

Existence of Positive Radial Solutions for Semilinear Elliptic Equations in the Annulus

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In this paper we study the existence of positive radial solutions for the Dirichlet problem

$$\Delta u(x) + f(u(x)) = 0, \quad x \in \Omega \tag{1}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{2}$$

where Ω is an annulus in \mathbb{R}^n ; i.e., $\Omega = \{x \in \mathbb{R}^n \mid 0 < \lambda < |x| < R\}$.

The existence of solutions for this problem in general domains has been widely studied. Most of these results are based on variational methods, and one finds in these works the following two limitations:

(a) it is often required that $f(0) \geq 0$ and

(b) the growth of f at infinity cannot exceed that of u^k with $k \leq (n+2)/(n-2)$.

We find positive radial solutions of (1), (2) avoiding these restrictions whenever possible; namely, we require no conditions on $f(0)$ and no upper bound for k . However, in the case of Ω an n -ball, requirement (b) is a necessary condition for existence of solutions, since from a well-known theorem of Pohozaev's [3] we have that if $f(u) = u^k$, then k has to satisfy $k < (n+2)/(n-2)$ in order to have positive solutions of (1), (2). Smoller and Wasserman [4, 5] have proved the existence of radial positive solutions on a ball when $f(0) = O(u^k)$ as $u \rightarrow +\infty$, $0 < k < n/(n-2)$, with no conditions on $f(0)$.

Now, if Ω is an annulus, Pohozaev's theorem does not apply anymore since the annulus is not a star shaped-domain. Therefore, there are no "natural" constraints for the growth of f .

In fact, if Ω is an annulus, $\Omega = \{x \in \mathbb{R}^n \mid \lambda < |x| < R\}$, $n \geq 2$, and f satisfies:

$$f(u) = O(u^k) \quad \text{as } u \rightarrow +\infty \text{ with } k > -1,$$

we can prove the existence of positive radial solutions for a wide range of domains, as well as some non-existence results. For example, let $f_0 = \lim_{s \rightarrow 0} f(s)/s$. If $f_0 = 0$ and $k > 1$ or $f_0 = +\infty$ and $k < 1$ (e.g., $f(u) = u^k$, $k \neq 1$), there is a radial positive solution in any annulus.

Another example would be: assume $f(0) < 0$ and $k < 1$ or $f(0) = 0$ and $k \leq 1$; then there are constants $C \leq C'$ such that there is a radial positive solution of (1), (2) when $R - \lambda \geq C'$ and no radial positive solutions when $R - \lambda < C$.

In Theorem A, we prove the existence of solutions on "some" domains, and in Theorem B, we discuss the range for the domains in which existence is found, as well as some non-existence results. The different cases in Theorem B come from the behavior of f at infinity (k less, greater or equal to 1) and at 0.

In Theorem 30 we give an existence result for the exterior problem ($\Omega = \mathbb{R}^n - B_\lambda = \{x \in \mathbb{R}^n \mid \lambda < |x|\}$).

These results extend those of our previous note [1].

The main theorems can be stated as follows:

THEOREM A. *Let $\lambda > 0$. Given $n \geq 2$ and f a continuous real function satisfying:*

- (i) *there is a $A \geq 0$, such that $F(u) \leq 0$ for $u < A$ and $f(u) > 0$ for $u > A$, where $F(s) = \int_0^s f(t) dt$; and*
- (ii) *$f(u) = O(u^k)$ when $u \rightarrow +\infty$ and $k > -1$.*

Then, there are R 's such that a solution to problem (1), (2) exists which is radial and positive.

Let us define $f_0 = \lim_{s \rightarrow 0} f(s)/s$.

THEOREM B. *Let $\lambda > 0$. Given $n \geq 2$ and $f(u)$ a function as in Theorem A, then the following hold:*

Assume $A > 0$.

- (i) *If $f(0) < 0$ and $k < 1$ or $f(0) = 0$ and $k \leq 1$, there are constants $C_1 \leq C_2$ such that there is a radial solution to problem (1), (2) when $R - \lambda \geq C_2$ and no radial solutions when $R - \lambda < C_1$.*

- (ii) *If $f(0) < 0$ and $k = 1$, there are constants $C_1 \leq C_2 < C_3 \leq C_4$ such that there is a radial solution to problem (1), (2) when $C_3 > R - \lambda > C_2$ and no radial solutions when $R - \lambda < C_1$ or $R - \lambda > C_4$.*

(iii) If $f(0) < 0$ and $k > 1$, there are constants $C_1 \leq C_2$ such that there is a radial solution to problem (1), (2) when $R - \lambda \leq C_1$ and no radial solutions when $R - \lambda > C_2$.

(iv) if $f(0) = 0$ and $k > 1$, there is a radial solution to problem (1), (2) for any $R > \lambda$.

Assume $A = 0$.

(v) If $f_0 = 0$ and $k \leq 1$ or $f_0 < +\infty$ and $k < 1$, then there is a constant $C > 0$ such that there is a radial solution to problem (1), (2) if $R - \lambda > C$ and no radial solution if $R - \lambda < C$.

(vi) If $0 < f_0 < +\infty$ and $k = 1$, there are constants $0 < C_1 \leq C_2$ such that there is a radial solution to problem (1), (2) if $C_1 < R - \lambda < C_2$ and no radial solution if $R - \lambda < C_1$ or $R - \lambda > C_2$.

(vii) If $0 < f_0 < +\infty$ and $k > 1$ or $f_0 = +\infty$ and $k \geq 1$, there is a constant $C > 0$ such that there is a radial solution to problem (1), (2) if $0 < R - \lambda < C$ and no radial solution if $R - \lambda > C$.

(viii) If $f_0 = 0$ and $k > 1$ or $f_0 = +\infty$ and $k < 1$, there is a radial solution to problem (1), (2) for any $R > \lambda$.

Note. To prove Theorem A we only need condition (ii). If condition (i) is not satisfied, let $A = \min\{s_0 \mid f(s) > 0, s > s_0\}$. Then there is a D , $0 < D < A$, such that $F(D) > 0$ and $f(D) = 0$. The existence of solutions for large R 's and $p = \max\{u(r) \mid \lambda < r < R\}$ near D easily follows. The existence results in cases (i)–(iv) of Theorem B still hold and, since we pick up more solutions (we prove the existence of these solutions in a future paper), we could extend these results case by case depending on the behavior of f at 0.

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Since we are interested in radial functions $u = u(r)$, we write Eq. (1) in the following form, where $r = |x|$.

$$u''(r) + \frac{n+1}{r} u'(r) + f(u(r)) = 0, \quad \lambda < r < R, \quad (3)$$

with boundary conditions

$$u(\lambda) = u(R) = 0. \quad (4)$$

Smoller and Wasserman [4, 5] studied problem (1), (2) when Ω is a n -ball, D_R^n , and gave a description of existence of solutions as well as uniqueness and nondegeneracy. We adapt their methods to the case of the annulus.

Unlike in the case of the ball, since Ω is not simply connected, we cannot fully use the results of Gidas *et al.* [2] for positive solutions. All that we know is that the maximum of any positive solution occurs at a point $|x_0| < (\lambda + R)/2$ and that $du/dr < 0$ for $r > (\lambda + R)/2$.

