1, \mu_1)\, n(\theta_2, \mu_2) \cdots n(\theta_m, \mu_m)$ *standard on the Schubert variety* $X(\tau)$ if $\tau \geqslant \theta_1 \geqslant \mu_1 \geqslant \theta_2 \geqslant \mu_2 \geqslant \cdots \geqslant \theta_m \geqslant \mu_m$ in W/W_p.

Remark. A linear combination of standard monomials on $X(\tau)$ will also be referred to as a standard sum on $X(\tau)$.

Proposition 5.2. *Standard monomials on $X(\tau)$ of degree m are linearly independent.*

Proof. The philosophy of the proof is the same as in [9, 10]; we first derive some relations on $X(\tau)$ as given by the following lemmas.

Lemma 5.3. *Let $\tau = \tau_4$ or τ_2 and let $X(\phi)$ be the unique divisor in $X(\tau)$. Then on $X(\tau)$, we have (up to ± 1)*

(1) $q_{\tau,\phi}^3 = p_\tau p_\phi^2$,

(2) $r_{\tau,\phi}^3 = p_\tau^2 p_\phi$,

(3) $q_{\tau,\phi} r_{\tau,\phi} = p_\tau p_\phi$.

Proof. Now the zero set of p_τ on $X(\tau)$ is $X(\phi)$ and the order of vanishing on $X(\phi)$ of p_τ is nothing but the multiplicity of $X(\phi)$ in $X(\tau)$ (cf. Section 1). Now since $X(\phi)$ is a triple divisor in $X(\tau)$, we conclude that order of vanishing of p_τ on $X(\phi)$ is 3. Now the function $q^3_{\tau,\phi}$ vanishes up to order $\geqslant 3$ on $X(\phi)$. Now $X(\tau)$ being normal (cf. [14]) this implies

$$q^3_{\tau,\phi}/p_\tau = F, \qquad F \in H^0(X(\tau), L^2).$$

Now weight of $F = 3$ weight of $q_{\tau,\phi}$ − weight of $p_\tau = -(\tau(\omega) + 2\phi(\omega)) - (-\tau(\omega)) = -2\phi(\omega)$, which is an extremal weight in $H^0(X(\tau), L^2)$. Hence we conclude that $F = ap^2_\phi$, $a \in k^*$. Assuming $k = \mathbf{Q}$, we have

$$q^3_{\tau,\phi} = ap_\tau p^2_\phi, \qquad a \in \mathbf{Q}^*.$$

Clearing denominators, we may rewrite this (over $\mathbf{Z}$) as

$$bQ^3_{\tau,\phi} = cP_\tau P^2_\phi, \qquad b, c \in \mathbf{Z}.$$

Now the fact that P_τ, P_ϕ, $Q_{\tau,\phi}$ are all non-zero modulo p (for all prime p) (cf. Proposition 4.2) implies that $b = c = \pm 1$. Thus up to ± 1, we have $Q^3_{\tau,\phi} = P_\tau P^2_\phi$, and reduction modulo p gives

$$q^3_{\tau,\phi} = p_\tau p^2_\phi$$

over any field k. Next we prove the relation $q_{\tau,\phi} r_{\tau,\phi} = p_\tau p_\phi$. Now considering $q^2_{\tau,q} r_{\tau,\phi}$ and proceeding as above, we obtain that $q^2_{\tau,\phi} r_{\tau,\phi}$ is divisible by p_τ, say $q^2_{\tau,\phi} r_{\tau,\phi} = p_\tau F$, where F is a weight vector of weight = 2 weight of $q_{\tau,\phi}$ + weight of $r_{\tau,\phi}$ − weight of $p_\tau = -\frac{1}{3}(\tau(\omega) + 5\phi(\omega))$. Now assuming $k = \mathbf{Q}$ and using the surjectivity of $H^0(X(\tau), L) \otimes H^0(X(\tau), L) \to H^0(X(\tau), L^2)$ (over $\mathbf{Q}$; cf. [4]) we find that the only choice for the weights χ_1, χ_2 in $H^0(X(\tau), L)$ such that $\chi_1 + \chi_2 = -\frac{1}{3}(\tau(\omega) + 5\phi(\omega))$ is given by $\chi_1 = -\frac{1}{3}(\tau(\omega) + 2\phi(\omega))$ and $\chi_2 = -\phi(\omega)$, from which we conclude that F is $aq_{\tau,\phi} p_\phi$, $a \in \mathbf{Q}^*$. Thus over $\mathbf{Q}$,

$$q^2_{\tau,\phi} r_{\tau,\phi} = ap_\tau p_\phi q_{\tau,\phi},$$

which yields

$$q_{\tau,\phi} r_{\tau,\phi} = ap_\tau p_\phi, \qquad a \in \mathbf{Q}^*.$$

Clearing the denominator of a and proceeding as above we conclude $a = \pm 1$ and

$$Q_{\tau,\phi} R_{\tau,\phi} = \pm P_\tau P_\phi$$

and we obtain (3) over any field k by reducing modulo p. Next, to prove $r^3_{\tau,\phi} = p^2_\tau p_\phi$. Proceeding as above we obtain that $r^3_{\tau,\phi}$ is divisible by p_τ, say $r^3_{\tau,\phi} = p_\tau F$, where F is a weight vector of weight = weight of $r^3_{\tau,\phi}$ − weight of $p_\tau = -(\tau(\omega) + \phi(\omega))$. Assuming $k = \mathbf{Q}$, we have weight of $F = \chi_1 + \chi_2$ where

χ_1, χ_2 are weights in $H^0(X(\tau), L)$. Now the choices for χ_1, χ_2 so that $\chi_1+\chi_2=-(\tau(\omega)+\phi(\omega))$ are either $\chi_1=-\tau(\omega)$, $\chi_2=-\phi(\omega)$ or $\chi_1=-\frac{1}{3}(2\tau(\omega)+\phi(\omega))$, $\chi_2=-\frac{1}{3}(\tau(\omega)+2\phi(\omega))$. Now corresponding to the first choice, we obtain the vector $p_\tau p_\phi$ and corresponding to the second choice, we obtain the vector $r_{\tau,\phi}q_{\tau,\phi}$, which is $p_\tau p_\phi$ (up to ± 1) in view of (3) above. Thus we obtain (over $\mathbf{Q}$)

$$r^3_{\tau,\phi}=ap^2_\tau p_\phi.$$

Then proceeding as above we obtain (over $\mathbf{Z}$)

$$R^3_{\tau,\phi}=\pm P^2_\tau P_\phi.$$

Now reducing modulo p, we obtain

$$r^3_{\tau,\phi}=\pm p_\tau p_\phi$$

over any field k, which is relation (2). This completes the proof of Lemma 5.3.

COROLLARY 5.4. *The elements τ and ϕ being as in Lemma* 5.3, *we have* (*up to* ± 1)

(1) $q^2_{\tau,\phi}=r_{\tau,\phi}p_\phi$,

(2) $r^2_{\tau,\phi}=p_\tau q_{\tau,\phi}$.

Proof. The result follows rather trivially. For example, multiplying relation (1) of Lemma 5.3 by $r_{\tau,\phi}$ and using relation (3) of Lemma 3.3, we obtain

$$q^2_{\tau,\phi}p_\tau p_\phi=p_\tau p^2_\phi r_{\tau,\phi},$$

which yields

$$q^2_{\tau,\phi}=r_{\tau,\phi}p_\phi.$$

(Note that cancellation of $p_\tau p_\phi$ is justified since $X(\tau)$ is an integral scheme and p_τ and p_ϕ are non-zero on $X(\tau)$.) To obtain the second relation, using (3) of Lemma 5.3 we may rewrite relation (1) of Lemma 5.3 as

$$q^3_{\tau,\phi}=q_{\tau,\phi}r_{\tau,\phi}p_\phi.$$

And cancelling $q_{\tau,\phi}$, we obtain

$$q^2_{\tau,\phi}=r_{\tau,\phi}p_\phi.$$

LEMMA 5.5. *On $X(\tau_3)$, we have (up to ± 1)*

(a) $p^2_{\tau_3,\tau_2} = p_{\tau_3} p_{\tau_2}$,

(b) $p^2_{\tau_3,\tau_1} = p_{\tau_3} q_{\tau_2,\tau_1}$,

(c) $p_{\tau_3,\tau_1} p_{\tau_3,\tau_2} = p_{\tau_3} r_{\tau_2,\tau_1}$.

On $X(\tau_4)$ we have (up to ± 1)

(d) $q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_1} = p_{\tau_4} p_{\tau_3} p_{\tau_3,\tau_1}$.

The above relations are proved in the same spirit as Lemma 5.3. (While deriving (a), (b), (c), one observes that the order of vanishing of p_{τ_3} on $X(\tau_2)$ is 2 and thus one obtains quadratic relations as given by (a), (b), (c).)

Return to the Proof of Proposition 5.2. For $\tau = \tau_0$, the result is obvious. The proof of the result for $\tau = \tau_i$, $1 \leqslant i \leqslant 4$, essentially follows as a consequence of the various relations given in Lemmas 5.3 and 5.5 and Corollary 5.4. We shall carry out the proof for $X(\tau_4)$ in detail (which will illustrate the proof for $X(\tau_1)$, $X(\tau_2)$, and $X(\tau_3)$). Now let us suppose the result to be true for τ_i, $i \leqslant 3$. Let $F = 0$ be a linear combination of standard monomials on $X(\tau_4)$.

Writing $F = F_1 + F_2$, where $F_1 \mid_{X(\tau_3)} = 0$ and $F_2 \mid_{X(\tau_3)} \neq 0$ (in other words, F_2 is a linear combination of monomials standard on $X(\tau_3)$) and restricting the relation $F = 0$ to $X(\tau_3)$, we may conclude $F_2 = 0$ (using linear independence of standard monomials on $X(\tau_3)$). Thus we may assume

$$F = p_{\tau_4} G + r_{\tau_4,\tau_3} F_3 + q_{\tau_4,\tau_3} G_3 + p_{\tau_4,\tau_2} G_2 + p_{\tau_4,\tau_1} G_1 = 0, \qquad (*)$$

where G_i, $1 \leqslant i \leqslant 3$, is a standard sum on $X(\tau_i)$ and G (resp. F_3) is a standard sum on $X(\tau_4)$ (resp. $X(\tau_3)$). Multiplying $(*)$ by $q^2_{\tau_4,\tau_3}$ and using (1) of Lemma 5.3, (1) of Corollary 5.4, and (d) of Lemma 5.5, we obtain (up to ± 1)

$$\begin{aligned} p_{\tau_4} r_{\tau_4,\tau_3} p_{\tau_3} G + p_{\tau_4} p_{\tau_3} q_{\tau_4,\tau_3} F_3 + p_{\tau_4} p^2_{\tau_3} G_3 \\ + q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2} G_2 + p_{\tau_4} p_{\tau_3} p_{\tau_3,\tau_1} G_1 = 0. \end{aligned} \qquad (**)$$

Now on $X(\tau_4)$, $q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2}$ is divisible by p_{τ_4}, say

$$q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2} = p_{\tau_4} F,$$

where weight of $F =$ weight of $q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2} -$ weight of ρ_{τ_4} ($=$ weight of p_{τ_3}) and $F \in H^0(X(\tau_4), L^2)$. Hence, over $\mathbf{Q}$, F is a linear combination of $p_{\tau_3} p_{\tau_3,\tau_2}$, $p_{\tau_3} p_{\tau_4,\tau_1}$, and $q_{\tau_4,\tau_3} p_{\tau_3,\tau_1}$ (note that these are the only weight vectors in $H^0(X(\tau_4), L^2)$ (over $\mathbf{Q}$) of weight $=$ weight of p_{τ_3}). Then Let

$$q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2} = a p_{\tau_4} p_{\tau_3} p_{\tau_3,\tau_2} + b p_{\tau_4} p_{\tau_3} p_{\tau_4,\tau_1} + c p_{\tau_4} q_{\tau_4,\tau_3} p_{\tau_3,\tau_1}. \qquad (1)$$

