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ϕ-Summing Operators in Banach Spaces

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Let ϕ: $[0, \infty) \to [0, \infty)$ be a continuous subadditive strictly increasing function and $\phi(0)=0$. Let E and F be Banach spaces. A bounded linear operator $A: E \to F$ will be called ϕ-summing operator if there exists $\lambda > 0$ such that $\sum_{i=1}^{n} \phi \|Ax_i\| \leqslant \lambda \sup_{\|x^*\| \leqslant 1} \sum_{i=1}^{n} \phi |\langle x_i, x^* \rangle|$, for all sequences $\{x_1, ..., x_n\} \subseteq E$. We set $\prod^{\phi}(E, F)$ to denote the space of all ϕ-summing operators from E to F. We study the basic properties of the space $\prod^{\phi}(E, F)$. In particular, we prove that $\prod^{\phi}(H, H) = \prod^{p}(H, H)$ for $0 \leqslant p < 1$, where H is a Banach space with the metric approximation property. © 1987 Academic Press, Inc.

0. INTRODUCTION

Let ϕ: $[0, \infty) \to [0, \infty)$ be a continuous function. The function ϕ is called a modulus function if

(i) $\phi(x+y) \leqslant \phi(x) + \phi(y)$

(ii) $\phi(0)=0$

(iii) ϕ is strictly increasing.

The functions $\phi(x)=x^p$, $0<p\leqslant 1$ and $\phi(x)=\ln(1+x)$ are examples of modulus functions.

For Banach spaces E and F, a bounded linear operator $A: E \to F$ is called p-summing, $0 < p < \infty$, if there exists $\lambda > 0$ such that

$$\sum_{i=1}^{n} \|Ax_i\|^p \leqslant \lambda \sup_{\|x^*\| \leqslant 1} \sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p,$$

for all sequences $\{x_1, ..., x_n\} \subseteq E$. For $p=1$, this definition is due to

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Grothendieck [3], and for $p \neq 1$, the definition was given by Pietsch [6]. If $\prod^p(E, F)$ is the space of all p-summing operators from E to F, then it is well known [3, p. 293] that $\prod^p(E, F) = \prod^q(E, F)$ for $0 < p, q \leqslant 1$. If E and F are Hilbert spaces then $\prod^p(E, F) = \prod^q(E, F)$ for $0 < p < q < \infty$ [6, p. 302].

The object is to introduce ϕ-summing operators for modulus functions ϕ. The basic properties of these operators are studied. We, further, prove that ϕ-summing operators are p-summing for $0 < p \leqslant 1$, for Banach spaces having the metric approximation property.

Throughout this paper, $L(E, F)$ denotes the space of all bounded linear operators from E to F. The dual of E is E^*. The compact elements in $L(E, F)$ will be denoted by $K(E, F)$. The unit sphere of a Banach space E is denoted by $S(E)$. The set of complex numbers is denoted by $\mathbb{C}$.

1. $\prod^\phi(E, F)$

Let E and F be two Banach spaces and ϕ be a modulus function on $[0, \infty)$. Consider the following two spaces:

(i) $l^\phi\langle E\rangle = \{(x_n): \sup_{\|x^*\| \leqslant 1} \sum_n \phi\, |\langle x_n, x^*\rangle| < \infty,\ x_n \in E\}$.

(ii) $l^\phi(F) = \{(x_n): \sum_n \phi\, \|x_n\| < \infty,\ x_n \in E\}$.

For $x = (x_n) \in l^\phi\ \langle E\rangle$, we define

$$\|x\|_\varepsilon = \sup_{\|x^*\| \leqslant 1} \sum_n \phi\, |\langle x_n, x^*\rangle|,$$

and for $y = (y_n) \in l^\phi(F)$ we define

$$\|y\|_\pi = \sum_n \phi\, \|y_n\|.$$

It is a routine matter to verify the following result:

THEOREM 1.1. *The spaces* $(l^\phi\ \langle E\rangle, \|\ \ \|_\varepsilon)$ *and* $(l^\phi(F), \|\ \ \|_\pi)$ *are complete metric linear spaces.*

Remark 1.2. The spaces $l^\phi\langle E\rangle$ and $l^\phi(E)$ are generalizations of the spaces $l^p\langle E\rangle$ and $l^p(E)$ for $0 < p < 1$. We refer to [6, Chap. 16; 1] for a discussion of such spaces.