Now for radial solutions of (3), (4), a simple “phase-plane” analysis shows that there is a unique R_0 such that $u'(R_0) = 0$ and so $u'(r) > 0$ for $\lambda < r < R_0$ and $u'(r) < 0$ for $R_0 < r < R$. Instead of solving directly problem (3), (4) we will study the trajectories of (3) with initial data

$$u'(\lambda) = \alpha > 0, \quad u(\lambda) = 0 \tag{5}$$

and find the minimal $R(\alpha)$ for which $u(R(\alpha)) = 0$; see Fig. 1. Thus we will have a solution of (1), (2) in the annulus $\Omega = \{x \in \mathbb{R}^n \mid \lambda < |x| < R(\alpha)\}$. Thus we introduce a new parameter, the initial velocity α . We will think of (3), (5) as a system

$$u' = v \tag{6}$$

$$v' = -\frac{n-1}{r} v - f(u), \quad r > \lambda$$

with initial data

$$u(\lambda) = 0, \quad v(\lambda) = \alpha. \tag{7}$$

We define an “energy function” $H(r)$ by

$$H(r) = \frac{(v(r))^2}{2} + F(u(r)), \tag{8}$$

where $F(s) = \int_0^s f(t) dt$. Then, on a trajectory of (6), we see that H is decreasing; namely,

$$H'(r) = v(r) v'(r) + v(r) f(u(r)) = -\frac{n-1}{r} (v(r))^2 \leq 0.$$

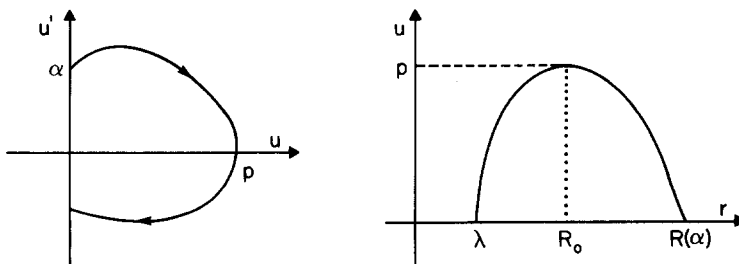


FIGURE 1

We have the strict inequality everywhere except when $v(r)=0$. A first result.

LEMMA 1. *If $(u(r), v(r))$ solves (3), (4), then v can be zero only once. (This means that for solutions of (3), (4) there is a unique local maximum.)*

Proof. If this is not the case, in the phase-plane we see that the trajectory of (6) will cross itself for $u > 0$ (Fig. 2a) or hit $v=0$ tangentially (Fig. 2b). In the first case we would have $r_2 > r_1 > \lambda$ with $u(r_1) = u(r_2) = u_0 > 0$ and $v(r_1) = v(r_2) = v_0$; i.e., $H(r_1) = H(r_2)$. Therefore, $v(r) = 0$ for $r_1 < r < r_2$ and so $v'(r) = 0$ and $f(u(r)) = f(u_0) = 0$ for $r_1 < r < r_2$. This means that the trajectory (u, v) is trapped at a rest point and thus it cannot solve (6), (7). The second case leads to $v'(r_0) = 0$ and a rest point again. ■

We now make the following assumptions on f :

f is a continuous real function on \mathbb{R}^+ which satisfies:

$$\text{there is an } A \geq 0 \text{ such that } F(u) \leq 0 \text{ if } 0 \leq u < A \text{ and } f(u) > 0 \text{ if } u > A, \tag{9}$$

$$\text{there are } b > 0, \text{ constants } d_1, d_2 > 0 \text{ and } k > -1 \text{ such that for } u \geq b \text{ we have } d_1 u^k \leq f(u) \leq d_2 u^k. \tag{10}$$

If $u(r)$ is going to be a solution of (3), (4) we need $p = \max\{u(r) \mid \lambda < r < R\} > A$. This follows from the energy function H ; namely at $r = R$, $H(R) = v^2(R)/2 \geq 0$, thus $H(r) > H(R) \geq 0$ for $r < R$ and in particular for r_0 such that $u(r_0) = p$ we get $H(r_0) = F(p) > 0$, and therefore $p > A$.

In the next two sections we study the trajectories of (6), (7). First we will show that they reach the maximum $(u(R_0))$ (Sect. 2), and then that they go from the maximum to $R(\alpha)$ (Sect. 3).

We prove all the results assuming $n > 2$. For $n = 2$ the same proofs are valid with only a slight variation.

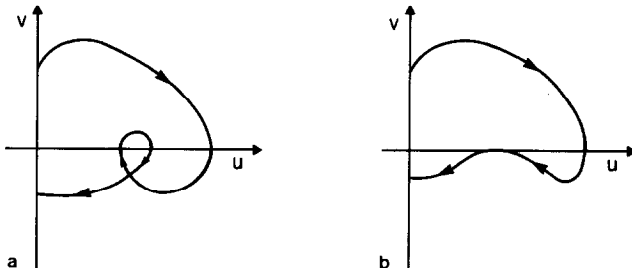


FIGURE 2

2

In what follows C will stand for a generic constant.

PROPOSITION 2. *Let f satisfy (9). We define $e = \max\{f(u) \mid u \leq A\}$. Then, if $\alpha > (2(n-2)A/\lambda) + ((3n-7)\lambda e/n)$, there is a t_1 such that $u(t_1) = A$ and*

$$t_1 \leq \lambda(1 + O(1/\alpha)). \quad (11)$$

Also we have

$$v(t_1) \geq \alpha(1 + O(1/\alpha)) \quad (12)$$

(if $n = 2$ we would need $\alpha > (A/\lambda \log 2) + (3\lambda e/2n \log 2)$).

Proof. If $A = 0$, then $t_1 = \lambda$ and $v(t_1) = \alpha$.

Let $A > 0$. If for r , $\lambda < r < 2\lambda$, $u(r) < A$, from (6) we would have

$$-(r^{n-1}v)' = r^{n-1}f(u) \leq r^{n-1}e.$$

We integrate twice from λ to r to get

$$r^{n-1}v \geq -e \frac{(r^n - \lambda^n)}{n} + \lambda^{n-1}\alpha;$$

i.e.,

$$v(r) \geq -\frac{er}{n} + \frac{\lambda^{n-1}}{r^{n-1}} \left(\frac{\lambda e}{n} + \alpha \right) \quad (13)$$

$$\begin{aligned} u(r) &\geq -e \frac{(r^2 - \lambda^2)}{2n} + \frac{\lambda^{n-1}}{n-2} \left(\frac{\lambda e}{n} + \alpha \right) \left(\frac{1}{\lambda^{n-2}} - \frac{1}{r^{n-2}} \right) \\ &\geq (r - \lambda) \left\{ \frac{-e}{2n} (r + \lambda) + \frac{\lambda}{r} \left(\frac{\lambda e}{n} + \alpha \right) \frac{1}{n-2} \right\}, \end{aligned} \quad (14)$$

and for $r = 2\lambda$, we get the contradiction $u(2\lambda) > A$.

Therefore, there is a $t_1 < 2\lambda$ with $u(t_1) = A$.

From (14), since $t_1 < 2\lambda$, we can write

$$A \geq -\frac{3e\lambda^2}{2n} + \frac{\lambda^{n-1}}{n-2} \left(\frac{\lambda e}{n} + \alpha \right) \left(\frac{1}{\lambda^{n-2}} - \frac{1}{t_1^{n-2}} \right)$$

Solving the above inequality for t_1 , we obtain (11).