Claim. $a \neq 0$. For, if $a=0$, then we would obtain $q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2} = b p_{\tau_4} p_{\rho_3} p_{\tau_4,\tau_1} + c p_{\tau_4} q_{\tau_4,\tau_3} p_{\tau_3,\tau_1}$. Now the order of vanishing of q_{τ_4,τ_3} on $X(\tau_3)$ is 1 (using the relation $q^3_{\tau_4,\tau_3} = p_{\tau_4} p^2_{\tau_3}$ and the fact that the order of vanishing of p_{τ_4} on $X(\tau_3)$ is 3). Hence this would imply that the order of vanishing of p_{τ_4,τ_2} on $X(\tau_3)$ is $\geqslant 2$ (since the order of vanishing of the right-hand side is $\geqslant 4$). Hence $p^2_{\tau_4,\tau_2}$ is divisible by p_{τ_4} and we would obtain (by weight considerations) that

$$p^2_{\tau_4,\tau_2} = d p_{\tau_4} r_{\tau_2,\tau_1}.$$

But this last relation cannot hold since the order of vanishing of the left-hand side on $X(\tau_3)$ is $\geqslant 4$ while that of right-hand side is 3. (Note that $r_{\tau_2,\tau_1} |_{X(\tau_3)} \neq 0$). Hence $a \neq 0$. (Note that incidentally we have also shown that the order of vanishing of p_{τ_4,τ_2} on $X(\tau_3)$ is 1). Now substituting the expression for $q^2_{\tau_4,\tau_3} p_{\tau_4,\tau_2}$ (as given by (1)) in (**), cancelling p_{τ_4}, and restricting to $X(\tau_3)$ we obtain

$$p^2_{\tau_3} G_3 + a p_{\tau_3} p_{\tau_3,\tau_2} G_2 + p_{\tau_3} p_{\tau_3,\tau_1} G_1 = 0,$$

where the left-hand side is a sum of standard monomials on $X(\tau_3)$. Also there cannot be any mutual cancellations among the three terms on the left-hand side. Hence we conclude that $G_i = 0$, $i = 1, 2, 3$ (note that $a \neq 0$), since by the induction hypothesis, standard monomials on $X(\tau_3)$ are linearly independent. Hence the relation $F=0$ reduces to

$$p_{\tau_4} G + r_{\tau_4,\tau_3} F_3 = 0.$$

Multiplying this relation by q_{τ_4,τ_3} and using Corollary 5.4 we obtain

$$p_{\tau_4} q_{\tau_4,\tau_3} G + p_{\tau_4} p_{\tau_3} F_3 = 0.$$

Now cancelling p_{τ_4} and restricting to $X(\tau_3)$ we obtain $p_{\tau_3} F_3 = 0$ on $X(\tau_3)$; i.e., $F_3 = 0$. And hence $F=0$ is reduced to $p_{\tau_4} G = 0$; i.e., $G=0$. Thus the original relation $F=0$ that we started with has to be the trivial relation.

This completes the proof of linear independence of standard monomials on $X(\tau_4)$. The proof for $X(\tau_5)$ is rather immediate. To make it very precise, we may start with a relation $F=0$ on $X(\tau_5)$, where F is a sum of standard monomials (of degree m, say) such that each monomial in F starts with p_{τ_5}. Thus $F = p_{\tau_5} G$. Now $F=0$ implies $G=0$ (since p_{τ_5} is not zero on $X(\tau_5)$) and we are through by induction on m. (When $m=1$, the only non-zero monomial of deg m is p_{τ_5} and the result is obvious in this case.)

This completes the proof of Proposition 5.2.

6. Basis for $H^0(X(\tau), L^m)$ by Means of Standard Monomials

First we want to recall the following result from [14]. Let L be a positive line bundle on G/B. (We may assume L is very ample; otherwise, we shall replace B by the parabolic subgroup Q such that L is very ample on G/Q.) Let $L = L(\lambda)$, where $\lambda = \sum_{i=1}^{n} a_i \omega_i$, n = rank of G. Let V_λ be the irreducible G-module (over $\mathbf{Q}$) with highest weight λ and let e be a highest weight vector in V_λ. Let $V_{\mathbf{Z}} = U_{\mathbf{Z}} e$ and for $\tau \in W$, let $V_{\mathbf{Z}}(\tau) = U_{\mathbf{Z}} e_\tau$, where e_τ is the τ-translate of e. We have a canonical closed immersion (cf. [9] or [14])

$$j_{\mathbf{Z}}: G_{\mathbf{Z}}/B_{\mathbf{Z}} \hookrightarrow \mathbf{P}(V_{\mathbf{Z}}^*).$$

Hence over any field k we obtain an immersion

$$j_\tau: X(\tau) \hookrightarrow \mathbf{P}(V_{\mathbf{Z}}^*)(k) \qquad (= (V_{\mathbf{Z}} \otimes k - (0))/k^*).$$

Now if N_τ denotes the image of

$$V_{\mathbf{Z}}(\tau) \otimes k \to V_{\mathbf{Z}} \otimes k$$

then we have (cf. [14]) that

$$N_\tau = \text{subspace of } V_{\mathbf{Z}} \otimes k \text{ spanned by } X(\tau)$$

(cf. the map j_τ above). Hence under the map

$$V_{\mathbf{Z}}^* \otimes k \to H^0(X(\tau), L)$$

induced by j_τ, the image of $V_{\mathbf{Z}}^* \otimes k$ can be identified with N_τ^*. Now *suppose the map $H^0(G_{\mathbf{Z}}/B_{\mathbf{Z}}, L_{\mathbf{Z}}) \to H^0(X_{\mathbf{Z}}(\tau), L_{\mathbf{Z}})$ is surjective.* Then we obtain that $N_\tau^* = H^0(X_{\mathbf{Z}}(\tau), L_{\mathbf{Z}}) \otimes k$ (recall that $H^0(G_{\mathbf{Z}}/B_{\mathbf{Z}}, L_{\mathbf{Z}}) \approx V_{\mathbf{Z}}^*$) and hence we get that dim N_τ^* (and hence dim N_τ) is the same in all characteristics; i.e., *$V_{\mathbf{Z}}(\tau)$ is a direct summand in $V_{\mathbf{Z}}$* (which is Demazure's conjecture (cf. [4]). In fact the sufjectivity of $H^0(G_{\mathbf{Z}}/B_{\mathbf{Z}}, L_{\mathbf{Z}}) \to H^0(X_{\mathbf{Z}}(\tau), L_{\mathbf{Z}})$ is equivalent to Demazure's conjecture (cf. [14]).

Now to prove that standard monomials of degree m form a basis for $H^0(X(\tau), L^m)$, we shall first prove this result in characteristic zero and as a consequence we shall also obtain the surjectivity of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}}^m) \to H^0(X_{\mathbf{Z}}(\tau), L^m)$. Hence by the above observation, this would prove Demazure's conjecture in this case and as a consequence we would obtain $H^1(X(\tau), L^m) = 0$ in all characteristics (cf. [4]). Hence dim $H^0(X(\tau), L^m)$ in any characteristic would equal its dimension in characteristic zero and thus we would obtain that in any characteristic, the # {standard monomials of degree m on $X(\tau)$} = dim $H^0(X(\tau), L^m)$. This toghether with linear independence of standard monomials (cf. Proposition 5.2) would prove the required result. Thus from now on (in this section) we shall assume $k=\mathbf{Q}$. The philosophy of the proof of the result that standard monomials of degree m generate $H^0(X(\tau), L^m)$ is the same as in [10]. We conclude results for $X(\tau)$ by assuming the results for $X(\phi)$ where $X(\phi)$ is a moving divisor in $X(\tau)$ and using Demazure's one-step construction (cf. [4, 10, 14]). To make it very precise, let $\tau = s_\alpha \phi$ and let $SL_{2,\alpha}$ denote the copy of SL_2 in G associated to α. Let B_α be the Borel subgroup in $SL_{2,\alpha}$ given by $B_\alpha = B \cap SL_{2,\alpha}$. Now for the canonical action of $SL_{2,\alpha}$ on G/B (induced by the canonical action of G on G/B) $X(\tau)$ remains stable. And observing that any Schubert variety in G/B is stable under the action of B_α, we set (cf. [10, 14]) $Z_\phi = SL_{2,\alpha} \times^{B_\alpha} X(\phi)$. Let p denote the canonical map

$$p\colon Z_\phi \to \mathbf{P}^1 = SL_{2,\alpha}/B_\alpha$$

(p is a fibration over $\mathbf{P}^1$ with fiber $X(\phi)$) and let Ψ be the canonical map

$$\Psi\colon SL_{2,\alpha} \times^{B_\alpha} X(\phi) \to X(\tau) \hookrightarrow G/B.$$

Then for any line bundle L on G/B, we have (cf. [10, 14])

$$H^i(Z_\phi, \Psi^*(L)) \simeq H^i(X(\tau), L) \qquad \text{for all } i. \tag{1}$$

Now, for any B_α-object M on $X(\phi)$, we can associate a canonical object $\tilde{M}$ on Z_ϕ (namely $\tilde{M} = SL_{2,\alpha} \times^{B_\alpha} M$). If H_τ (resp. H_ϕ) denotes the zero set of p_τ (resp. p_ϕ) in $X(\tau)$ (resp. $X(\phi)$) then we have (cf. [10]) on Z_ϕ

$$\Psi^*(I(H_\tau)) \approx \widetilde{I(H_\phi)}^{(-n)}, \tag{2}$$

where $n = (\phi(\omega), \alpha^*)$ and $I(H_\phi)^{(-n)} = I(H(\phi) \otimes p^*(O_{\mathbf{P}^1}(-n))$ (here $I(H_\tau)$ (resp. $I(H_\phi)$ denotes the ideal sheaf of H_τ (resp. H_ϕ) in $X(\tau)$ (resp. $X(\phi)$); further we have (cf. [10]), on Z_ϕ

$$\widetilde{I((H_\phi)_{\text{red}})}^{(-1)} \simeq I(H'_\tau) \tag{3}$$

$$H^0((H_\tau)_{\text{red}}, L) \simeq H^0(H'_\tau, \psi^*(L)), \tag{4}$$

where H'_τ is the reduced subscheme of Z_ϕ underlying $\Psi^{-1}(H_\tau)$.