A linear operator $T: l^\phi\langle E\rangle \to l^\phi(F)$ will be called *metrically bounded* if there is a $\lambda > 0$ such that

$$\|Tx\|_\pi \leqslant \lambda\, \|x\|_\varepsilon$$

for all $x=(x_n)\in l^\phi\langle E\rangle$. Clearly every metrically bounded operator is continuous. We let $L^\phi(E,F)$ denote the space of all metrically bounded operator from $l^\phi\langle E\rangle$ into $l^\phi(F)$. For $T\in L^\phi(E,F)$, we set $\|T\|_\phi=\inf\{\lambda\colon \|Tx\|_\pi\leqslant\lambda\,\|x\|_\varepsilon,\ x\in l^\phi\langle E\rangle\}$. The proof of the following result is similar to the proof in case of Banach spaces, [7, p. 185], and it will be omitted.

THEOREM 1.3. *The space* $(L^\phi(E,F),\|\ \|_\phi)$ *is a complete metric linear space.*

DEFINITION 1.4. Let E and F be two Banach spaces. Then, a bounded linear operator $T\colon E\to F$ is called ϕ-summing if there is $\lambda>0$ such that

$$\sum_1^N \phi\,\|Tx_n\|\leqslant\lambda\sup_{\|x^*\|\leqslant 1}\sum_1^N\phi\,|\langle x_n,x^*\rangle| \tag{$*$}$$

for all sequences $\{x_1,\ldots,x_n\}\subseteq E$.

The definition is a generalization of the definition of p-summing operators for $0\leqslant p\leqslant 1$. We refer to [6] for a full study of p-summing operators $0<p<\infty$.

Let $\prod^\phi(E,F)$ be the set of all ϕ-summing operators from E to F. Every $T\in\prod^\phi(E,F)$ defines an element $\hat{T}\in L^\phi(E,F)$ via:

$$\hat{T}\colon l^\phi\,\langle E\rangle\to l^\phi(E)$$

$$\hat{T}((x_n))=((Tx_n)).$$

For $T\in\prod^\phi(E,F)$ we define the ϕ-summing metric of T as: $\|T\|_\phi=\|\hat{T}\|_\phi$. Hence $\|T\|_\phi=\inf\{\lambda\colon *\text{ holds}\}$. The definition of ϕ-summing operators together with Theorem 1.2 implies:

THEOREM 1.5. $(\prod^\phi(E,F),\|\ \|_\phi)$ *is a complete metric linear space.*

THEOREM 1.6. *Let* $A\in\Pi^\phi(E,F)$, $B\in L(G,E)$, *and* $D\in L(F,H)$. *Then* $AB\in\prod^\phi(G,E)$ *and* $DA\in\prod^\phi(E,H)$. *Further,* $\|AB\|_\phi\leqslant(\|B\|+1)\,\|A\|_\phi$ *and* $\|DA\|_\phi\leqslant(\|D\|+1)\,\|A\|_\phi$.

Proof. The proof follows from the fact that for all $a>0$, $\phi(at)\leqslant(a+1)\,\phi(t)$ which is a consequence of the monotonocity and subaddivity of ϕ. Q.E.D.

Let $B_1(E^*)$ be the unit ball of E^* equibbed with the w^*-topology, and M be the space of all regular Borel measures on $B_1(E^*)$. The unit sphere of M is denoted by $S(M)$.

THEOREM 1.6. *Let $A \in L(E, F)$. The followings are equivalent:*

(i) $A \in \prod^{\phi}(E, F)$.

(ii) *There exists $\lambda > 0$ and $\nu \in S(M)$ such that*

$$\phi \|Ax\| \leqslant \lambda \int_{B_1(E^*)} \phi |\langle x, x^* \rangle| \, d\nu(x^*).$$

Proof. (ii) → (i) This is evident.