Inequality (12) follows by combining (11) and (13). ■

PROPOSITION 3. *Let f satisfy (9), (10). If $\alpha > (2(n-2)A/\lambda) + ((3n-7)\lambda\epsilon/n)$, there is a $R_0 > t_1$ such that $v(R_0) = 0$.*

Proof. For t_1 we have $v(t_1) > 0$. Then once we hit $u = A$ we can go a bit further to $t_\epsilon = t_1 + \epsilon$, $\epsilon > 0$, with $v(t_\epsilon) > 0$ and $u(t_\epsilon) = A + \epsilon$; i.e., $f(u(t_\epsilon)) > 0$. Also since $H'(r) \leq 0$ and $k > -1$, there is a u^* such that $F(u^*) = H(t_1)$ and $u(r) \leq u^*$ for $r > t_1$.

We have $f(u) \geq m_\epsilon > 0$, for $u^* \geq u \geq A + \epsilon$. So

$$-(r^{n-1}v)' = r^{n-1}f(u) \geq m_\epsilon r^{n-1},$$

and integrating from t_ϵ to r we obtain

$$-r^{n-1}v(r) + t_\epsilon^{n-1}v(t_\epsilon) \geq (r^n - t_\epsilon^n) m_\epsilon/n.$$

Then,

$$v(r) \leq v_\epsilon \left(\frac{t_\epsilon}{r}\right)^{n-1} + \frac{m_\epsilon}{n} \left(\frac{t_\epsilon^n}{r^{n-1}} - r\right),$$

where the right-hand side goes to $-\infty$ when r increases.

Therefore $v(r)$ becomes 0 for some r ; i.e., there is a R_0 such that $v(R_0) = 0$. ■

It is clear that R_0 depends continuously on α , thus $R_0 = R_0(\alpha)$. We denote $p = u(R_0)$ or $p(\alpha) = u(R_0(\alpha))$.

In Section 3 we will need to consider large p 's. The next proposition proves that we can make p as large as we need by choosing α large enough.

PROPOSITION 4. *If there is an $R_0(\alpha)$ such that for $(u(r), v(r))$ satisfying (6) we have $v(R_0(\alpha)) = 0$ and $u(R_0(\alpha)) = p > A$; then*

$$p(\alpha) \rightarrow +\infty \quad \text{when } \alpha \rightarrow +\infty$$

Proof. For $r \in [t_1, R_0]$, from (6) we have $v'(r) < 0$ since $f(u(r)) \geq 0$ and $v(r) \geq 0$. Therefore $0 \leq v(r) \leq v(t_1)$ and

$$0 \leq \frac{v(r)}{r} \leq \frac{v(t_1)}{t_1}.$$

Let $M_p = \sup\{f(s) \mid A < s < p\}$, then

$$v'(r) = -\frac{n-1}{r} v(r) - f(u(r)) \geq -v(t_1) \frac{n-1}{t_1} - M_p.$$

Multiplying both sides by $v(r)$ and integrating from t_1 to R_0 , we get

$$-v^2(t_1)/2 \geq -\left(v(t_1) \frac{n-1}{t_1} + M_p\right)(p-A).$$

Solving the above inequality for $t_1 v(t_1)$ we have

$$t_1 v(t_1) \leq (n-1)(p-A) + [(n-1)^2(p-A)^2 + 2M_p(p-A)t_1^2]^{1/2}. \quad (15)$$

This, together with (11) and (12), proves the proposition. ■

Now, we will focus our attention at the bounds for R_0 when α is large, namely $p(\alpha) > b$. We define t_b by $u(t_b) = b$ and $v(t_b) \geq 0$. f will satisfy (10). We have two cases; namely $k \geq 1$ and $k < 1$.

PROPOSITION 5. *If $k > -1$, then there is a constant $C > 0$ such that*

$$R_0 - t_b < Cp^{(1-k)/2}. \quad (16)$$

Proof. Since $H'(v) \leq 0$, we have $H(r) \geq H(R_0)$ for $t_b \leq r \leq R_0$ and so

$$\frac{v^2(r)}{2} + F(u(r)) \geq F(p).$$

Thus from (10)

$$v(r) \geq (2\{F(p) - F(u(r))\})^{1/2} \geq \left(\frac{2d_1}{k+1}\right)^{1/2} (p^{k+1} - u^{k+1})^{1/2}$$

and

$$C \frac{v(r)}{[p^{k+1} - u^{k+1}]^{1/2}} \geq 1.$$

Integrating from t_b to R_0 gives

$$R_0 - t_b \leq C \int_{t_b}^{R_0} \frac{v(r)}{[p^{k+1} - u^{k+1}]^{1/2}} dr.$$

Let $u^{(k+1)/2} = p^{(k+1)/2} \sin \theta$; i.e.,

$$u = p(\sin \theta)^{2/(k+1)},$$

and

$$du = p(\sin \theta)^{(1-k)/(k+1)} \cos \theta \frac{2}{k+1} d\theta.$$

Let us define $b_0 = \arcsin(b/p)^{(k+1)/2}$. Then

$$R_0 - t_b \leq Cp^{(1-k)/2} \int_{b_0}^{\pi/2} (\sin \theta)^{(1-k)/(k+1)} d\theta.$$

Since $(1-k)/(1+k) > -1$, the integral is convergent and this completes the proof. ■

PROPOSITION 6. *Assume f satisfy (10), let $M(b) = \max\{f(u) \mid u \leq b\}$. Then if $\alpha > (2(n-2)b/\lambda) + ((3n-7)\lambda M(b)/n)$, there is a t_b such that $u(t_b) = b$ and*

$$t_b \leq \lambda(1 + O(1/\alpha)). \tag{17}$$

Also we have

$$v(t_b) \geq \alpha(1 + O(1/\alpha)).$$

Proof. The proof is similar to the one in Proposition 2. ■

COROLLARY 7. *If $k > 1$, then for large α , $R_0 - \lambda \leq O(1/\alpha)$.*

Proof. From Propositions 5 and 6

$$R_0 - \lambda = R_0 - t_b + t_b - \lambda \leq Cp^{(1-k)/2} + O(1/\alpha).$$

From (15) and (12), for large α and $k > 1$, we have

$$\alpha \leq Cp^{(1+k)/2}.$$

Thus

$$p^{(1-k)/2} \leq C\alpha^{(1-k)/(1+k)} = O(1/\alpha).$$

Therefore

$$R_0 - \lambda \leq O(1/\alpha). \quad \blacksquare$$

PROPOSITION 8. *If $k \leq 1$, then for large α $R_0 - \lambda \leq Cp^{(1-k)/2}$.*

Proof. From (16) and (17),

$$R_0 - \lambda = R_0 - t_b + t_b - \lambda \leq Cp^{(1-k)/2} + O(1/\alpha) \leq Cp^{(1-k)/2}. \quad \blacksquare$$

3

In this section we will show that there are trajectories of (6) with initial data

$$u(R_0) = p, \quad v(R_0) = 0 \quad (18)$$

such that there is an R so that $u(R) = 0$; cf. Fig. 3.