Notation. For any sheaf $\mathscr{F}$ on Z, we set

$$\mathscr{F}^{(m)} = \mathscr{F} \otimes p^*(O_{\mathbf{P}^1}(m)).$$

PROPOSITION 6.1. *The base field being* $\mathbf{Q}$,

$$h^0(X(\tau), L^m) = \#\{\textit{standard monomials on } X(\tau) \textit{ of degree } m\}.$$

To prove Proposition 6.1, we first prove the following

LEMMA 6.2. *Let $s_m(\tau)$ denote $\#\{$standard monomials on $X(\tau)$ of degree $m\}$. We have (assuming $m \geqslant 1$)*

$$s_m(\tau_i) = s_{m-1}(\tau_i) + s_m(\tau_{i-1}), \qquad i = 1 \text{ or } 5$$
$$s_m(\tau_2) = s_{m-1}(\tau_2) + s_m(\tau_1) + 2s_{m-1}(\tau_1)$$
$$s_m(\tau_3) = s_{m-1}(\tau_3) + s_m(\tau_2) + s_{m-1}(\tau_2) + s_{m-1}(\tau_1)$$
$$s_m(\tau_4) = s_{m-1}(\tau_4) + s_m(\tau_3) + 2s_{m-1}(\tau_3) + s_{m-1}(\tau_2) + s_{m-1}(\tau_1).$$

Proof. The proof is rather immediate. For instance, to see why the last relation is true, we group the standard monomials on $X(\tau_4)$ of degree m as those that are non-zero (and hence standard) on $X(\tau_3)$ and those that are zero on $X(\tau_3)$; the former set obviously has cardinality $= s_m(\tau_3)$; and the latter set can be expressed as the disjoint union of standard monomials starting with p_{τ_4}, r_{τ_4,τ_3}, q_{τ_4,τ_3}, p_{τ_4,τ_2}, and p_{τ_4,τ_1}, respectively, and it is clear that the cardinalities of the corresponding sets are $s_{m-1}(\tau_4)$, $s_{m-1}(\tau_3)$, $s_{m-1}(\tau_3)$, $s_{m-1}(\tau_2)$, and $s_{m-1}(\tau_1)$, respectively.

Proof of Proposition 6.1. We shall prove this using the ideal theoretic results described above. Since the ideal theory of the Schubert varieties (in the present case) does not admit a uniform discussion, we prove the required result for each Schubert variety separately. The result is obviously true for $X(\tau_0)$.

The Result for $X(\tau_1)$. Now we have the exact sequence

$$0 \to L^{-1} \to O_{X_1} \to O_{X_0} \to 0,$$

where $X_0 = X(\tau_0)$, $X_1 = X(\tau_1)$. Hence

$$h^0(X_1, L^m) = h^0(X_0, L^m) + h^0(X_1, L^{m-1})$$

(the higher cohomologies vanish since $k = \mathbf{Q}$ (cf. [4])). Now the above equation is satisfied with h^0 replaced by s (cf. Lemma 6.2 above) and $h^0(X_0, L^m) = s_m(X_0)$, and we may assume (by induction on m) that $h^0(X_1, L^{m-1}) = s_{m-1}(X_1)$. Hence $h^0(X_1, L^m) = s_m(X_1)$.

The Result for $X(\tau_2)$. Denoting $\tau_2 = \tau$ and $\tau_1 = \phi$, we have $\tau = s_\alpha \phi$, where $\alpha = \alpha_1$ and m ($=$ multiplicity of $X(\phi)$ in $X(\tau)$) is 3. Hence (in view of (2) above)

$$\Psi^*(L^{-1}) = \widetilde{I(H(\phi))}^{(-3)} \qquad \text{on} \quad Z_\phi.$$

Now

$$\begin{array}{ccccc} \widetilde{I(H(\phi))}^{(-3)} \subset & \widetilde{I(H(\phi))}^{(-2)} \subset & \widetilde{I(H(\phi))}^{(-1)} = I(H'(\tau)). \\ \| & \| & \| \\ K_0 & K_1 & K_2 \end{array}$$

(Note that $H(\phi)$ is reduced.) Now $K_1/K_0 \approx O_{X(\phi)}(L^{-1}) \approx K_2/K_1$

$$(\text{using } 0 \to O_{\mathbf{P}^1}(-3) \to O_{\mathbf{P}^1}(-2) \to k \to 0)$$
$$(\text{resp. } 0 \to O_{\mathbf{P}^1}(-2) \to O_{\mathbf{P}^1}(-1) \to k \to 0))$$

and $K_0 \approx \Psi^*(L^{-1})$.

Hence considering

$$0 \to K_0 \to O_Z \to O_Z/K_0 \to 0,$$
$$0 \to K_1/K_0 \to O_Z/K_0 \to O_Z/K_1 \to 0,$$
$$0 \to K_2/K_1 \to O_Z/K_1 \to O_Z/K_2 \to 0,$$

tensoring the above sequences with $\Psi^*(L^m)$ $(m \geqslant 1)$ and writing down the cohomology exact sequence (noting that the higher cohomology groups for $K_0 \otimes \psi^*(L^m)$ and $K_i/K_{i-1} \otimes \psi^*(L^m)$, $i = 1, 2$, vanish because of the above identifications and our assumption that $k = \mathbf{Q}$ (cf. [4])), we obtain (using (1) and (4) in the beginning of this section)

$$h^0(X(\tau), L^m) = h^0(X(\tau), L^{m-1}) + h^0(X(\phi), L^m) + 2h^0(X(\phi), L^{m-1}). \quad (*)$$

(Note that $H(\tau)_{\text{red}} = X(\phi)$.) Now the above relation is satisfied with h^0 replaced by s (cf. Lemma 6.2); also, the result is true for $X(\phi)$ and the result may be assumed to be true for $m-1$ (by induction on m). Hence we obtain the result for m.

The result for $X(\tau_3)$. First we note that as a consequence of the result for $X(\tau_2)$ we have a nice filtration of $I((H_{\tau_2})_{\text{red}})$ given by

$$I_0 = (p_\tau) \subset I_1 = (p_\tau, r_{\tau,\phi}) \subset I_2 = (p_\tau, r_{\tau,\phi}, q_{\tau,\phi}),$$

where $\tau = \tau_2$, $\phi = \tau_1$; we see easily that the ideals I_r, $r = 0, 1, 2$, in $R_\tau = \oplus_{m \geqslant 0} H^0(X(\tau), L^m)$ are B-stable. As B-modules we have the isomorphisms

$$I_1/I_0 \approx \chi_0 \otimes R_\phi(-1) \quad (1)$$

$$I_0/I_1 \approx \chi_1 \otimes R_\phi(-1), \quad (2)$$

where χ_0 (resp. χ_1) represents the 1-dim l B-module induced by the character $\chi_0\colon B\to \mathbf{G}_m$ (resp. $\chi_1\colon B\to \mathbf{G}_m$), where χ_0 (resp. χ_1) is the weight of $r_{\tau,\phi}$ (resp. $q_{\tau,\phi}$).

Proof. To prove (1), for instance, first we note that we have a map

$$g\colon I_1/I_0\to R_\phi(-1)$$
$$r_{\tau,\phi}G\mapsto G. \tag{3}$$

To see that the above map is well-defined, we are required to show that if $r_{\tau,\phi}G\in I_0$, then $G\,|_{X(\phi)}=0$. Let $r_{\tau,\phi}G\in I_0$, say $r_{\tau,\phi}G=p_\tau H$. Multiplying by $q_{\tau,\phi}$ (and using Corollary 5.4) we obtain

$$p_\tau\, p_\phi G=p_\tau q_{\tau,\phi}H.$$

Cancelling p_τ and restricting to $X(\phi)$ we obtain $G\,|_{X(\phi)}=0$, as required. Similarly we see that the map

$$f\colon I_2/I_1\to R_\phi(-1)$$
$$q_{\tau,\phi}F\mapsto F$$

is well-defined. (As above if $q_{\tau,\phi}F\in I_1$, say,

$$q_{\tau,\phi}F=r_{\tau,\phi}G+p_\tau H.$$

Then multiplying by $r_{\tau,\phi}$ (and using Corollary 5.4) we obtain

$$p_\tau\, p_\phi F=p_\tau q_{\tau,\phi}G+p_\tau r_{\tau,\phi}H.$$

Concelling p_τ and restricting to $X(\phi)$, we obtain $F\,|_{X(\phi)}=0$.) Now the surjectivity of the above two maps is obvious. To see they are injective, let K_1, K_2 denote the respective kernels. For any ideal I in R_τ, let $\mathscr{I}$ denote the associated sheaf on $X(\tau)$. The considering

$$0\to\mathscr{I}_0\to O_{X(\tau)}\to O_{X(\tau)}/\mathscr{I}_0\to 0,$$
$$0\to\mathscr{I}_1/\mathscr{I}_0\to O_{X(\tau)}/\mathscr{I}_0\to O_{X(\tau)}/\mathscr{I}_1\to 0,$$
$$0\to\mathscr{I}_2/\mathscr{I}_1\to O_{X(\tau)}/\mathscr{I}_1\to O_{X(\tau)}/\mathscr{I}_2\to 0,$$

and tensoring with L^m ($m\gg 0$), we obtain

$$h^0(X(\tau), L^m)\geqslant h^0(X(\tau), L^{m-1})+h^0(X(\phi), L^{m-1})$$
$$+\ h^0(X(\phi), L^{m-1})+h^0(H_2, L^m),$$

where H_2 is the subscheme of $X(\tau)$ defined by the ideal I_2 (in particular, $h^0(H_2, L^m) \geqslant h^0(H(\tau)_{\text{red}}, L^m) = h^0(X(\phi), L^m)$. In the above relation, the inequality would be strict if either K_1 or K_2 were nonzero. But then we would obtain

$$h^0(X(\tau), L^m) > h^0(X(\tau), L^{m-1} + h^0(X(\phi), L^{m-1}) \\ + h^0(X(\phi), L^{m-1}) + h^0(X(\phi), L^m),$$

which would then contradict the relation $(*)$ above. Thus we conclude that both K_1 and K_2 are zero and thus we obtain the isomorphisms (1) and (2) above. The B-module identifications are also clear; for instance, looking at the B-actions, we have

$$bg(x) = \chi_0(b^{-1})\, g(bx)$$
$$bf(x) = \chi_1(b^{-1}) f(bx), \qquad b \in B.$$

Now that the above inequality is an equlity, we obtain that $H_2 = (H_\tau)_{\text{red}}$, i.e., $I_2 = I((H_\tau)_{\text{red}})$.

Now $X(\tau_2)$ is a double divisor in $X(\tau_3)$ and hence

$$\Psi^*(L^{-1}) \approx \widetilde{I(H_{\tau_2})}^{(-2)} \qquad \text{on} \quad Z_{\tau_2}$$

(where Ψ is the map $Z_{\tau_2} = X(\tau_2) \times^{B_\alpha} SL_{2,\alpha} \to X(\tau_3)$ and $\alpha = \alpha_2$). Further proceeding as in [10], we obtain

$$\widetilde{I_1/I_0} \approx \Psi^*(L^{-1})^{(m)} \qquad \text{on} \quad Z_{\tau_1},$$

where $m = -(\text{weight of } r_{\tau_2,\tau_1}, \alpha^*) = \frac{1}{3}(2\tau_2(\omega) + \tau_1(\omega), \alpha^*) = \frac{1}{3}(4-1) = 1$. Thus

$$\widetilde{I_1/I_0} = \Psi^*(L^{-1})^{(1)} \qquad (\text{on } Z_{\tau_1}).$$

In a similar way, we obtain that

$$\widetilde{I_2/I_1} = \Psi^*(L^{-1}) \qquad (\text{on } Z_{\tau_1}).$$

(The corresponding $m = \frac{1}{3}(\tau_2(\omega) + 2\tau_1(\omega), \alpha_2^*) = 0$.) Now

$$\widetilde{I(H_{\tau_2})}^{(-2)} \hookrightarrow \widetilde{I(H_{\tau_2})}^{(-1)} = \tilde{I}_0^{(-1)} \hookrightarrow \tilde{I}_1^{-1} \hookrightarrow \tilde{I}_2^{(-1)}$$

and proceeding as in the case of $X(\tau_2)$ we obtain (noting that for any m, $H^i(Z_{\tau_1}, \Psi^*(L^m) \otimes \widetilde{I_2/I_1}^{(-1)}) = 0$, for all i, since in this case, the fiber space