(i) → (ii) Let $A \in \prod^{\phi}(E, F)$ and $\lambda = \|A\|_{\phi}$. For every finite sequence $\{x_1, ..., x_N\} \subseteq E$, define the map:

$$Q: S(M) \to \mathbb{C}$$

$$Q(\mu) = \sum_{n=1}^{N} \phi \|Ax_n\| - \lambda \sum_{1}^{N} \int_{B_1(E^*)} \phi |x_n, x^* \rangle| \, d\mu \cdots. \qquad (**)$$

Clearly, the function Q is convex. Further, there is a point $\mu_0 \in S(M)$ such that $Q(\mu_0) < 0$. Indeed choose $\mu_0 =$ the dirac measure at x_0^*, where

$$\sum_{1}^{N} \phi |\langle x_n, x_0^* \rangle| = \sup_{\|x^*\| \leqslant 1} \sum_{1}^{N} \phi |\langle x_n, x^* \rangle|.$$

Further, if $\{Q_1, ..., Q_r\}$ is a collection of such functions defined by (**), then for any $a_1, ..., a_r, \sum_1^r a_k = 1$, there is Q defined in a similar way, such that $\sum_1^r a_k Q_k(\mu) \leqslant Q(\mu)$ for all $\mu \in S(M)$. Hence the collection of functions on $S(M)$ defined by (**) satisfies Fan's lemma [6, p. 40]. Consequently there is a measure ν in $S(M)$ such that $Q(\nu) \leqslant 0$ for all Q defined by (**). In particular if Q is defined by (**) with associated sequence $\{x\}$, $x \in E$, we get

$$\phi \|Ax\| \leqslant \lambda \int_{B_1(E^*)} \phi |\langle x, x^* \rangle| \, d\nu. \qquad \text{Q.E.D.}$$

Remark 1.7. The proof of Theorem 1.6 is similar to the proof of Theorem 17.3.2. in [6], where $\phi(t) = t^p$, $0 < p \leqslant 1$. We included the detailed proof here for completeness and to include modulus functions.

2. $\prod^{\phi}(H, H) = \prod^{p}(H, H)$, $0 \leqslant p \leqslant 1$

Let m be the Lebesgue measure on $I = [0, 1]$. For the modulus function ϕ, set L^{ϕ} to denote the space of all measurable functions f on $[0, 1]$ for which $\int_0^1 \phi |f(t)| \, dm(t) < \infty$. For $f \in L^{\phi}$ we define $\|f\|_{\phi} = \phi^{-1} \int_0^1 \phi |f(t)| \, dm(t)$. The function $\| \ \|_{\phi}$ is not a metric on L^{ϕ}. However, we can define a topology via: $f_n \to f$ in L^{ϕ} if

$\phi^{-1}\int\phi\,|f_n-f|\,dm(t)\to 0$. It is not difficult to prove that such a topology makes L^ϕ a topological vector space. In case $\phi(t)=t^p$, $0<p\leqslant 1$, L^ϕ is a quasi-normed space [4, p. 159]. If $\phi(t)=t/(1+t)$, we write L^0 for L^ϕ.

The concept of ϕ-summing operators is still valid for operators $T: E\to L^\phi$, where E is a Banach space.

DEFINITION 2.1. Let E be a Banach space. A linear map $T: E\to L^\phi$ is called ϕ-decomposable if there is a function ψ: $[0, 1]\to E^*$ such that

(i) The function $\langle x, \psi(t)\rangle$ is m-measurable and

$$(Tx)(t)=\langle x, \psi(t)\rangle \text{ a.e.m.} \quad \text{for all} \quad x\in E.$$

(ii) There exists $f\in L^1$ such that $\|\psi(t)\|\leqslant f(t)$ a.e.m.

This definition is due to Kwapien [5] for $\phi(t)=t^p$. In [5], the function f in (ii) is assumed to belong to L^p. Since $L^\phi\subseteq L^0$ for all modulus functions ϕ, the following lemma is immediate:

LEMMA 2.2. *Every ϕ-decomposable map $T: E\to L^\phi$ is 0-decomposable.*

THEOREM 2.3. *Let E be any Banach space. If a linear map $T: E\to L^\phi$ is ϕ-decomposable, then T is ϕ-summing.*

Proof. Let ψ: $[0, 1]\to E^*$ be as in Definition 2.1 and $\{x_1, ..., x_N\}\subseteq E$. Then

$$\begin{aligned}\sum_1^N\phi\,\|Tx_n\|_\phi&=\sum_1^N\phi\left[\phi^{-1}\int_0^1\phi\,|\langle x_n, \psi(t)\rangle|\,dm(t)\right]\\&\leqslant\sum_1^N\int_0^1(\|\psi(t)\|+1)\,\phi\left|\left\langle x_n, \frac{\psi(t)}{\|\psi(t)\|}\right\rangle\right|\,dm(t)\\&\leqslant(\|f\|_1+1)\sup_{\|x^*\|\leqslant 1}\sum_1^N\phi\,|x_n, x^*\rangle|.\end{aligned}$$

Q.E.D.