We begin with a comparison theorem which will be needed in what follows. Thus consider the systems of equations associated to (6),

$$\begin{aligned} \bar{u}' &= \bar{v}, & \bar{u}(T) &= B \\ \bar{v}' &= -\frac{n-1}{r} \bar{v} - M, & \bar{v}(T) &= q \end{aligned} \quad (19)$$

$$\begin{aligned} z' &= w, & z(T) &= B \\ w' &= -\frac{n-1}{T} w - M, & w(T) &= q. \end{aligned} \quad (20)$$

THEOREM 9. *Suppose that $f(u) > M$ for $0 \leq u \leq B$, $u(T) = B > 0$ and $v(T) = q < 0$ for some $T > R_0$. Then $\bar{u}(r) \geq u(r)$ for $r \geq T$ on $0 \leq u \leq B$ and if $\bar{v}(r) < 0$, $T \leq r \leq T_1$, then $z(r) \geq \bar{u}(r)$ on this range.*

Proof. We only sketch the proof since it can be found in [4, Theorem 5]. Let $h(r) = \bar{u}(r) - u(r)$, then $h(T) = h'(T) = 0$ and $h''(r) = -(n-1)h'(r)/r - M + f(u)$. Thus $h''(T) = 0$ and if $h'(r_1) = 0$ for some $r_1 > T$, $0 \leq u \leq B$, then $h''(r_1) > 0$. It follows that $h'(r) > 0$ for all $r > T$ on $0 \leq u \leq B$ and so $\bar{u}(r) > u(r)$ if $r > T$ on $0 \leq u \leq B$.

Also if $g(r) = z(r) - \bar{u}(r)$, then $g(T) = g'(T) = g''(T) = 0$ and $g'''(r) = -(n-1)w'(r)/T + (n-1)\bar{v}(r)/r - (n-1)\bar{v}(r)/r^2$. Thus $g'''(T) > 0$ and so $g'(r) > 0$ and $g(r) > 0$ for $T < r \leq T_1$. ■

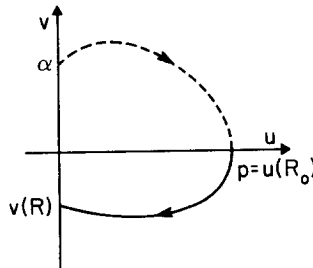


FIGURE 3

LEMMA 10. Suppose that $0 < m(p) \leq f(u) \leq M(p)$, when $B \leq u \leq p$. Then for any trajectory $(u(\cdot, p), v(\cdot, p))$ of (6), (7), there is a $T > R_0$ with $u(T, p) = B$.

Proof. For $B \leq u \leq p$ we have

$$r^{n-1}m(p) \leq r^{n-1}f(u) = -(r^{n-1}v)' \leq r^{n-1}M(p).$$

Thus integrating from R_0 to r , we have

$$\frac{m(p)}{n} (r^n - R_0^n) \leq -r^{n-1}v \leq \frac{M(p)}{n} (r^n - R_0^n).$$

We divide by r^{n-1} and integrate a second time to get

$$\begin{aligned} \frac{m(p)}{n} \left\{ \frac{r^2}{2} - \frac{R_0^2}{2} - \frac{R_0^n}{n-2} \left(\frac{1}{R_0^{n-2}} - \frac{1}{r^{n-2}} \right) \right\} &\leq p - u(r) \\ &\leq \frac{M(p)}{n} \left\{ \frac{r^2}{2} - \frac{R_0^2}{2} - \frac{R_0^n}{n-2} \left(\frac{1}{R_0^{n-2}} - \frac{1}{r^{n-2}} \right) \right\}. \end{aligned}$$

Since the first term goes to $+\infty$ when $r \rightarrow +\infty$, we see that the lemma follows. ■

We choose B such that $F(B) \geq 0$ and $B \geq b$. Define $q = v(T)$ where as before $u(T) = B > A$.

LEMMA 11. (i) If $k \geq 0$, then $-q/T$ is bounded away from zero independently of α for large α .

(ii) If $0 > k > -1$, then there is a constant C such that $-q/T \geq Cp^k$.

Proof. (i) Let $k \geq 0$. Then there is a $m > 0$ such that $f(u) \geq m$ for $u \geq B$.

We integrate $-(vr^{n-1})' = r^{n-1}f(u) \geq r^{n-1}m$ from R_0 to T to get

$$-qT^{n-1} \geq \frac{m}{n} (T^n - R_0^n)$$

and

$$-\frac{q}{T} \geq \frac{m}{n} \left(1 - \left(\frac{R_0}{T} \right)^n \right). \tag{21}$$

Also

$$H(T) - H(R_0) = \frac{q^2}{2} - [F(p) - F(B)] = -(n-1) \int_{R_0}^T \frac{v^2}{r} dr,$$

and since $(v(r))^2 \leq 2[F(p) - F(u(r))] \leq 2[F(p) - F(B)]$ for $R_0 \leq r \leq T$ we have

$$\frac{q^2}{2} \geq [F(p) - F(B)] \left\{ 1 - 2(n-1) \frac{T - R_0}{R_0} \right\}$$

and

$$\left(\frac{q}{T}\right)^2 \geq \frac{2[F(p) - F(B)]}{R_0^2} \left(1 - 2(n-1) \frac{T - R_0}{R_0}\right) \left(\frac{R_0}{T}\right)^2. \quad (22)$$

Now, whenever $q/T \rightarrow 0$ from (21) we get $R_0/T \rightarrow 1$ and then the right-hand side in (22) would behave like $2[F(p) - F(B)]/C$ when $k \geq 1$ (by Corollary 7) or like Cp^{k+1}/p^{1-k} when $k < 1$ (by Proposition 8). In either case we reach a contradiction. Therefore, there is a $C > 0$ such that $|q/T| \geq C$.

(ii) Let $0 > k > -1$. In this case, $f(u) \geq d_1 p^k$ for $B \leq u \leq p$. As above, if $R_0/T \rightarrow 1$ we would have $|q/T| \geq Cp^k$. Thus from (21) we obtain $-q/T \geq Cp^k$. ■

We now state the following theorem which can be found in [4]. It will give the existence of solutions when $k \geq 0$. A similar result for $0 > k > -1$ will be given in Theorem 18.

THEOREM 12. *Suppose that $f(u) \geq m > 0$ for $u \geq B$. Then*

- (i) *for any $p > B$, there is a $T > R_0$ such that $u(T, p) = B$;*
- (ii) *let $q = v(T, p)$, if $-qT \rightarrow +\infty$ as $p \rightarrow +\infty$, then the problem (3), (18) has a solution with $R = R(p)$.*

Proof. Part (i) follows from Lemma 10. For (ii) consider the system (20). We shall show that if $w(r_1) = 0$ and $-qT$ is sufficiently large, then $z(r_1) < 0$. Thus from Theorem 9 we conclude that $u(R, p) = 0$ for some $R \leq r_1$.

Suppose $f(u) \geq \mu$, $\mu < 0$, and set

$$\beta = (n-1)/T, \quad \delta = (n-1)q/(T\mu) = \beta q/\mu.$$

Equation (20) can be explicitly integrated as

$$w(r) = q \exp(-\beta(r-T)) - [1 - \exp(-\beta(r-T))] \mu/\beta.$$

Thus for r_1 such that $w(r_1) = 0$

$$r_1 - T = \ln(1 + \delta)/\beta.$$

Furthermore,

$$\begin{aligned} z(r_1) - B &= \int_T^{r_1} w(s) ds \\ &= -(\exp(-\beta(r_1 - T)) - 1) q/\beta - \mu(r_1 - T)/\beta \\ &\quad - (\exp(-\beta(r_1 - T)) - 1) \mu/\beta^2, \end{aligned}$$

so

$$z(r_1) = B + \frac{qT}{n-1} \left[1 - \frac{\ln(1 + \delta)}{\delta} \right].$$

If $\phi(\delta) = 1 - \ln(1 + \delta)/\delta$ then $\phi(0) = 0$ and $\phi'(0) > 0$ for $\delta > 0$. Since $\delta = (n - 1) q/(T\mu)$, it follows from part (i) of Lemma 11 that $\phi(\delta)$ is bounded away from zero. Since $qT \rightarrow -\infty$ as $p \rightarrow +\infty$ we see that $z(r_1) < 0$ for large p . This completes the proof. ■

At this point we only need to prove that $-qT \rightarrow +\infty$ as $p \rightarrow +\infty$. To do this we follow [5] and we choose B so that also $B \geq A + 2\varepsilon$.