$p\colon Z_{\tau_1} \to \mathbf{P}^1$ splits (since $X(\tau_1)$ is stable under the canonical $SL_{2,\alpha}$ action (here $\alpha = \alpha_2$)); and also noting that

$$\begin{aligned} H^i(Z_{\tau_1}, \Psi^*(L^m) \otimes \widetilde{I_1/I_0}^{(-1)})) &= H^i(Z_{\tau_1}, \Psi^*(L^{m-1})) \\ &= H^i(X(\tau_1), L^{m-1})), \end{aligned}$$

$$\begin{aligned} h^0(X(\tau_3), L^m) = {} & h^0(X(\tau_3), L^{m-1}) + h^0(X(\tau_2), L^{m-1}) \\ & + h^0(X(\tau_1), L^{m-1}) + h^0((H_{\tau_3})_{\text{red}}, L^m). \end{aligned}$$

(Now the above relation is satisfied by s replacing h^0 (cf. Lemma 6.2) (note that $(H_{\tau_3})_{\text{red}} = X(\tau_2)$). And we conclude (using induction on m and the fact that the result holds for $X(\tau_i)$, $i \leqslant 2$) that

$$h^0(X(\tau_3), L^m) = s_m(\tau_3).$$

The Result for $X(\tau_4)$. Proceeding as above, we first prove that if I_0, I_1, I_2 denote the ideals in R_{τ_3} $(= \oplus_{m \geqslant 0} H^0(X(\tau_3), L^m))$ generated by $\{p_{\tau_3}\}$, $\{p_{\tau_3}, p_{\tau_3,\tau_2}\}$, and $\{p_{\tau_3}, p_{\tau_3,\tau_2}, p_{\tau_3,\tau_1}\}$, respectively, then I_0, I_1, and I_2 are B-stable and we have the B-module isomorphisms

$$\begin{aligned} I_1/I_0 &\approx \chi_0 \otimes R_{\tau_2}(-1) \\ p_{\tau_3,\tau_2} F &\mapsto F. \end{aligned} \tag{4}$$

$$\begin{aligned} I_2/I_1 &\approx \chi_1 \otimes R_{\tau_1}(-1) \\ p_{\tau_3,\tau_1} G &\mapsto G. \end{aligned} \tag{5}$$

Further we have

$$I_2 = I((H_{\tau_3})_{\text{red}}) \qquad (= I(X(\tau_2)) \tag{6}$$

where χ_0 (resp. χ_1) denotes the 1 dim$l$ B-module induced by the character $\chi_0 =$ weight of p_{τ_3,τ_2} (resp. weight of p_{τ_3,τ_1}). Now $X(\tau_3)$ is a triple divisor in $X(\tau_4)$ and hence

$$\Psi^*(L^{-1}) \approx \widetilde{I(H_{\tau_3})}^{(-3)} \qquad \text{on} \quad Z_{\tau_3}.$$

Also

$$\begin{aligned} \widetilde{I_1/I_0} &\approx \Psi^*(L^{-1}) && (\text{on } Z_{\tau_2}) \\ \widetilde{I_2/I_1} &\approx \Psi^*(L^{-1})^{(2)} && (\text{on } Z_{\tau_1}). \end{aligned}$$

Hence for any $m \geqslant 0$,

$$H^i(Z_{\tau_2}, \psi^*(L^m) \otimes \widetilde{I_1/I_0}^{(-1)}) = 0 \qquad \text{for all } i$$

(since the fibration $p: Z_{\tau_2} \to \mathbf{P}^1$ is trivial (as $X(\tau_2)$ is stable for the canonical $S_{2,\alpha}$ action where $\alpha = \alpha_1$), and

$$H^i(Z_{\tau_1}, \Psi^*(L^m) \otimes \widetilde{I_2/I_1}^{(-1)}) = H^i(Z_{\tau_1}, \Psi^*(L^{m-1})^{(1)}).$$

Considering

$$0 \to \theta_{\mathbf{P}^1} \to \theta_{\mathbf{P}^1}(1) \to k \to 0,$$

we have (for any m)

$$\to H^i(Z_{\tau_1}, \Psi^*(L^m)) \to H^i(Z_{\tau_1}, \Psi^*(L^m)^{(1)}) \to H^i(X(\tau_1), L^m) \to$$

where $H^i(Z_{\tau_1}, \Psi^*(L^m)) \approx H^i(X(\tau_2), \Psi^*(L^m))$, and hence proceeding as above we have

$$\widetilde{I(H_{\tau_3})}^{(-3)} \subset \widetilde{I(H_{\tau_3})}^{(-2)} \subset \widetilde{I(H_{\tau_3})}^{(-1)} = \tilde{I}_0^{(-1)} \subset \tilde{I}_1^{(-1)} \subset \tilde{I}_2^{(-1)}$$

and writing down the exact sequence (as above) we obtain

$$\begin{aligned} h^0(X(\tau_4), L^m) = {} & h^0(X(\tau_4), L^{m-1}) + 2h^0(X(\tau_3), L^{m-1}) \\ & + h^0(X(\tau_2), L^{m-1}) + h^0(X(\tau_1), L^{m-1}) \\ & + h^0((H_{\tau_4})_{\mathrm{red}}, L^m). \end{aligned}$$

Now the above equation is satisfied by replacing h^0 by s (cf. Lemma 6.2). (Note that $(H_{\tau_4})_{\mathrm{red}} = X(\tau_3)$.) And hence we conclude (using induction on m and the fact that the result holds for $X(\tau_i)$, $i \leqslant 3$) that

$$h^0(X(\tau_4), L^m) = s_m(\tau_4).$$

Finally, the result for $X(\tau_5)$ is rather immediate, since in this case the hyperplane section is reduced (since multiplicity of $X(\tau_4)$ in $X(\tau_5) = 1$ (cf. Proposition 1.4)) and hence we have the exact sequence

$$0 \to L^{-1} \to O_{X(\tau_5)} \to O_{X(\tau_4)} \to 0,$$

which gives rise to the cohomology exact sequence

$$\to H^i(X(\tau_5), L^{m-1}) \to H^i(X(\tau_5), L^m) \to H^i(X(\tau_4), L^m) \to$$

which yields

$$h^0(X(\tau_5), L^m) = h^0(X(\tau_5), L^{m-1}) + h^0(X(\tau_4), L^m)$$

(since the higher cohomology groups vanish (cf. [4]) taking $m \geqslant 1$ (note that $k = \mathbf{Q}$)). We also have

$$s_m(\tau_5) = s_{m-1}(\tau_5) + s_m(\tau_4),$$

from which we conclude (using the result for $X(\tau_5)$ and induction on m) that

$$h^0(X(\tau_5), L^m) = s_m(\tau_5).$$

This completes the proof of Proposition 6.1.

THEOREM 6.3. *For $X(\tau)$ in G/P and k arbitrary we have*

(a) *Distinct standard monomials on $X(\tau)$ of degree m form a basis for $H^0(X(\tau), L^m)$.*

(b) $H^i(X(\tau), L^m) = 0$, $i \geqslant 1$, $m \geqslant 0$.

Proof. In view of Propositions 5.2 and 6.1 we obtain that standard monomials on $X(\tau)$ of degree m form a basis for $H^0(X(\tau), L^m)$ (k being $\mathbf{Q}$) and hence we obtain a $\mathbf{Z}$ basis for $H^0(X_{\mathbf{Z}}(\tau), L^m_{\mathbf{Z}})$ by means of standard monomials and hence also obtain that the canonical map

$$H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L^m_{\mathbf{Z}}) \to H^0(X_{\mathbf{Z}}(\tau), L^m_{\mathbf{Z}})$$

is surjective. And from this (in view of the remarks made in the beginning of this section) Demazure's conjecture follows and hence we obtain (b). Now (b) implies that dim $H^0(X(\tau), L^m)$ is the same in all characteristics and hence (using Proposition 6.1) we obtain

$$h^0(X(\tau), L^m) = s_m(\tau)$$

in all characteristics. From this and linear independence of standard monimials on $X(\tau)$ (cf. Proposition 5.2), (a) follows.

Remark 6.4. As a consequence of standard monimial theory on $X(\tau_4)$ we have (as in the proof of Proposition 6.1) that if I_0, I_1, I_2, I_3, I_4 are the ideals in R_{τ_4}, where

$$I_0 = (p_{\tau 4})$$

$$I_1 = (p_{\tau_4}, r_{\tau 4, \tau_3})$$

$$I_2 = (p_{\tau_4}, r_{\tau 4, \tau_3}, q_{\tau 4, \tau_3})$$

$$I_3 = (p_{\rho 4}, r_{\tau 4, \tau_3}, q_{\tau 4, \tau_3}, p_{\tau 4, \tau_2})$$

$$I_4 = (p_{\tau_4}, \tau_{\tau 4, \tau_3}, q_{\tau 4, \tau_3}, p_{\tau 4, \tau_2}, {}_{\tau 4, \tau_1})$$

then I_j, $0 \leqslant j \leqslant 4$, are B-stable and we have B-module isomorphisms

$$I_1/I_0 \approx \chi_0 \otimes R_{\tau_3}(-1) \tag{1}$$

$$I_2/I_1 \approx \chi_1 \otimes R_{\tau_3}(-1) \tag{2}$$

$$I_3/I_2 \approx \chi_2 \otimes R_{\tau_2}(-1) \tag{3}$$

$$I_4/I_3 \approx \chi_3 \otimes R_{\tau_1}(-1). \tag{4}$$

Further, we have

$$I_4 = I(H(\tau_4)_{\text{red}}), \tag{5}$$

where χ_i denotes the 1 diml B-module associated to the character χ_i, where

$$\begin{aligned} \chi_i &= \text{wt of } r_{\tau_4,\tau_3} && \text{if} \quad i=1 \\ &= \text{wt of } q_{\tau_4,\tau_3} && \text{if} \quad i=2 \\ &= \text{wt of } p_{\tau_4,\tau_2} && \text{if} \quad i=3 \\ &= \text{wt of } p_{\tau_4,\tau_1} && \text{if} \quad i=4 \end{aligned}$$

Remark 6.5. Note that Theorem 6.3 in particular verifies the conjecture of Section 2 for $\chi = \omega_2$ (G being of type $\mathbf{G}_2$) (note that the conjecture of Section 2 holds for $\chi = \omega_1$, since it holds for all fundamental weights of classical type).

Remark 6.6. For any monomial F on $X(\tau)$, its unique expression as a sum of standard monomials will be referred to as the standard sum for F on $X(\tau)$.

7. Cohen–Macaulayness of $\widehat{X(\phi_i)}$ and $\widehat{X(\tau_i)}$, $0 \leqslant i \leqslant 5$

In this section, we prove that the cones $X(w)$ for the embedding

$$X(w) \subset G/p \hookrightarrow \mathbf{P}(H^0(G/p, L)),$$

are Cohen–Macaulay. Let R_i be the homogeneous co-ordinate ring of $\widehat{X(\tau_i)}$ for the above embedding. Now to prove $\widehat{X(\tau_i)}$ is Cohen–Macaulay, it is enough to prove that $X(\tau_i)$ is Cohen–Macaulay at its vertex. We now prove the following.

PROPOSITION 7.1. *The sequence* $\{p_{\tau_i}, p_{\tau_{i-1}}, \ldots, p_{\tau_0}\}$ *is* R_i*-regular.*

Proof. We prove the result for $X(\tau_i)$ by assuming it to be true for $X(\tau_j)$, $0 \leqslant j < i$. (The result is obviously true for $X(\tau_0)$, since R_0 is the polynomial ring in one variable.) We shall now carry out the details for $X(\tau_4)$ (which will illustrate the proof for the other cases also).

