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On Reflexivity of Operators

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Let $\mathscr{H}$ be a separable, complex Hilbert space of (finite or infinite) dimension larger than one, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, let $\mathscr{A}_T$ (resp. $\mathscr{W}_T$) denote the smallest subalgebra of $\mathscr{L}(\mathscr{H})$ that contains T and $1_{\mathscr{H}}$ and is closed in the ultraweak (resp. weak operator (WOT)) topology. The algebra $\mathscr{A}_T$ is called the *dual algebra generated by T*. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be *reflexive* if $\operatorname{Alg}\operatorname{Lat}(\mathscr{W}_T) = \mathscr{W}_T$. For several years, the question of which operators are reflexive has been studied (cf. [2, 4, 9, 11, 12]), and, of course, one is always searching for conditions on an operator T that will be equivalent to reflexivity. Recently, in [9] some progress was made in this direction by using properties $(\mathbf{B}_{m,n})$ and $(\mathbf{B}_{\tilde{m},n})$ (to be defined below) associated with the dual algebra $\mathscr{A}_T$. In particular, the main theorem of [9] states that if $T \in \mathscr{L}(\mathscr{H})$ and $\mathscr{A}_T$ is WOT-closed and has property $(\mathbf{B}_{\tilde{2},3})$ then T is reflexive. Easy examples show that there exist operators T with property $(\mathbf{B}_{1,2})$ that are not reflexive, so, as noted in [8], the question remains open whether the conclusion still holds if $(\mathbf{B}_{\tilde{2},3})$ is replaced by $(\mathbf{B}_{2,2})$ or $(\mathbf{B}_{2,3})$. Related to this problem is also the question whether a dual algebra $\mathscr{A}_T$ with property $(\mathbf{B}_{m,n})$ necessarily has property $(\mathbf{B}_{\tilde{m},n})$.

On the other hand, necessary and sufficient conditions that an algebraic operator be reflexive are known (cf. [11, 12]), so a natural test question arises: Given an algebraic operator T, which of the properties $(\mathbf{B}_{m,n})$ does $\mathscr{A}_T$ have?

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In this note we completely answer this question, and thereby discover that at least for algebraic operators, $\mathscr{A}_T$ having property $(\mathbf{B}_{2,2})$ is sufficient to imply reflexivity (Corollary 6). We also give an example to show that having property $(\mathbf{B}_{1,2})$ does not, in general, imply the possession of the property $(\tilde{\mathbf{B}}_{1,2})$, thereby settling the second question posed above.

Recall that the ultraweak topology on $\mathscr{L}(\mathscr{H})$ is the weak* topology inherited by $\mathscr{L}(\mathscr{H})$ as the dual space of the Banach space $\mathscr{C}_1(\mathscr{H})$ of trace-less operators on $\mathscr{H}$, and the bilinear functional associated with this duality is given by

$$\langle A, L\rangle = \operatorname{tr}(AL), \qquad A \in \mathscr{L}(\mathscr{H}), L \in \mathscr{C}_1(\mathscr{H}).$$

If $T \in \mathscr{L}(\mathscr{H})$ then $\mathscr{A}_T$ is the dual space of the quotient space $Q_T = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}_T$, where ${}^{\perp}\mathscr{A}_T$ is the preannihilator of $\mathscr{A}_T$ in $\mathscr{C}_1(\mathscr{H})$, and the duality between $\mathscr{A}_T$ and Q_T is given by the pairing

$$\langle A, [L]\rangle = \operatorname{tr}(AL), \qquad A \in \mathscr{A}_T, \qquad [L] \in Q_T,$$

where $[L]$ is the image in Q_T of the operator L in $\mathscr{C}_1(\mathscr{H})$ (cf. [10] for more details). If x and y are vectors in $\mathscr{H}$, then we write $x \otimes y$ for the rank-one operator in $\mathscr{C}_1(\mathscr{H})$ defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathscr{H}$. We are now ready to introduce the properties $(\mathbf{B}_{m,n})$ and $(\tilde{\mathbf{B}}_{m,n})$ which are defined and used in [9, 8]. We remark that these properties are natural analogs of some properties $(\mathbf{A}_n)$ which recently have been studied extensively (cf. [2, 5, 6, 8]).