Let us note that T depends on p and R_0 . Sometimes we will write $T = T(p)$, but since $R_0 = R_0(\alpha)$ and $p = p(\alpha)$ we have obviously $T = T(\alpha)$.

LEMMA 13. *If $k \geq 1$, there is a constant $C > 0$ such that*

$$T - R_0 \leq CB^{(1-k)/2}. \tag{23}$$

If $k < 1$, there is a constant $C > 0$ such that

$$T - R_0 \leq Cp^{(1-k)/2}.$$

Proof. From (10) we have $f(u) \geq d_1 u^k$.

We define $C(p, k) = d_1 B^{k-1}$ if $k \geq 1$ and $C(p, k) = d_1 p^{k-1}$ if $k < 1$. Then $f(u) \geq C(p, k) u$ for $B \leq u \leq p$. We make the following change of variables:

$$s = ar, \quad \text{where } a = (C(p, k))^{1/2}.$$

Then $u(r) = w(s)$, $aR_0 = S_0$, $aT = S$, $g(w) = a^{-2}f(u) \geq w$ and Eq. (3) is written as

$$w''(s) + \frac{n-1}{s} w'(s) + g(w(s)) = 0.$$

We define θ by $\tan \theta = w'/w$. Then for $S_0 \leq s < S$, $-\pi/2 < \theta \leq 0$. We differentiate to get

$$\begin{aligned} \theta &= \frac{w''w - (w')^2}{w^2 + (w')^2} = \frac{-((n-1)/s)ww' - wg(w) - (w')^2}{w^2 + (w')^2} \\ &\leq -\frac{n-1}{2s} \sin 2\theta - \cos^2 \theta - \sin^2 \theta = -1 - \frac{n-1}{2s} \sin 2\theta. \end{aligned}$$

If $S > S_0 + (\pi + n - 1)$, then for $n - 1 < s - S_0 < \pi + n - 1$ we have

$$\frac{s}{n-1} \geq \frac{s-S_0}{n-1} \geq 1 \geq -\sin 2\theta,$$

and so

$$1 \geq -(\sin 2\theta) \frac{n-1}{s}; \quad \text{i.e., } \theta \leq -\frac{1}{2}.$$

Thus

$$\theta((\pi + n - 1) + S_0) - \theta((n - 1) + S_0) = \int_{S_0+n-1}^{S_0+\pi+n-1} \theta \, ds \leq -\frac{1}{2}\pi.$$

This contradicts $0 > \theta(s) > -\pi/2$. Therefore $S - S_0 \leq \pi + n - 1$. So $T - R_0 \leq a^{-1}(S - S_0)$ and (23) follows with constant $C = (\pi + n - 1)d_1^{-1/2}$. ■

COROLLARY 14. *If $k < 1$, then $T < Cp^{(1-k)/2}$.*

Proof. This follows from Proposition 8 and Lemma 13. ■

LEMMA 15. *For $p \geq 2B$, $u(R_0) = p$, we define $\tau(p)$ by $u(\tau(p)) = p/2$. Then*

$$\tau(p) \geq (Cp^{1-k} + R_0^2)^{1/2}. \quad (24)$$

Proof. The existence of $\tau(p)$ follows from Lemma 10 for $B = p/2$. As in Lemma 10 (now $M(p) = d_2 p^k$ if $k \geq 0$ and $M(p) = d_2 2^{-k} p^k$ if $k < 0$)

$$\frac{p}{2} \leq \frac{Cp^k}{n} \left\{ \frac{\tau(p)^2}{2} - \frac{R_0^2}{2} - \frac{R_0^2}{n-2} \left(\frac{1}{R_0^{n-2}} - \frac{1}{\tau(p)^{n-2}} \right) \right\} \leq \frac{Cp^k}{2n} (\tau(p)^2 - R_0^2),$$

and (24) follows. ■

LEMMA 16. *If f satisfies (9) and (10), then for large p 's (i.e., $B < p/2$)*

$$-qT^{n-1} \geq Cp^k(\tau(p)^n - R_0^n), \quad (25)$$

where C is a constant.

Proof. We integrate $-(r^{n-1}v) = r^{n-1}f(u(r))$ from R_0 to T ; this gives

$$\begin{aligned} -qT^{n-1} &= -\int_{R_0}^T (r^{n-1}v) dr = \int_{R_0}^T r^{n-1}f(u(r)) dr \\ &\geq \int_{R_0}^{\tau(p)} r^{n-1}f(u(r)) dr \geq d_1 \int_{R_0}^{\tau(p)} u^k r^{n-1} dr \\ &\geq Cp^k \int_{R_0}^{\tau(p)} r^{n-1} dr = Cp^k(\tau(p)^n - R_0^n). \quad \blacksquare \end{aligned}$$

THEOREM 17. *Suppose f satisfies (9) and (10) with $k \geq 0$. Then there are solutions of (3)–(4) with $R = R(\alpha)$ (or $R(p)$).*

Proof. We write $-qT = -qT^{n-1}/T^{n-2}$ and use (25) and (24),

$$-qT \geq C \frac{1}{T^{n-2}} [(Cp^{1-k} + R_0^2)^{n/2} - R_0^n] p^k.$$

We claim that for $k \geq 1$ (see below)

$$(Cp^{1-k} + R_0^2)^{n/2} - R_0^n \geq \text{const } R_0^{n-2} p^{1-k}.$$

It follows from this that

$$-qT \geq Cp^k p^{1-k} \left(\frac{R_0}{T}\right)^{n-2} = \Psi(k, p). \tag{26}$$

Since $R < Cp^{(1-k)/2}$, if $k < 1$ we have a better estimate

$$(Cp^{1-k} + R_0^2)^{n/2} - R_0^n \geq (Cp^{1-k})^{n/2} \geq Cp^{(1-k)n/2}.$$

Thus

$$-qT \geq Cp^{k+(1-k)n/2} T^{-(n-2)} = \Psi(k, p).$$

We shall show that $\Psi(p) \rightarrow +\infty$ when $p \rightarrow +\infty$ (i.e., $\alpha \rightarrow +\infty$). From (23), Corollary 7, Lemma 13, and Corollary 14,

$$\Psi(p) > Cp.$$

The existence of solutions follows from Theorem 11. \blacksquare

Proof of the claim. If $f(x, b) = (x + b)^{n/2} - b^{n/2}$ with $x > 0$ and $b > 0$, then

$$f(a, b) = f(0, b) + af_x(\theta, b) = a(\theta + b)^{(n-2)/2} n/2 \geq ab^{(n-2)/2} n/2.$$

Now the claim is proved with $a = Cp^{1-k}$ and $b = R_0^2$. \blacksquare

THEOREM 18. *Suppose f satisfies (9) and (10) with $0 > k > -1$. Then there are solutions of (3), (4) with $R = R(\alpha)$ (or $R(p)$).*

Proof. Let B, q, T be as above and $\mu < 0$ such that $\mu \leq f(u)$ for $0 \leq u \leq B$. Set $\delta = (n-1)q/\mu T$.