Before we state the next theorem, we should remark that the topology on L^ϕ generated by the gauget $\|f\|_\phi=\phi^{-1}\int\phi\,|f|\,dm$, is equivalent to the topology generated by the metric $|||f|||_\phi=\int\phi\,|f|\,dm$. Consequently, the bounded sets in both topologies coincide.

THEOREM 2.4. *Let $T\in L(E, F)$ such that $T^*\in\prod^\phi(F^*, E^*)$. If F has the metric approximation property, then for any continuous linear map $\gamma: F\to L^\phi$, the map γT is ϕ-decomposable.*

Proof. First, we claim that there exists an $M>0$ such that for all

$x_1, x_2, ..., x_n \in E$, $\|x_i\| \leq 1$ and for all measurable disjoint sets $A_1, ..., A_n$ in $[0, 1]$ we have

$$\sum_{i=1}^{n} \int_{A_i} \phi\, |\gamma T(x_i)(t)|\, dt \leq M. \qquad (*)$$

By the remark preceding the theorem and the assumption that F has the metric approximation property, it is enough to prove $(*)$ for operators $\gamma = \sum_{i=1}^{k} y_i' \otimes 1_{B_i}$, $y_i' \in F^*$ and B_i measurable in $[0, 1]$. One can take B_i to be disjoint of equal length and $\bigcup_{i=1}^{k} B_i = [0, 1]$.

Let $\gamma = \sum_{j=1}^{k} y_i' \otimes 1_{B_i}$, B_i disjoint in $[0, 1]$ and $m(B_i) = 1/k$, $y_i' \in F^*$, for $i = 1, ..., k$. If $x_1, ..., x_n \in E$, with $\|x_i\| \leq 1$ and if $A_1, ..., A_n$ are disjoint measurable subsets in $[0, 1]$, then

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{A_i} \phi\, |\gamma T(x_i)(t)|\, dm(t) \\
&\quad = \sum_{i=1}^{n} \int_{A_i} \phi \left| \sum_{j=1}^{k} \langle Tx_i, y_j \rangle\, 1_{B_j}(t) \right| dm(t) \\
&\quad \leq \sum_{i=1}^{n} \sum_{j=1}^{k} \phi\, |\langle Tx_i, y_j' \rangle|\, m(B_j \cap A_i) && \text{(since } \phi \text{ is subadditive)} \\
&\quad \leq \sum_{j=1}^{k} \phi\, \|T^* y_j'\| \cdot \sum_{i} m(B_j \cap A_i) \\
&\quad \leq \sum_{j=1}^{k} \frac{1}{k} \phi\, \|T^* y_j'\| && \text{(since } A_i\text{'s are disjoint)} \\
&\quad \leq \lambda \sup_{\substack{\|x^*\| \leq 1 \\ x^* \in F}} \sum_{j=1}^{k} \phi\, |\langle y_j', x^* \rangle|\, m(B_j) && \text{(since } T^* \in \textstyle\prod^{\phi}(F^*, E^*)) \\
&\quad = \lambda \sup_{\|x^*\| \leq 1} \int \phi\, |\gamma x^*(t)|\, dm(t).
\end{aligned}$$

Since γ is continuous, by the remark preceding the theorem we get $\sup_{\|x^*\| \leq 1} \int \phi\, |\gamma x^*(t)\, dm(t) \leq M$ for some $M > 0$, and $(*)$ is proved.

It follows from $(*)$ that the image of the unit ball of E under γT is bounded in the lattice L^{ϕ}. If $g \in L^{\phi}$ such that $\gamma T(x) \leq g$ for all $x \in E$, $\|x\| \leq 1$, then the function $\theta(t) = \gamma Tx(t)/g(t)$ if $g(t) \neq 0$ and $\theta(0) = 0$, is an element of L^{∞}. Consequently, the linear map

$$S: E \to L^{\infty},$$

$$S(x) = \gamma Tx \mid g$$

is continuous and $\|S\| \leqslant 1$. Hence, by the lifting theorem, there exists $Q: [0, 1] \to (L^\infty)^*$ such that the function $\langle Q(t), f \rangle$ is m-measurable a.e., for all $f \in L^\infty$, and $f(t) = \langle Q(t), f \rangle$ a.e. Further $\|Q(t)\| = 1$ for all $t \in [0, 1]$. Now, consider the function $\psi: [0, 1] \to E^*$ defined by $\psi(t) = g(t) \cdot S^*(Q(t))$. It is not difficult to see that ψ is the function needed for γT to be ϕ-decomposable, noting that $g \in L^\infty \subseteq L^\phi$. Q.E.D.