Now, obviously p_{τ_4} is a non-zero divisor in R_4 (since $X(\tau)$ for any τ is an integral scheme). For simplicity let us denote R_4 by R.

(i) *p_{τ_3} is a non-zero divisor in $R/(p_{\tau_4})$.* In view of Theorem 6.3, the nonstandard monomials $p_{\tau_3} p_{\tau_4,\tau_1}$ and $p_{\tau_3} p_{\tau_4,\tau_2}$ can be expressed as sums of standard monomials and we have (by weight considerations) (up to ± 1) on $X(\tau_4)$

$$p_{\tau_3} p_{\tau_4,\tau_1} = q_{\tau_4,\tau_3} p_{\tau_3,\tau_1} \tag{1}$$

$$p_{\tau_3} p_{\tau_4,\tau_2} = q_{\tau_4,\tau_3} p_{\tau_3,\tau_2}. \tag{2}$$

Now, if possible let $F^{p_{\tau_3}} \in (p_{\tau_4})$. Then since p_{τ_3} is a non-zero divisor in R_3, we obtain that $F|_{X(\tau_3)} = 0$; hence we may assume (modulo (p_{τ_4}))

$$F = p_{\tau_4,\tau_1} F_1 + p_{\tau_4,\tau_2} F_2 + q_{\tau_4,\tau_3} F_3 + r_{\tau_4,\tau_3} G_3$$

(sum of standard monomials), where F_1, F_2, F_3, G_3 are sums of standard monomials on $X(\tau_1)$, $X(\tau_2)$, $X(\tau_3)$, $X(\tau_3)$, respecmtively. Hence (using (1) and (2) above) we have

$$F^{p_{\tau_3}} = q_{\tau_4,\tau_3} p_{\tau_3,\tau_1} F_1 + q_{\tau_4,\tau_3} p_{\tau_3,\tau_2} F + q_{\tau_4,\tau_3} p_{\tau_3} F_3 + r_{\tau_4,\tau_3} p_{\tau_3} G_3.$$

(The right-hand side is a sum of standard monomials.) But now, in the expression as a sum of standard monomials for any element in (p_{τ_4}) every term starts with p_{τ_4} (since p_{τ_4} is the greatest among the standard monomials on $X(\tau_4)$ of degree 1); from this and the above expression for $F^{p_{\tau_3}}$ it is clear that $F^{p_{\tau_3}}$ cannot belong to (p_{τ_4}) unless $F_1 = 0 = F_2 = F_3 = G_3$, in which case F itself is 0. Thus p_{τ_3} is a non-zero divisor in $R/(p_{\tau_4})$.

(ii) *p_{τ_2} is a non-zero divisor in $R/(p_{\tau_4}, p_{\tau_3})$.* As above (since p_{τ_2} is a non-zero divisor in $R_3/(p_{\tau_3})$) if $F^{p_{\tau_2}} \in (p_{\tau_4}, p_{\tau_3})$, we may assume (modulo (p_{τ_4}, p_{τ_3})) that $F|_{X(\tau_3)} = 0$, i.e.,

$$F = p_{\tau_4,\tau_1} F_1 + p_{\tau_4,\tau_2} F_2 + q_{\tau_4,\tau_3} F_3 + r_{\tau_4,\tau_3} H_3.$$

Let $F^{p_{\tau_2}} = G p_{\tau_3} + H p_{\tau_4}$. Then since $F|_{X(\tau_3)} = 0$, we have $G|_{X(\tau_3)}$ is also 0. Hence we may assume

$$G = p_{\tau_4,\tau_1} G_1 + p_{\tau_4,\tau_2} G_2 + q_{\tau_4,\tau_3} G_3 + r_{\tau_4,\tau_3} M_3$$

(where the right-hand side is a sum of standard monomials). Now in view of Theorem 6.3 (and weight considerations) we have (up to ± 1)

$$r_{\tau_4,\tau_3} p_{\tau_4,\tau_1} = p_{\tau_4} p_{\tau_3,\tau_1} \tag{3}$$

$$r_{\tau_4,\tau_3} p_{\tau_4,\tau_2} = a p_{\tau_4} p_{\tau_3,\tau_2} + b p_{\tau_4} p_{\tau_4,\tau_1}, \qquad a, b \in k. \tag{4}$$

(Note that in (4) above, $a \neq 0$. For, if $a = 0$, then we would obtain

$$r_{\tau_4,\tau_3}\, p_{\tau_4,\tau_2} = b p_{\tau_4}\, p_{\tau_4,\tau_1},$$

which would then imply that the order of vanishing of p_{τ_4,τ_2} on $X(\tau_3)$ is at least 2 (recall—cf. proof of Proposition 5.1—that the order of vanishing of p_{τ_4,τ_2} on $X(\tau_3)$ is 1).) Now multiplying $Fp_{\tau_2} = p_{\tau_3}\, G + p_{\tau_4} H$ by r_{τ_4,τ_3}, cancelling p_{τ_4} (using (3) and (4) above), and restricting to $X(\tau_3)$ we obtain.

$$(p_{\tau_3,\tau_1} F_1 + a p_{\tau_3,\tau_2} F_2)\, p_{\tau_2} \in (p_{\tau_3}).$$

But now using the fact that p_{τ_2} is a non-zero divisor in $R_3/(p_{\tau_3})$, we conclude that $p_{\tau_3,\tau_1} F_1 + p_{\tau_3,\tau_2} F_2 \in (p_{\tau_3})$ in R_3. But then in the standard expression for any element in (p_{τ_3}) as a sum of standard monomials on $X(\tau_3)$, each term starts with p_{τ_3}. Hence in view of linear independence of standard monomials, we conclude that $F_1 = 0 = F_2$. Hence modulo (p_{τ_3}) we may assume

$$F = q_{\tau_4,\tau_3} F_3 + r_{\tau_4,\tau_3} H_3.$$

Again proceeding as above (i.e., multiplying $p_{\tau_2} F$ by r_{τ_4,τ_3}, cancelling p_{τ_4}, and restricting to $X(\tau_3)$), we obtain

$$p_{\tau_3}\, p_{\tau_2} F_3 = (p_{\tau_3,\tau_1} G_1 + a p_{\tau_3,\tau_2} G_2 + p_{\tau_3} G_3)\, p_{\tau_3}.$$

Hence

$$F_3\, p_{\tau_2} = p_{\tau_3,\tau_1} G_1 + a p_{\tau_3,\tau_2} G_2 + p_{\tau_3} G_3, \tag{†}$$

i.e., $F_3\, p_{\tau_2} - p_{\tau_3,\tau_1} G_1 - a p_{\tau_3,\tau_2} G_2 \in (p_{\tau_3})$ in R_3.

Now, we may assume that when F_3 is expressed as a sum of standard monomials on $X(\tau_3)$, no term involves p_{τ_3,τ_1} (using the relation $q_{\tau_4,\tau_3}\, p_{\tau_3,\tau_1} = p_{\tau_3}\, p_{\tau_4,\tau_1}$ (cf. (1) above) and the fact that we are considering F modulo (p_{τ_4}, p_{τ_3})). Hence $F_3\, p_{\tau_2}$ is a sum of standard monomials where no term involves p_{τ_3,τ_1} or p_{τ_3}. But then $F_3\, p_{\tau_2} - p_{\tau_3,\tau_1} G_1 - a p_{\tau_3,\tau_2} G_2$ (being a sum of standard monomials with no term involving p_{τ_3}) cannot belong to (p_{τ_3}) in R_3, unless it is zero (since p_{τ_3} is the greatest among standard monomials of degree 1 on $X(\tau_3)$). Thus we obtain.

$$F_3\, p_{\tau_2} = p_{\tau_3,\tau_1} G_1 + a p_{\tau_3,\tau_2} G_2,$$

which then implies (cf. (†) above) that $G_3 = 0$. Restricting the above relation to $X(\tau_2)$ we obtain $F_3 \mid_{X(\tau_2)} = 0$, i.e., $F_3 \in (p_{\tau_3,\tau_1}, p_{\tau_3,\tau_2}, p_{\tau_3})$ (in R_3). But then $q_{\tau_4,\tau_3} F_3 \in (p_{\tau_3})$ (cf. (1), (2) above). Hence modulo (p_{τ_3}, p_{τ_4}) we may assume $F_3 = 0$ (which also implies $G_1 = G_2 = 0$) and thus

$$F = r_{\tau_4,\tau_3} H_3$$

and

$$G = r_{\tau_4,\tau_3} M_3.$$

(F_3 now being 0, (†) above yields $G_1 = 0 = G_2 = G_3$, since the right-hand side of (†) is a standard sum on $X(\tau_3)$.) Now multiplying $Fp_{\tau_2} = p_{\tau_3} G + p_{\tau_4} H$ by q_{τ_4,τ_3}, cancelling p_{τ_4} (using $r_{\tau_4,\tau_3} q_{\tau_4,\tau_3} = p_{\tau_4} p_{\tau_3}$), and restricting to $X(\tau_3)$ we obtain

$$p_{\tau_3} H_3 p_{\tau_2} = M_3 P^2_{\tau_3},$$

i.e., $H_3 p_{\tau_2} \in (p_{\tau_3})$ in R_3. But now the fact that p_{τ_2} is a non-zero divisor in $R_3/(p_{\tau_3})$ implies that $H_3 \in (p_{\tau_3})$. Hence (modulo (p_{τ_4}, p_{τ_3})) we obtain $F = 0$. This completes the proof of the assertion that p_{τ_2} is a non-zero divisor in $R_4/(p_{\tau_4}, p_{\tau_3})$.

(iii) *p_{τ_1} is a non-zero divisor in $R/(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$.* Let $Fp_{\tau_1} \in (p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$. Then, as above, since p_{τ_1} is a non-zero divisor in $R_3/(p_{\tau_3}, p_{\tau_2})$, we may assume

$$F = p_{\tau_4,\tau_1} F_1 + p_{\tau_4,\tau_2} F_2 + q_{\tau_4,\tau_3} F_3 + r_{\tau_4,\tau_3} H_3,$$

where the right-hand side is a sum of standard monomials (in particular, we have, F_1 standard on $X(\tau_1)$, F_2 on $X(\tau_2)$, etc.). Further in view of the relations $q_{\tau_4,\tau_3} p_{\tau_3,\tau_i} = p_{\tau_4,\tau_i} p_{\tau_3}$, $i = 1, 2$, we may assume that when F_3 is expressed as a sum of standard monomials, no term involves p_{τ_3,τ_1} or p_{τ_3,τ_2} (since we are looking at F modulo $(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$). Now Fp_{τ_1} with the above expression substituted for F remains a standard sum and hence if $Fp_{\tau_1} \in (p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$, then $Fp_{\tau_1} \in (p_{\tau_2})$ (since any standard monomial H on $X(\tau_4)$ remains standard on multiplication with p_{τ_4} and also on multiplication with p_{τ_3} as along as it does not involve p_{τ_4,τ_i}, $i = 1, 2$: and if $H = G p_{\tau_4,\tau_i} F_i$ (a standard monomial), $i = 1, 2$, then $Hp_{\tau_3} = G q_{\tau_4,\tau_3} p_{\tau_3,\tau_i} F_i$ (a standard monimial)). Let

$$Fp_{\tau_1} = Mp_{\tau_2}.$$

Since $F|_{X(\tau_3)} = 0$, we obtain $M|_{X(\tau_3)} = 0$, say

$$M = p_{\tau_4,\tau_1} G_1 + p_{\tau_4,\tau_2} G_2 + q_{\tau_4,\tau_3} G_3 + r_{\tau_4,\tau_3} M_3,$$

the right-hand side being a standard sum. Now, multiplying $Fp_{\tau_1} = Mp_{\tau_2}$ by r_{τ_4,τ_3}, cancelling p_{τ_4} (using the various relations above), and restricting to $X(\tau_3)$ we obtain

$$(p_{\tau_3,\tau_1} F_1 + a p_{\tau_3,\tau_2} F_2 + p_{\tau_3} F_3) p_{\tau_1} = (p_{\tau_3,\tau_1} G_1 + a p_{\tau_3,\tau_2} G_2 + p_{\tau_3} G_3) p_{\tau_2} \quad (*)$$

(where a is as in (4) above).