From the proof of Theorem 12 we see that in order to have existence of solutions we need to prove that

$$-\frac{qT}{n-1} \left(1 - \frac{\ln(1+\delta)}{\delta} \right) \rightarrow +\infty \quad \text{as } \alpha \rightarrow +\infty.$$

From Lemma 11 we have $\delta \geq Cp^k$, and then

$$\frac{\ln(1+\delta)}{\delta} \leq \frac{\ln(1+Cp^k)}{Cp^k} = 1 - Cp^k + O(Cp^{2k}).$$

Thus

$$-\frac{qT}{n-1} \left(1 - \frac{\ln(1+\delta)}{\delta} \right) \geq -\frac{qT}{n-1} [Cp^k + O(Cp^{2k})].$$

As in Theorem 17, $-qT \geq Cp$. Therefore, if $k > -1$,

$$-\frac{qT}{n-1} \left(1 - \frac{\ln(1+\delta)}{\delta} \right) \geq Cp^{1+k} \rightarrow +\infty \quad \text{as } \alpha \rightarrow +\infty. \quad \blacksquare$$

With this result we have completed the proof of Theorem A.

Remarks. (i) The above argument extends the existence results of Smoller and Wasserman [5] for the n -ball to $-1 < k < n/(n-2)$.

(ii) The cut-off $k > -1$ seems to be optimal for the methods we use. For $k < -1$, $F(s) = \int_0^s f(t) dt$ is a bounded function, and, since we give no condition at 0, if $f(u)$ is negative and large for u near 0 we will have $F(s) \leq 0$ for all $s > 0$. Therefore there is no positive solution of (3), (4).

In order to prove Theorem B we need to observe the behavior of $R(\alpha)$ both when α is large and when α is "small" (i.e., α bounded). We will do this in the next series of lemmas.

PROPOSITION 19. *Let f satisfy (9), (10) with $k \geq 1$. Given α^* large enough, there is a constant $C > 0$ such that $R(\alpha) \leq C$ for $\alpha > \alpha^*$.*

(In Lemma 27 we will give a better estimate when $k > 1$.) To prove this proposition we need

LEMMA 20. $v(R(\alpha)) \rightarrow +\infty$ when $\alpha \rightarrow +\infty$.

Proof. Since $H(r) \leq H(T)$ for $r \geq T$. We have

$$v^2(r) \leq q^2 + 2[F(B) - F(u(r))] \leq q^2 + \tilde{C},$$

where

$$\tilde{C} = 2 \max_{0 \leq s \leq B} [F(B) - F(s)] = 2[F(B) - F(A)].$$

Therefore

$$|v(r)| \leq (q^2 + \tilde{C})^{1/2},$$

if $T(\alpha) \leq r \leq R(\alpha)$. Also,

$$\begin{aligned} & \frac{v^2(R(\alpha))}{2} - \frac{q^2}{2} - F(B) \\ &= H(R(\alpha)) - H(T) = - \int_T^{R(\alpha)} \frac{n-1}{r} v^2(r) dr \\ &\geq - \frac{n-1}{T} \int_T^{R(\alpha)} v^2(r) dr \geq \frac{n-1}{T} (q^2 + \tilde{C})^{1/2} \int_T^{R(\alpha)} v(r) dr \\ &= -B(n-1) \frac{(q + \tilde{C})^{1/2}}{T}. \end{aligned}$$

Thus

$$v^2(R(\alpha)) \geq q^2 - 2B(n-1) \frac{(q^2 + \tilde{C})^{1/2}}{T} + 2F(B). \tag{27}$$

and

$$v^2(R(\alpha)) \geq \left| \frac{q}{T} \right| \left\{ |qT| - 2B(n-1) \left(1 + \frac{\tilde{C}}{q^2} \right)^{1/2} \right\} + 2F(B).$$

By Lemma 11 and Theorem 17 (proof) we see that the r.h.s. goes to $+\infty$ with α . This proves the lemma. ■

Proof of Proposition 19. Choose α^* so that $R(\alpha)$ exists for $\alpha \geq \alpha^*$ and (from the last lemma)

$$\min_{\alpha \geq \alpha^*} \{v^2(R(\alpha)), v^2(R(\alpha)) - 2F(B)\} \geq v^2(R(\alpha)) - \tilde{C} > 0.$$

For $T \leq r \leq R(\alpha)$, we have

$$\frac{v^2(r)}{2} + F(u(r)) \leq \frac{q^2}{2} + F(B),$$

i.e., $|v(r)| \leq (q^2 + 2[F(B) - F(u(r))])^{1/2} \leq (q^2 + C)^{1/2}$ and

$$\frac{v^2(r)}{2} + F(u(r)) \geq \frac{v^2(R(\alpha))}{2};$$

i.e.,

$$|v(r)| \geq (v^2(R(\alpha)) - 2F(u(r)))^{1/2} \geq (v^2(R(\alpha)) - \tilde{C})^{1/2} > 0.$$

Then

$$H(R) - H(T) = - \int_T^R \frac{n-1}{r} v^2(r) dr \geq - \frac{n-1}{T} B(q^2 + \tilde{C})^{1/2}$$

and

$$H(R) - H(T) = - \int_T^R \frac{n-1}{r} v^2(r) dr \leq - \frac{n-1}{R(\alpha)} [v^2(R(\alpha)) - \tilde{C}](R - T).$$

Thus

$$\frac{B(q^2 + \tilde{C})^{1/2}}{T} \geq \frac{R(\alpha) - T}{R(\alpha)} (v^2(R(\alpha)) - \tilde{C})$$

and

$$\frac{R(\alpha) - T}{R(\alpha)} \leq \frac{C}{T} \frac{(q^2 + \tilde{C})^{1/2}}{v^2(R(\alpha)) - \tilde{C}}.$$

From (27), for large α , $v(R(\alpha))^2 - \tilde{C} \geq Cq^2$ and $q \rightarrow +\infty$; thus

$$\frac{R(\alpha) - T}{R(\alpha)} \leq C \frac{|q|}{Tq^2} = \frac{C}{|Tq|},$$

and therefore

$$R < T \left(\frac{|qT|}{|qT| - C} \right) \quad \text{when } |qT| \rightarrow +\infty. \quad (28)$$

(Recall that T is bounded follows from Corollary 14.) ■

LEMMA 21. *Let $\alpha^* > 0$ be given. If $A > 0$ and $f(0) < 0$ then there are constants C_1, C_2 such that if $R(\alpha)$ exists, we have $C_1 < R(\alpha) < C_2$ for $\alpha \leq \alpha^*$.*

Proof. For $\alpha < \alpha^*$ if $R(\alpha)$ exists, we will have $v(r) < (\alpha^2 - 2F(A))^{1/2}$ and $p(\alpha) > A$. Thus $R_0(\alpha) - \lambda > A(\alpha^2 - 2F(A))^{-(1/2)} > C_1$ for all $\alpha < \alpha^*$.

Also if there is a sequence $\{\alpha_m\} \subset (0, \alpha^*)$ such that $R(\alpha_m)$ converges to $+\infty$, then since $v(R(\alpha_m)) \in (-\alpha^*, 0)$ we can find a subsequence $\{\alpha_j\}$ which converges to $\alpha \leq \alpha^*$ and such that $(u(r, \alpha), v(r, \alpha)) \rightarrow (0, v^*)$ when $r \rightarrow +\infty$. This is impossible since the only rest point is $(A, 0)$.