Before we prove the next result, we need the following two lemmas:

LEMMA 2.5. *Let $T: L^\phi \to L^2$ be a continuous linear operator. Then $\|Tf\| \leqslant \lambda \int \phi |f(t)| dm(t)$ for all $f \in L^\phi$ for which $\int \phi |f(t)|\, dm(t) = |||f|||_\phi \leqslant 1$.*

Proof. First we prove it for $f \in L^\phi$, $|||f||| = 1$. If the inequality $\|Tf\| \leqslant \lambda\, |||f|||_\phi$ is not true, then we can find a sequence (f_n) such that $|||f_n|||_\phi = 1$ but $\|Tf_n\| > n\, |||f_n|||_\phi$. Then the sequence $f_n/n \to 0$ in L^ϕ, but $\|T(f_n/n)\| > 1$, which contradicts the continuity of T.

Now, let $f \in L^\phi$, $|||f|||_\phi < 1$. Then one can find an $\alpha > 1$ such that $|||\alpha f|||_\phi = 1$. Hence

$$\begin{aligned} \|Tf\| &= \frac{1}{\alpha}\, \|T\alpha f\| \\ &\leqslant \frac{\lambda}{\alpha}\, |||\alpha f|||_\phi \\ &\leqslant \lambda \frac{\alpha + 1}{\alpha}\, |||f|||_\phi \\ &\leqslant 2\lambda\, |||f|||_\phi. \end{aligned}$$

Q.E.D.

It should be remarked that for every $r > 0$ there exists $\lambda > 0$ such that $\|Tf\| \leqslant \lambda\, |||f|||_\phi$ for all $f \in L^\phi$, $|||f|||_\phi \leqslant r$.

LEMMA 2.6. *Let $T: L^2 \to L^\phi$ be p-summing operator. Then $ST: L^2 \to L^2$ is p-summing for continuous operators $S: L^\phi \to L^2$.*

Proof. Using Lemma 2.5 and the argument in the proof of Theorem 1.6, the result follows. Q.E.D.

Now we prove:

THEOREM 2.7. *Let ϕ be any modulus function. Then $\prod^\phi(L^2, L^2) \subseteq \prod^2(L^2, L^2)$.*

Proof. Let $T: L^2 \to L^2$ be ϕ-summing operator. By Theorem 2.4, $\gamma T^*: L^2 \to L^2 \to L^\phi$ is ϕ decomposable for all continuous linear operators $\gamma: L^2 \to L^\phi$. In particular, we can choose $\gamma(f) = \int f(t)\, dx_t$ [2, 5], where (x_t)

is a symmetric stable process on $([0, 1], m)$ with exponent 2. This makes γ an isomorphic embedding of L^2 into L^ϕ and also into L^0. Hence $\gamma T^*: L^2 \to L^0$ is zero decomposable. Using Theorem 3 in [5], we get $T^*: L^2 \to L^2$ is zero summing. By Lemma 2.6, $\gamma T^*: L^2 \to L^0$ is zero decomposable. Another application of Theorem 3 in [5]: we get $T: L^2 \to L^2$ is zero-summing. However, every zero-summing map is 2-summing, [5]. Hence $T \in \prod^2(L^2, L^2)$.

THEOREM 2.8. *For any modulus function* ϕ, $\prod^2(L^2, L^2) \subseteq \prod^\phi(L^2, L^2)$.

Proof. Let $T: L^2 \to L^2$ be 2-summing operator. If γ is the isomorphic embedding of L^2 into L^ϕ as in Theorem 2.7, then using Theorem 3 in [5], we get

$$\gamma T: L^2 \to L^2 \to L^\phi$$

is ϕ-decomposable. By Theorem 2.3, γT is ϕ-summing. Using Lemma 2.5, we get $T: L^2 \to L^2$ is ϕ-summing. Q.E.D.

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