Hence $p_{\tau_3,\tau_1}F_1 + ap_{\tau_3,\tau_2}F_2 \in (p_{\tau_3}, p_{\tau_2})$ in R_3 (since p_{τ_1} is a non-zero divisor in $R_3/(p_{\tau_3}, p_{\tau_2})$). We may assume that when F_2 is expressed as a standard sum, no term involves r_{τ_2,τ_1} (in view of the fact that $p_{\tau_4,\tau_2}r_{\tau_2,\tau_1} = p_{\tau_4,\tau_1}p_{\tau_2}$ and that we are considering F modulo $(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$). From these observations, it is clear that $p_{\tau_3,\tau_1}F_1 + ap_{\tau_3,\tau_2}F_2$ cannot belong to (p_{τ_3}, p_{τ_2}) unless $F_1 = 0 = F_2$. Thus we obtain

$$F = q_{\tau_4,\tau_3}F_3 + r_{\tau_4,\tau_3}H_3.$$

Then (*) yields

$$p_{\tau_3}F_3\,p_{\tau_1} = (p_{\tau_3,\tau_1}G_1 + ap_{\tau_3,\tau_2}G_2 + p_{\tau_3}G_3)\,p_{\tau_2}. \qquad (**)$$

This then implies that $p_{\tau_3,\tau_1}p_{\tau_2}G_1 + ap_{\tau_3,\tau_2}p_{\tau_2}G_2 \in (p_{\tau_3})$ in R_3; i.e., $p_{\tau_3,\tau_2}r_{\tau_2,\tau_1}G_1 + ap_{\tau_3,\tau_2}p_{\tau_2}G_2 \in (p_{\tau_3})$ in R_3. Now $p_{\tau_3,\tau_2}r_{\tau_2,\tau_1}G_1 + p_{\tau_3,\tau_2}p_{\tau_2}G_2$, being a sum of (distinct) standard monomials on $X(\tau_3)$, cannot belong to (p_{τ_3}) in R_3, unless $G_1 = 0 = G_2$ (since p_{τ_3} is the greatest monomial of deg 1 in R_3, in the standard sum for any element in (p_{τ_3}) each term begins with p_{τ_3}). Hence we obtain from (**) above that $F_3 \in (p_{\tau_3}, p_{\tau_2})$ in R_3 (using the fact that p_{τ_1} is a non-zero divisor in $R_3/(p_{\tau_3}, p_{\tau_2})$). This then implies (modulo $(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$) that $F = r_{\tau_4,\tau_3}H_3$ and $M = r_{\tau_4,\tau_3}M_3$ and we have

$$r_{\tau_4,\tau_3}H_3\,p_{\tau_1} = r_{\tau_4,\tau_3}M_3\,p_{\tau_2},$$

i.e.,

$$H_3\,p_{\tau_1} = M_3\,p_{\tau_2}.$$

Hence $H_3 \in (p_{\tau_3}, p_{\tau_2})$ in R_3 (since p_{τ_1} is a non-zero divisor in $R_3/(p_{\tau_3}, p_{\tau_2})$). Thus modulo $(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$, $F = 0$, proving that p_{τ_1} is a non-zero divisor in $R/(p_{\tau_4}, p_{\tau_3}, p_{\tau_2})$.

(iv) *p_{τ_0} is a non-zero divisor in $R/(p_{\tau_4}, p_{\tau_3}, p_{\tau_2}, p_{\tau_1})$.* Let $Fp_{\tau_0} \in (p_{\tau_4}, p_{\tau_3}, p_{\tau_2}, p_{\tau_1})$. Then, as above, we may assume

$$F = p_{\tau_4,\tau_1}F_1 + p_{\tau_4,\tau_2}F_2 + q_{\tau_4,\tau_3}F_3 + r_{\tau_4,\tau_3}H_3$$

(since p_{τ_0} is a non-zero divisor in $R_3/(p_{\tau_3}, p_{\tau_2}, p_{\tau_1})$). Now Fp_{τ_0} remains a standard sum and any monomial H (standard on $X(\tau_4)$) on multiplication with p_{τ_4} or p_{τ_1} remains standard and Hp_{τ_3} is standard, if H does not involve p_{τ_4,τ_i}, $i = 1, 2$, and if $H = Gp_{\tau_4,\tau_i}M_i$, then $Hp_{\tau_3} = Gq_{\tau_4,\tau_3}p_{\tau_3,\tau_i}M_i$ (standard). From these facts we conclude (as above) that $Fp_{\tau_0} \in (p_{\tau_2})$, say

$$Fp_{\tau_0} = p_{\tau_2}M.$$

Then we may write

$$M = p_{\tau_4,\tau_1}G_1 + p_{\tau_4,\tau_2}G_2 + q_{\tau_4,\tau_3}G_3 + r_{\tau_4,\tau_3}M_3$$

(since $M|_{X(\tau_3)}=0$, as $F|_{X(\tau_3)}=0$). As above, multiplying by r_{τ_4,τ_3}, cancelling p_{τ_4} (using the various relations), and restricting to $X(\tau_3)$ we obtain

$$(p_{\tau_3,\tau_1}F_1+ap_{\tau_3,\tau_2}F_2+p_{\tau_3})\,p_{\tau_0}=(p_{\tau_3,\tau_1}G_1+ap_{\tau_3,\tau_2}G_2+p_{\tau_3}G_3)\,p_{\tau_2} \qquad (\dagger\dagger)$$

(where a is as in (4) above). Hence $p_{\tau_3,\tau_1}F_1+ap_{\tau_3,\tau_2}F_2\in(p_{\tau_3},p_{\tau_2},p_{\tau_1})$ in R_3. Now noting that any monomial standard on $X(\tau_3)$ remains standard on multiplication with p_{τ_3} or p_{τ_1} and using the relations

$$p_{\tau_3,\tau_1}p_{\tau_2}=p_{\tau_3,\tau_2}r_{\tau_2,\tau_1} \qquad \text{and} \qquad p_{\tau_4,\tau_2}r_{\tau_2,\tau_1}=p_{\tau_4,\tau_1}p_{\tau_2}$$

we conclude that $F_1=0=F_2$ (modulo (p_{τ_3},p_{τ_2})) in R_3. Hence we obtain $F=q_{\tau_4,\tau_3}F_3+r_{\tau_4,\tau_3}H_3$ and from $(\dagger\dagger)$ we obtain that $p_{\tau_3,\tau_1}G_1p_{\tau_2}+p_{\tau_3,\tau_2}G_2p_{\tau_2}\in(p_{\tau_3})$ in R_3; i.e., $p_{\tau_3,\tau_2}r_{\tau_2,\tau_1}G_1+p_{\tau_3,\tau_2}p_{\tau_2}G_2\in(p_{\tau_3})$ in R_3. Hence $G_1=0=G_2$ (noting that p_{τ_3} is the largest element among monomials of deg 1 standard on $X(\tau_3)$). Hence $(\dagger\dagger)$ yields

$$p_{\tau_3}F_3\,p_{\tau_0}=p_{\tau_3}G_3\,p_{\tau_2}.$$

Hence $F_3\in(p_{\tau_3},p_{\tau_2},p_{\tau_1})$ in R_3 (since p_{τ_0} is a non-zero divisor in $R_3/(p_{\tau_3},p_{\tau_2},p_{\tau_1})$. Hence (modulo $(p_{\tau_4},p_{\tau_3},p_{\tau_2},p_{\tau_1})$) we obtain

$$F=r_{\tau_4,\tau_3}H_3$$

and this in turn implies $M=r_{\tau_4,\tau_3}M_3$.
And hence $Fp_{\tau_0}=Mp_{\tau_2}$ yields that $H_3\in(p_{\tau_3},p_{\tau_2},p_{\tau_1})$ (since p_{τ_0} is a non-zero divisor in $R_3/(p_{\tau_3},p_{\tau_2},p_{\tau_1})$). Thus we obtain $F=0$ $(\mathrm{mod}(p_{\tau_4},p_{\tau_3},p_{\tau_2},p_{\tau_1}))$ proving that p_{τ_0} is non-zero divisor in $R/(p_{\tau_4},p_{\tau_3},p_{\tau_2},p_{\tau_1})$.

This completes the proof of Proposition 7.1 for $X(\tau_4)$ (and as already observed the same philosophy carries over to $X(\tau_i)$, $i\leqslant 3$).

Finally the result for $X(\tau_5)$ is rather immediate; for, we have p_{τ_5} is a non-zero divisor in R_5 and $(p_{\tau_4},p_{\tau_3},p_{\tau_2},p_{\tau_1},p_{\tau_0})$ is a regular sequence in $R_4(=R_5/(p_{\tau_5}))$.

This completes the proof of Proposition 7.1.

Now Proposition 7.1, together with the fact that dim $X(\tau_i)=i$, yields the following

THEOREM 7.2. *The cones* $\widehat{X(\tau_i)}$ *and* $\widehat{X(\phi_i)}$ *are Cohen–Macaulay for* $0\leqslant i\leqslant 5$.

(One may note that the proof of the result for $\widehat{X(\phi_i)}$ is completely analogous to that of $\widehat{X(\tau_i)}$; it is even simpler, since we work with just one equation, namely $p^2_{\phi_3,\phi_2}=p_{\phi_3}p_{\phi_2}$ (up to ± 1)).

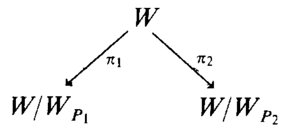
THEOREM 7.3. *The cones* $\widehat{X(\tau_i)}$ *and* $\widehat{X(\phi_i)}$, $0\leqslant i\leqslant 5$, *are normal.*

This follows in view of Theorem 7.3 *and Chevalley's result* (*cf.* [3]) *that Schubert varieties are non-singular in* codim 1.

8. Standard Monomial Theory on G/B

P_1 and P_2 continue as above. However, if the notations are reversed, so that, for example, P_1 rather than P_2 denotes the maximal parabolic subgroup corresponding to α_2, the results to follow remain substantially true. Following [9, 11], we define *weakly standard* and *standard* monomials on $X(w)$ of multidegree (m_1, m_2), $m_1, m_2 \in \mathbf{Z}^+$ as follows.

Definition 8.1. Given $\theta \in W/W_{P_1}$ and $\mu \in W/W_{P_2}$, we define $\theta \geqslant \mu$, if $\pi_2(\pi_1^{-1}(\theta)) \geqslant \mu$ in W/W_{P_2}, under the canonical maps



Definition 8.2. Given $\mathscr{m} = (m_1, m_2) \in (\mathbf{Z}^+)^2$, by a *young diagram of type* $\mathscr{m}$ or *multidegree* $\mathscr{m}$ on G/B (or W) we mean a sequence (θ, δ), $\theta = (\theta_{ij})$, $\delta = (\delta_{ij})$, where $\theta_{ij}, \delta_{ij} \in W/W_{P_i}$ and either θ_{ij} is δ_{ij} or θ_{ij} is connected to δ_{ij} by a *multiple path* (by which we mean ther exists a chain $\tau_0 = \theta_{ij} > \tau_1 > \cdots > \tau_r = \delta_{ij}$ such that $X(\tau_i)$ is a multiple divisor in $X(\tau_{i-1})$), $1 \leqslant j \leqslant m_i$, $1 \leqslant i \leqslant 2$).