Therefore $R(\alpha)$ is bounded from above when $\alpha \leq \alpha^*$. ■

LEMMA 22. *If $A > 0$ and $f(0) = 0$, then given any M we can find an α such that $R(\alpha)$ exists and $R(\alpha) > M$.*

Proof. We know that there is an α^* such that for any $\alpha > \alpha^*$, $R(\alpha) < +\infty$ exists. If there is a solution for any $\alpha > 0$ we can choose α so close to 0 as to have $\sup_{\lambda < r \leq M} |u(r, \alpha) - u(r, 0)| < A/2$ by continuity of u . Then $u(r, \alpha) < A/2$ for $\lambda < r \leq M$ and therefore $R(\alpha) > M$.

If there are α 's such that $R(\alpha)$ does not exist, let us call $\alpha_1 = \inf\{\alpha^* \mid R(\alpha) \text{ exists for all } \alpha > \alpha^*\} > 0$.

Let us suppose that $R(\alpha) < M$ for all $\alpha > \alpha_1$ and that there is an $\varepsilon > 0$ such that $-v(R(\alpha)) > \varepsilon$ for all $\alpha > \alpha_1$. Then we can find a sequence $\{\alpha_m\}$ which converges to α_1 and such that $R(\alpha) \rightarrow R(\alpha_1) < M$ and $v(R(\alpha)) \rightarrow v(R(\alpha_1)) < -\varepsilon < 0$. Therefore, by transversality, there is an $\alpha_2 < \alpha_1$ such that $R(\alpha)$ exists for $\alpha > \alpha_2$. This contradicts the definition of α_1 .

Thus $v(R(\alpha)) \rightarrow 0$ when $\alpha \rightarrow \alpha_1$ and we have $(u(r, \alpha_1), v(r, \alpha_2)) \rightarrow (0, 0)$ as $r \rightarrow R(\alpha_1)$. This is impossible since $(0, 0)$ is a rest point and can only be reached at infinite time.

Therefore for any M there is a α such that $R(\alpha) > M$. ■

Let us recall that $f_0 = \lim_{s \rightarrow 0} f(s)/s$.

LEMMA 23. *If $A = 0, f(0) = 0$ and $f_0 = 0$, then given any M there is an α such that $R(\alpha) - \lambda > M$.*

Proof. First, since $f(u) > 0$ for $u > 0$, we have that the domain of $R(\alpha)$ is $(0, +\infty)$.

Given $M > 0$, we can find u^* such that $f(u) < u/M$ for $0 < u < u^*$. Then if $\alpha^* = (2F(u^*))^{1/2}$ we will have $p(\alpha) < u^*$ when $\alpha < \alpha^*$. For this α we have, when $R_0 < r < R(\alpha)$,

$$-(r^{n-1}v)' = r^{n-1}f(u) \leq \frac{u}{M} r^{n-1} \leq \frac{P}{M} r^{n-1}.$$

Integrating from R_0 to r gives

$$-r^{n-1}v \leq \frac{p}{nM} (r^n - R_0^n) \leq \frac{p}{nM} r^n,$$

so that

$$-v \leq \frac{pr}{nM},$$

and integrating again, we find

$$p \leq \frac{p}{2nM} (R(\alpha)^2 - R_0^2).$$

Thus

$$2nM \leq R^2(\alpha) - R_0^2 \leq R^2(\alpha) - \lambda^2. \quad \blacksquare$$

LEMMA 24. *If $A = 0$ and $f_0 = +\infty$, then given any $\varepsilon > 0$ there is an α such that $R(\alpha) - \lambda < \varepsilon$.*

Proof. Given $\varepsilon > 0$ we can find a u^* such that $f(u) \geq u/\varepsilon^2$ for $u < u^*$. Let $\alpha^* = (2F(u^*))^{1/2}$. Then for $\alpha < \alpha^*$, we have $u(r, \alpha) < u^*$ when $\lambda < r < R(\alpha)$.

For $\lambda < r < R_0$ we have

$$\frac{v^2(r)}{2} \geq F(p) - F(u(r)) \geq \frac{(p^2 - u^2)}{2\varepsilon^2},$$

so

$$v(r) \geq \frac{(p^2 - u^2)^{1/2}}{\varepsilon}$$

and

$$\varepsilon \frac{v(r)}{(p^2 - u^2)^{1/2}} \geq 1.$$

Integrating the above from λ to R_0 we get $R_0 - \lambda \leq \varepsilon\pi/2$.

Also when $R_0 < r < R(\alpha)$ an argument similar to that in Lemma 13 gives $R(\alpha) - R_0 \leq (\pi + n - 1)\varepsilon$. \blacksquare

LEMMA 25. *If $A = 0$, $f(0) = 0$, and $0 < f_0(0) < +\infty$, then given $\alpha^* > 0$, there are constants C_1, C_2 such that $C_1 < R(\alpha) - \lambda < C_2$ for any $\alpha \leq \alpha^*$.*

Proof. Let u^* be such that $2F(u^*) = \alpha^2$. Then $u(r, \alpha) < u^*$ for $\lambda < r < R(\alpha)$ when $\alpha < \alpha^*$.

We can find $M > m > 0$ such that $m^2u \leq f(u) \leq Mu$ for $0 < u \leq u^*$. As in Lemma 24,

$$R(\alpha) - \lambda \leq \frac{C}{m},$$

and as in Lemma 23,

$$R^2(\alpha) - \lambda^2 \geq \frac{2n}{M}.$$

Then

$$R(\alpha) - \lambda \geq \frac{n}{M} \left(\lambda + \frac{C}{m} \right)^{-1}. \quad \blacksquare$$

Now we consider large α 's.

LEMMA 26. *If $k \leq 1$, then there is a constant $C > 0$ such that $R(\alpha) - \lambda \geq Cp^{(1-k)/2}$ for large values of α .*

Proof. For $R_0 < r < T(\alpha)$,

$$\frac{v^2(r)}{2} + F(u(r)) < F(p) = Cp^{k+1}.$$

Then, since $F(u(T)) > 0$, we have $-v(r) \leq Cp^{(k+1)/2}$; and so

$$\frac{-v(r)}{Cp^{(k+1)/2}} \leq 1.$$

Integrating from R_0 to $T(\alpha)$ we have

$$T(\alpha) - R_0 \geq Cp^{-(k+1)/2}(p - B) \geq Cp^{(1-k)/2},$$

so that

$$R(\alpha) - \lambda > T(\alpha) - R_0 \geq Cp^{(1-k)/2}. \quad \blacksquare$$

LEMMA 27. *If $k > 1$, given any $\varepsilon > 0$, there is an α such that $R(\alpha)$ exists and $R(\alpha) - \lambda < \varepsilon$.*

Proof. Let thus set p^* such that $(\pi + n - 1) d_1^{-1/2}(p^*)^{(1-k)/2} < \varepsilon/4$. Then, from Lemma 13, $T - R_0 < \varepsilon/4$ for $p > p^*$. From (28), $R(\alpha) - T(\alpha) \rightarrow 0$ when

$\alpha \rightarrow +\infty$. Thus, there is an α^* such that $R(\alpha) - T(\alpha) < \varepsilon/4$ for $\alpha > \alpha^*$, and therefore there is a large α such that

$$R(\alpha) - R_0(\alpha) = R(\alpha) - T(\alpha) + T(\alpha) - R_0(\alpha) < \frac{\varepsilon}{2}.$$

Now by Theorem 2 in [2] we know that

$$R_0(\alpha) < \frac{R(\alpha) + \lambda}{2},$$

so

$$R(\alpha) - \lambda < 2\{R(\alpha) - R_0(\alpha)\} < \varepsilon. \quad \blacksquare$$

LEMMA 28. *If $k = 1$, then given α^* large, there are constants C_1, C_2 such that $C_1 < R(\alpha) - \lambda < C_2$ for any $\alpha > \alpha^*$.*

Proof. From Lemma 26 we get the existence of C_1 . The existence of C_2 follows from Proposition 19. \blacksquare

Proof of Theorem B. (iii) Let $\mathcal{G} = \{\alpha \geq 0 \mid \text{there is an } R(\alpha) < +\infty\}$. It is clear that \mathcal{G} is closed and as in Proposition 19, there is an α^* such that $[\alpha^*, +\infty) \subseteq \mathcal{G}$.

Moreover, the set $\{R(\alpha) \mid \alpha \geq \alpha^*\}$ is bounded. Also, since for $\alpha = 0$, $H(\lambda, \alpha) = H(\lambda, 0) = 0$; then $H(r) < 0$ for $r > \lambda$. Thus $0 \notin \mathcal{G}$, and so there is an $\alpha_1 > 0$ such that $\mathcal{G} \subseteq [\alpha_1, +\infty)$. For $\alpha \in [\alpha_1, \alpha^*] \cap \mathcal{G}$, let us suppose that the set $\{R(\alpha)\}_\alpha$ is not bounded; i.e., we have a sequence $\{\alpha_m\} \subset [\alpha_1, \alpha^*] \cap \mathcal{G}$ such that $R(\alpha_m)$ goes to $+\infty$ with m . Since $|v(R(\alpha_m))| \leq \alpha_m$ we can find a subsequence $\{\alpha_j\} \subseteq \{\alpha_m\}$ which converges to say $\bar{\alpha}$ and $v(R(\alpha_j))$ converges to some \bar{v} . Thus the trajectory $((u(r, \bar{\alpha}), v(r, \bar{\alpha}))$ goes to $(0, \bar{v})$ when r goes to $+\infty$. This is impossible since there are no rest points on the line $u = 0$.

Therefore $R(\alpha)$ is bounded when $\alpha \in \mathcal{G}$ and this gives the existence of C_2 . Now let us call $C_1 = \max\{R(\alpha) - \lambda \mid \alpha \geq \alpha^*\}$. From Lemma 27 we see that $R([\alpha^*, +\infty)) = (0, C_1]$.

The rest of the theorem is proved similarly. Note that when $A = 0$, $\mathcal{G} = (0, +\infty)$. \blacksquare

4. FURTHER RESULTS

PROPOSITION 29. *If, in addition to (9) and (10), f satisfies*

$$F(A) < 0.$$

Then, given an $\alpha^ > 0$, there is a constant $\delta = \delta(f, \alpha^*)$ such that if*

$(u(\cdot, \alpha), v(\cdot, \alpha))$ is a trajectory of (6), (7) with $H(R_0) \geq 0$ and $\alpha < \alpha^*$, then $t_1 - \lambda \leq \delta$. (Note that this will always hold for solutions of (3), (4).)

Proof. Let α^* be given. We define $Q > A$ such that $F(Q) = 0$.

We consider an $\alpha < \alpha^*$ such that $H(R_0(\alpha)) \geq 0$. Since $H'(r) < 0$ for $\lambda < r < R_0$, we see that

$$H(r) \leq \frac{(\alpha)^2}{2} \leq \frac{(\alpha^*)^2}{2} \quad \text{for } \lambda < r < R_0,$$

and in particular

$$v^2(r) \leq \alpha^2 - 2F(u(r)) \leq C(\alpha^*).$$

Let us define T_0 by $u(T_0) = Q$. Then for $\lambda < r < T_0$ we have

$$v^2(r) + 2F(u(r)) \geq 2F(Q) + v^2(T_0) \geq 0$$

and

$$\frac{v(r)}{(-2F(u(r)))^{1/2}} \geq 1.$$

Thus, integrating from t_1 to T_0 we obtain

$$T_0 - t_1 \leq \int_A^Q \frac{ds}{(-2F(s))^{1/2}} < C,$$

since $-F(s) \cong f(Q)(Q - s)$, for s near Q , and $f(Q) > 0$.

Also

$$\begin{aligned} H(t_1) - H(T_0) &= (n-1) \int_{t_1}^{T_0} \frac{v^2}{r} dr \geq \frac{n-1}{T_0} \int_{t_1}^{T_0} v^2 dr \\ &\geq \frac{n-1}{T_0} \int_{t_1}^{T_0} (-2F(u)) dr \geq \frac{n-1}{T_0} \int_{T_1}^{T_0} \frac{-2F(u) v(r)}{v(r)} dr \\ &\geq \frac{n-1}{T_0 C(\alpha^*)} \int_A^Q (-2F(s)) ds = \frac{C}{T_0}. \end{aligned}$$

And for $\lambda < r < t_1$,

$$v^2(r) \geq 2H(t_1) - 2F(u(r)) \geq \frac{C}{T_0} - 2F(u(r)) \geq \frac{C}{T_0}.$$

Hence

$$Cv(r) T_0^{1/2} \geq 1;$$

and integrating from λ to t_1 ,

$$t_1 - \lambda \leq ACT_0^{1/2} \leq C(t_1 + C)^{1/2}.$$

Therefore for $\alpha \leq \alpha^*$, t_1 remains bounded. ■

THEOREM 30. *If $f(0) = 0$ and $F(A) < 0$, then there is a radial positive solution for the exterior problem:*

$$\begin{aligned} \Delta u(x) + f(u(x)) &= 0, & x \in \Omega &= \{x \in \mathbb{R}^n \mid |x| > \lambda > 0\} \\ u(x) &= 0, & x \in \partial\Omega \end{aligned}$$

and

$$u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

Proof. From Theorem A we know the existence of positive solutions for (3), (4).

By the last proposition, if there is a solution to (3), (4) then $t_1 - \lambda$ is bounded.

Also by continuity of the flow if α is very small, then t_1 is very large; i.e., for any $\varepsilon_0 > 0$ there is an ε such that if $\alpha < \varepsilon$, then

$$|u(1/\varepsilon, 0) - u(1/\varepsilon, \alpha)| \leq \varepsilon_0.$$

Therefore, we conclude the existence of a value \mathbf{a} such that if $\alpha \leq \mathbf{a}$ then $\alpha \notin \mathcal{G}$ (as defined in proof of Theorem B). Let us define $\alpha_1 = \inf\{\alpha \mid (\alpha, +\infty) \subset \mathcal{G}\}$.

Obviously, $\alpha_1 \notin \mathcal{G}$ and the only possibility is $(u(r, \alpha_1), v(r, \alpha_1)) \rightarrow (0, 0)$ as $r \rightarrow +\infty$ with $u(r, \alpha_1) > 0$ when $\lambda < r < +\infty$; i.e., $u(r, \alpha_1)$ is a solution to the exterior problem. ■

Note. Actually, we can find at least as many solutions as the number of connected components of \mathcal{G} .

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