Definition 8.3. A young diagram (θ, δ) is said to be a *young diagram on* $X(w)$ (or just w) where $w \in W$, if $w^{(i)} \geqslant \theta_{ij}$, $1 \leqslant j \leqslant m_i$, $1 \leqslant i \leqslant 2$, $w^{(i)}$ denotes the projection of w on W/W_{P_i} under $W \to W/W_{P_i}$.

Definition 8.4. A young diagram (θ, δ) is said to be *weakly standard* if

$$\theta_{11} \geqslant \delta_{11} \geqslant \theta_{12} \geqslant \cdots \geqslant \delta_{1m_1} \geqslant \theta_{21} \geqslant \delta_{21} \geqslant \cdots \geqslant \delta_{2m_2}$$

(where the order $\delta_{1m_1} > \theta_{21}$ is as given by Definition 8.1).

Definition 8.5. A young diagram (θ, δ) is said to be *weakly standard on* $X(w)$ if (θ, δ) is weakly standard and (θ, δ) is a young diagram on w.

Definition 8.6. A young diagram (θ, δ) is said to be *standard* if there exists a sequence (α, β) which we call a *defining sequence for* (θ, δ) such that

(1) $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$, $\alpha_{ij}, \beta_{ij} \in W$, $1 \leqslant j \leqslant m_i$, $1 \leqslant i \leqslant 2$.

(2) Each α_{ij} (resp. β_{ij}) is a lift for θ_{ij} (resp. δ_{ij}) under $W \to W/W_{P_i}$.

(3) $\alpha_{11} \geqslant \beta_{11} \geqslant \alpha_{12} \geqslant \cdots \geqslant \beta_{1m_1} \geqslant \alpha_{21} \geqslant \beta_{21} \geqslant \cdots \geqslant \beta_{2m_2}$ (as elements of W).

DEFINITION 8.7. A young diagram (θ, δ) is said to be *standard on* $X(w)$, if there exists a defining sequence (α, β) for (θ, δ) such that $w \geqslant \alpha_{11}$ (in W).

Remark 8.8. It is obvious that standard implies weakly standard; but the converse is not true. For example, the young diagram (Φ_2, τ_2) of type $(1, 1)$ is weakly standard but not standard on $X(\phi_3)$.

DEFINITION 8.9. Given a union of Schubert varieties, say $Z = \bigcup X(w_i)$, a young diagram is said to be standard on Z if it is standard on some component of Z.

Remark 8.10. Given a young diagram of degree $\mathscr{m}$ standard on $X(\tau)$, we have a unique minimal defining sequence (θ^-, λ^-) for F which is independent of τ (cf. [9, Corollary 11.2′]).

Remark 8.11. Now $X(\tau_i)$ (resp. $X(\phi_i)$), $0 \leqslant i \leqslant 5$, being saturated for the canonical projection $G/B \to G/P_1$ (resp. $G/B \to G/P_2$), it can be easily seen that young diagrams weakly standard on $X(\tau_i)$ are standard on $X(\tau_i)$ (resp. $X(\phi_i)$) for the ordering (P_1, P_2) (resp. (P_2, P_1) of the maximal parabolics.

DEFINITION 8.12. Given a monomial of multidegree (m_1, m_2) on $X(w)$, we call it *standard on* $X(w)$ if the associated young diagram is standard on $X(w)$.

In this section we shall show that (G *being of type* $\mathbf{G}_2$) given $\mathscr{m} = (m_1, m_2)$, if $L = L(\lambda)$, where $\lambda = m_1\omega_1 + m_2\omega_2$, then monomials of type $\mathscr{m}$ standard on Z (a union of Schubert varieties) form a basis for $H^0(Z, L)$. Our result is independent of the ordering of the maximal parabolic subgroups. For any $X(\tau)$ in G/B, let $X(\bar{\tau})$ denote its projection under the canonical map $G/B \to G/P_1$. Now given $X(\tau)$ in G/B, we find that there exists a unique Schubert subvariety $X(w)$ of codim 1 in $X(\tau)$ such that $X(\bar{w})$ is of codim 1 in $X(\bar{\tau})$ (in G/P_1). Also, the multiplicity of $X(\bar{w})$ in $X(\bar{\tau})$ is always one except when $\tau = \tau_4$ or ϕ_3.

PROPOSITION 8.13. *Distinct standard monomials of degree* $\mathscr{m} = (m_1, m_2)$ *on* Z (*a union of Schubert varieties*) *are linearly independent.*

Proof. If Z is not irreducible, then $Z = X(\tau_i) \cup X(\phi_i)$, $1 \leqslant i \leqslant 5$; hence whether Z is irreducible or not, the projection of Z on G/P_1 is irreducible; to be very precise

$$\begin{aligned} \text{Projection of } Z \text{ on } G/P_1 &= X(\phi_i) \qquad \text{if} \quad Z = X(\tau_i) \cup X(\phi_i) \\ &\qquad\qquad\qquad\quad \text{or} \quad X(\phi_i) \\ &= X(\phi_{j-1}) \qquad \text{if} \quad Z = X(\tau_j), \end{aligned}$$

where $1 \leqslant i \leqslant 5$ and $1 \leqslant j \leqslant 6$, where $\tau_6 = w_0$, the unique element of maximal length in W. (If $Z = X(\mathrm{Id})$, the corresponding projection is $X(\phi_0)$.) Let us denote the projection of Z on G/P_1 by $X(\phi)$. Now let $F = \sum a_i F_i$ be a linear combination of monomials of degree m standard on Z, and if possible let $F = 0$. We distinguish the following two cases.

Case 1. $Z = X(\tau_j)$, $0 \leqslant j \leqslant 6$. Now if there is at least one F_i which starts with $p_{\alpha,\beta}$ with $\alpha < \phi$, then restricting $F = 0$ to $X(\tau_{j-1})$ we may conclude the corresponding a_i to be zero (note that if $F_i \mid_{X(\tau_{j-1})} \neq 0$, then F_i remains standard on $X(\tau_{j-1})$ (cf. Remark 8.11). Thus we may assume that each F_i in $F = \sum a_i F_i$ starts with $p_{\alpha,\beta}$ where $\alpha = \phi$. Again, if $\phi \neq \phi_3$ (i.e., if $j \neq 4$), then β has to be α; in other words, each term in F_i starts with p_ϕ, and hence cancelling p_ϕ (and using induction on m_1) we conclude that $a_i = 0$ for all i; i.e., the relation $\sum a_i F_i = 0$ has to be the trivial relation. If $\phi = \phi_3$ (i.e., if $j = 4$), then writing

$$F = \sum a_i p_{\phi_3} G_i + \sum b_k p_{\phi_3,\phi_2} G_k = 0$$

(note that G_k is standard on $X(\tau_3)$) we obtain (by multiplying the above relation by p_{ϕ_3,ϕ_2} and using the result that $p^2_{\phi_3,\phi_2} = p_{\phi_3} p_{\phi_2}$ on $X(\tau_4)$), on $X(\tau_4)$

$$p_{\phi_3} \sum a_i p_{\phi_3,\phi_2} G_i + p_{\phi_3} \sum b_{jk} p_{\phi_2} G_k = 0.$$

Now cancelling p_{ϕ_3} and restricting to $X(\tau_3)$, we obtain $b_k = 0$ for all k (since each G_k is standard on $X(\tau_3)$). Hence the relation $F = 0$ becomes $p_{\phi_3} F' = 0$, where $F' = \sum a_i G_i$. Hence cancelling p_{ϕ_3} and using induction on m_1, we conclude $a_i = 0$ for all i. Thus the relation $F = 0$ has to be the trival relation.

Case 2. $Z = X(\tau_j) \cup X(\phi_j)$. As above, if $F = \sum a_i F_i = 0$, there is at least one F_i which starts with $p_{\alpha,\beta}$, where $\alpha < \phi$ $(= \phi_j)$, then such an F_i is standard on $X(\tau_j)$ and hence restricting $F = 0$ to $X(\tau_j)$ we may conclude the corresponding $a_i = 0$ (by case 1). Thus we may assume each F_i starts with $p_{\alpha,\beta}$, where $\alpha = \phi$ $(= \phi_j)$. Now $X(\tau_j)$. has projection $X(\phi_{j-1})$ on G/P_1 and hence in this case (i.e., every $\alpha = \phi$) we conclude that each F_i is standard on $X(\phi_j)$. Thus we may assume F is a standard sum on $X(\phi_j)$. If $j \neq 3$, then we obtain $\alpha = \beta = \phi$ and $F = p_\phi F' = 0$; now, cancelling p_ϕ and using induction on m_1, we are through. If $j = 3$, then writing

$$F = \sum a_i p_{\phi_3} G_i + \sum b_k p_{\phi_3,\phi_2} G_k$$

and proceeding as in case 1 we obtain

$$\sum b_k G_k = 0 \qquad \text{on} \quad X(\phi_2).$$

But now by the induction hypothesis the result is true for $X(\phi_2)$ and hence $b_k = 0$ for all k. Hence $F = 0$ is reduced to $F = p_{\phi_3} \sum a_i G_i$, where each G_i is a standard sum on $X(\phi_3)$. Hence restricting $F=0$ to $X(\phi_3)$ and cancelling P_{ϕ_3} (and using induction on m_1) we conclude that $a_i=0$ for all i. Thus the relation $F=0$ has to be the trivial relation.

This completes the proof of Proposition 8.13.

LEMMA 8.14. *Let $Z=X(\tau_i)\cup X(\phi_i)$ $1\leqslant i\leqslant 5$; given $m=(m_1, m_2)$, let $s(Z, m)=\#\{$monomials of degree m standard on $Z\}$. Then we have*

$$s(Z, m)=s(X(\tau_i), m)+s(X(\phi_i), m)-s((X(\tau_i)\cap X(\phi_i))_{\mathrm{red}}, m).$$

Proof. Now in view of Definition 8.9, we have

$$\begin{aligned} s(Z, m)=s(X(\tau_i), m)+s(X(\phi_i), m)\\ -\ \#\{\text{monomials standard on both } X(\tau_i) \text{ and } X(\phi_i)\}. \end{aligned}$$

Now let F be a monomial of deg m that is standard on both $X(\tau_i)$ and $X(\phi_i)$. Let (θ^-, λ^-) be the unique minimal defining sequence for F (which depends only on F and not on $X(\tau_i)$ or $X(\phi_i)$ (cf. Remark 8.10)). Then $\theta_{11}^- <$ both τ_i and ϕ_i and hence $X(\theta_{11}^-)$ is contained in some irreducible component of $X(\tau_i)\cap X(\phi_i)$. Thus F is standard on $(X(\tau_i)\cap X(\phi_i))_{\mathrm{red}}$, from which the lemma is immediate.

PROPOSITION 8.15. *Given $m=(m_1, m_2)$, suppose $s(\tau, m)=h^0(X(\tau), L)$ for $\tau\in W$. Then we have*

(1) *$X(\tau)\cap X(\tau')$ is reduced for all $\tau, \tau'\in W$.*

(2) *$h^0(Z, L)=s(Z, m)$, where Z is a union of Schubert varieties.*

(3) *If $H^i(X, L)=0$, $i\geqslant 1$, $L\geqslant 0$, for Schubert varieties, the same result holds for unions of Schubert varieties.*

Proof. The proof is similar to that in [10] or [12] (using Lemma 8.14 above).

Now given $\tau\in W$, let H_τ be the zero set of $p_{\bar\tau}$ in $X(\tau)$ (where $X(\bar\tau)$ denotes the projection of $X(\tau)$ under $G/B\to G/P_1$). Set theoretically H_τ is a Schubert variety. To be precise, if $\tau=\tau_i$ $(i\geqslant 1)$ then $H_\tau=X(\tau_{i-1})$ (set theoretically) and if $\phi=\phi_i$, then $H_\tau=X(\phi_{i-1})$ (set theoretically). We now prove the following

LEMMA 8.16. *Given $m=(m_1, m_2)$, where $m_1\geqslant 1$, let $m'=(m_1-1, m_2)$ and let $L=L(\lambda)$, $L'=L(\lambda')$, where $\lambda=\sum_{i=1}^2 m_i\omega_i$ and $\lambda'=(m_1-1)\,\omega_1+m_2\omega_2$. Then we have (denoting $X=X(\tau)$ and $Y=H(\tau)_{\mathrm{red}}$)*

$$\begin{aligned} s(X, L)&=s(X, L')+s(Y, L) && \text{if } \tau\neq\tau_4 \text{ or } \phi_3\\ &=s(X, L')+s(Y, L')+s(y, L) && \text{if } \tau=\tau_4 \text{ or } \phi_3. \end{aligned}$$

Case 1. $\tau=\tau_i$, $1\leqslant i\leqslant 6$ (where $\tau_6=w_0$) *Suppose* $i\neq 4$ (so that $\bar{\tau}\neq\phi_3$). Then grouping the standard monomials on X of deg m as

(i) monomials starting with $p_{\bar{\tau}}$,

(ii) monomials not starting with $p_{\bar{\tau}}$,

we find easily that the cardinality of the first set is $s(X, L')$ while that of the second set is $s(Y, L')$ (note that if a monomial F standard on $X(\tau)$ does not start with $p_{\bar{\tau}}$, then $F|_Y\neq 0$ and further F remains standard on Y (cf. Remark 8.11). Thus the lemma follows in this case. If $i=4$ (so that $\bar{\tau}=\phi_3$) then we group the standard monomials on X of deg m as

(i) monomials starting with p_{ϕ_3},

(ii) monomials starting with p_{ϕ_3,ϕ_2},

(iii) monomials starting with p_α ($\alpha\leqslant\phi_2$).

Now it is clear that cardinality of the first set is $h^0(X, L')$, that of the second is $h^0(Y, L')$, and that of the third is $h^0(Y, L)$. Thus we obtain

$$h^0(X, L)=h^0(X, L')+h^0(Y, L')+h^0(Y, L).$$

Case 2. $\phi=\phi_i$, $1\leqslant i\leqslant 5$. If $i\neq 3$, then the proof is as in case 1. If $i=3$, then again we have the same kind of grouping and the result follows as in case 1 (one should observe that if $p_{\phi_3,\phi_2}F$ is standard on $X(\phi_3)$, then F remains standard on $X(\phi_2)$) (since $X(\phi_2)$ is the unique Schubert subvariety of $X(\phi)$ having the projection $X(\phi_2)$ on G/P_1).

This completes the proof of Lemma 8.16.

PROPOSITION 8.17. *Let Z be a union of Schubert varieties. Then taking $k=\mathbf{Q}$, we have*

$$h^0(Z, L)=s(Z, m),$$

where $m=(m_1, m_2)$ and $L=L(\lambda)$, $\lambda=a_1\omega_1+a_2\omega_2$.

Proof. In view of Proposition 8.15, it is enough to prove the result when Z is a Schubert variety. Also, we may assume $m_1\geqslant 1$. Then let $Z=X(\tau_i)$, $0\leqslant i\leqslant 6$ (or $X(\phi_i)$, $1\leqslant i\leqslant 5$). For simplicity, let us write $Z=X(\tau)$. If $X(\bar{\tau})$ is the projection of $X(\tau)$ on G/P_1 under $G/B\to G/P_1$, then as remarked earlier, H_τ, the zero set of $p_{\bar{\tau}}$ is a Schubert variety (set theoretically), namely the unique Schubert divisor $X(w)$ such that the projection $X(\bar{w})$ of $X(w)$ on G/P_1 is a Schubert divisor in $X(\bar{\tau})$. Now the multiplicity of $X(\bar{w})$ in $X(\bar{\tau})$ is 1 if $\tau\neq\tau_4$ or ϕ_3 and is 2 otherwise. Hence if

$\tau \neq \tau_4$ or ϕ_3, H_τ is reduced (cf. Proposition 1.4) and is $X(w)$. Hence we obtain the exact sequence

$$0 \to L_1^{-1} \to O_X \to O_Y \to 0,$$

where $X = X(\tau)$, $Y = X(w)$. Now tensoring the above exact sequence with L and writing down the cohomology exact sequence we have

$$h^0(X, L) = h^0(X, L') + h^0(Y, L),$$

where $L' = L(\lambda')$, $\lambda' = (m_1 - 1)\,\omega_1 + m_2\omega_2$. (Since $k = \mathbf{Q}$, the higher cohomology groups vanish (cf. [4]).) But now the above relation is satisfied by s replacing h^0 (cf. Lemma 8.16) and we may assume $h^0(X, L') = s(X, m')$ (by induction on m_1; when $m_1 = 1$, the result is true in view of Proposition 6.1, and $h^0(Y, L) = s(Y, m)$ by induction on dim X)). Hence we obtain $h^0(X, L) = s(X, m)$.

If $\tau = \tau_4$ or ϕ_3, then proceeding as in [10] (see also opening *remarks* in Section 6) we obtain that under

$$Z_\phi = X(w) \times^{B_\alpha} SL_{2,\alpha} \xrightarrow{\psi} X(\tau)$$

(where α is given by $\tau = s_\alpha w$)

$$\psi^*(L_1^{-1}) \approx \widetilde{I(H(\omega))}^{(-2)}$$

and proceeding as in Section 6 we obtain

$$\begin{aligned} h^0(X, L) &= h^0(X, L') + h^0(Y, L') + h^0((H_\tau)_{\text{red}}, L) \\ &= h^0(X, L') + h^0(Y, L') + h^0(Y, L) \end{aligned}$$

(since $(H_\tau)_{\text{red}} = Y$). Now the above relation is satisfied by s replacing h^0 (cf. Lemma 8.16). Hence we obtain $h^0(X, L) = s(X, m)$ (by induction on dim X and m_1).

THEOREM 8.18. *Let Z be union of Schubert varieties. Then*

(a) *Distinct standard monomials of degree m on Z form a basis for $H^0(Z, L)$*

(b) $H^i(Z, L) = 0$, $i \geq 1$, $L \geq 0$.

Proof. In view of the linear independence of standard monomials (cf. Proposition 8.13) to prove (a) it is enough to prove that

$$h^0(Z, L) = s(Z, m).$$

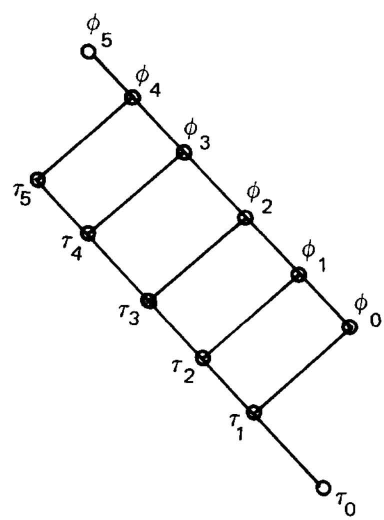
Also, in view of Proposition 8.15, to prove (a) and (b) we may assume Z is a Schubert variety, say $Z = X(\tau)$, $\tau \in W$. Now (as remarked in the beginning of Section 6), in view of Proposition 8.17, we obtain that over $\mathbf{Q}$, $H^0(X(\tau), L)$ has a basis by means of standard monomials of deg m and that the canonical map

$$H^0(G_{\mathbf{Z}}/B_{\mathbf{Z}}, L_{\mathbf{Z}}) \to H^0(X(\tau)_{\mathbf{Z}}, L_{\mathbf{Z}})$$

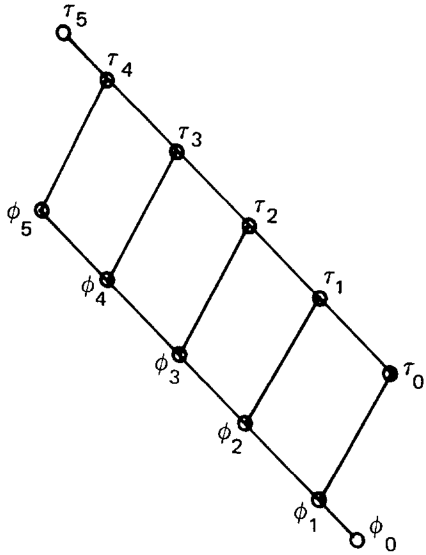
is surjective; from this, Demazure's conjecture follows and hence we obtain (b). Now (b) implies that dim $H^0(X(\tau), L)$ is independent of the characteristic of the base field and hence we obtain (a) (in view of Propositions 8.13 and 8.17).

THEOREM 8.19. *Let $w \in W$ and $R(w) = \oplus_{L \geq 0} (H^0(X(w), L)$. Then the ring $R(w)$ is Cohen–Macaulay.*

Proof. As in the proof of Proposition 5.1, it is enough to show that the ring $S(w) = (R(w))_{(0)}$ is Cohen–Macaulay. For, $2 \leq i \leq 5$, let $f_i = p_{\tau_i} + p_{\phi_{i-2}}$ and $g_i = p_{\phi_i} + p_{\tau_{i-2}}$. Then proceeding on the same lines as in Proposition 7.1, it can be easily shown that for $w = \tau_i$ (resp. ϕ_i), $2 \leq i \leq 5$, $\{p_{\phi_{i-1}}, f_i, f_{i-1}, \ldots, f_2, p_{\tau_1}, p_{\tau_0}\}$ (resp. $\{p_{\tau_{i-1}}, g_i, g_{i-1}, \ldots, g_2, p_{\phi_1}, p_{\phi_0}\}$) is an R-regular sequence (where $R = R(w)$). For $w = \tau_1$ (resp. ϕ_1), $\{p_{\phi_0}, p_{\tau_1}, p_{\tau_0}\}$ (resp. $\{p_{\tau_0}, p_{\phi_1}, p_{\phi_0}\}$) is an R-regular sequence. For $w = \text{Id}$, $\{p_{\tau_0}, p_{\phi_0}\}$ is an R-regular sequence and for $w = w_0$ (the unique element of maximal length) $\{p_{\phi_5}, p_{\phi_4}, f_5, \ldots, f_2, p_{\tau_1}, p_{\tau_0}\}$ (as well as $\{p_{\tau_5}, p_{\tau_4}, g_5, \ldots, g_2, p_{\phi_1}, p_{\phi_0}\}$) is an R-regular sequence. These are suggested by looking at the configuration



for the τ's and the configuration



for the ϕ's. (cf. Remark 8.11 and Definition 8.1).

THEOREM 8.20. *Let $w \in W$. Then the ring $R(w)$ is normal.*

This follows from Theorem 8.19 and Chevalley's result (cf. [3]) that Schubert varieties are non-singular in codim 1.

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