DEFINITION 1. Let $T \in \mathscr{L}(\mathscr{H})$ and let p and q be cardinal numbers satisfying $1 \leqslant p, q \leqslant \aleph_0$. We say that $\mathscr{A}_T$ has property $(\mathbf{B}_{p,q})$ if for every $p \times q$ system

$$\{[L_{ij}] \in Q_T : 0 \leqslant i < p, 0 \leqslant j < q\},$$

where the L_{ij}'s are finite rank operators, there exist sequences

$$\{x_i : 0 \leqslant i < p\} \qquad \text{and} \qquad \{y_j : 0 \leqslant j < q\}$$

of vectors in $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leqslant i < p, 0 \leqslant j < q.$$

Furthermore, if p and q are positive integers, we say that $\mathscr{A}_T$ has property $(\tilde{\mathbf{B}}_{p,q})$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever

$$\{[L_{ij}] \in Q_T : 0 \leqslant i < p, 0 \leqslant j < q\}$$

is a system as above and

$$\{[x_i' \otimes y_j']: 0 \leqslant i < p, 0 \leqslant j < q\}$$

is a system satisfying the inequalities

$$\| [L_{ij}] - [x_i' \otimes y_j'] \| < \delta, \qquad 0 \leqslant i < p, 0 \leqslant j < q,$$

there exist sequences

$$\{x_i: 0 \leqslant i < p\} \qquad \text{and} \qquad \{y_j: 0 \leqslant j < q\}$$

in $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leqslant i < p, 0 \leqslant j < q,$$

and

$$\| x_i' - x_i \| < \varepsilon, \qquad \| y_j' - y_j \| < \varepsilon, \qquad 0 \leqslant i < p, 0 \leqslant j < q.$$

We will often make use of the fact (cf. [8, Proposition 2.09]) that if T_1, $T_2 \in \mathscr{L}(\mathscr{H})$ and T_1 is similar to T_2, then $\mathscr{A}_{T_1}$ has property $(\mathbf{B}_{p,q})$ if and only if $\mathscr{A}_{T_2}$ has property $(\mathbf{B}_{p,q})$. If $T \in \mathscr{L}(\mathscr{H})$ and n is a positive integer, then $\mathscr{H}^{(n)}$ denotes the direct sum of n copies of $\mathscr{H}$, and $T^{(n)}$ denotes a direct sum of n copies of T acting on $\mathscr{H}^{(n)}$. If $T \in \mathscr{L}(\mathscr{H})$ then $x \in \mathscr{H}$ is a *separating vector* for $\mathscr{A}_T$ if $A \in \mathscr{A}_T$ and $Ax = 0$ imply that $A = 0$.

LEMMA 2. *Assume that $T \in \mathscr{L}(\mathscr{H})$ and that $x \in \mathscr{H}$ is a separating vector for $\mathscr{A}_T$ such that $\mathscr{A}_T x$ is a closed subspace of $\mathscr{H}$. Then $\mathscr{A}_{T^{(n)}}$ has property $(\mathbf{B}_{n,\aleph_0})$ for every positive integer n.*

Proof. Since x is separating for $\mathscr{A}_T$ and $\mathscr{A}_T x$ is closed, from [12, Theorem 5.1] it follows that every element $[L] \in Q_T$ can be written as $[L] = [x \otimes y]$ for some $y \in \mathscr{H}$. The map Φ: $\mathscr{A}_T \to \mathscr{A}_{T^{(n)}}$ defined by $\Phi(A) = A^{(n)}$ induces an isometric isomorphism ϕ: $Q_{T^{(n)}} \to Q_T$ with range Q_T, and such that $\phi^* = \Phi$ (cf. [8, Proposition 2.5]). Given a system $\{[\tilde{L}_{ij}]: 1 \leqslant i \leqslant n, 1 \leqslant j < \aleph_0\}$ in $Q_{T^{(n)}}$, we define $[L_{ij}] = \phi(\tilde{L}_{ij}]) \in Q_T$. Let $y_j^{(i)} \in \mathscr{H}$ be such that

$$[L_{ij}] = [x \otimes y_j^{(i)}], \qquad 1 \leqslant i \leqslant n, 1 \leqslant j < \aleph_0.$$

Let $\tilde{x}_i = 0 \oplus \cdots \oplus x \oplus \cdots \oplus 0$ be the vector in $\mathscr{H}^{(n)}$ such that x lies in the ith slot, and let $\tilde{y}_j = y_j^{(1)} \oplus \cdots \oplus y_j^{(n)}$, $1 \leqslant j < \aleph_0$. Since

$$\phi([\tilde{x}_i \oplus \tilde{y}_j]) = [x \oplus y_j^{(i)}] = \phi([\tilde{L}_{ij}]),$$

we conclude that

$$[\tilde{L}_{ij}] = [\tilde{x}_i \otimes \tilde{y}_j], \qquad 1 \leqslant i \leqslant n, 1 \leqslant j < \aleph_0. \quad \blacksquare$$

LEMMA 3. *Let $T \in \mathscr{L}(\mathscr{H})$ be a nilpotent operator of order n. Let m be a positive integer, and assume that rank $T^{n-1} \geqslant m$. Then there exist an invertible operator $S \in \mathscr{L}(\mathscr{H})$ and a subspace $\mathscr{M} \subset \mathscr{H}$ such that $\mathscr{M}$ is invariant under $S^{-1}TS$ and $S^{-1}TS \mid \mathscr{M} = J_n \oplus \cdots \oplus J_n$ (m copies of J_n), where J_n is an operator acting on a Hilbert space of dimension n whose matrix with respect to some orthonormal basis for the space is a nilpotent Jordan block of size n.*

Proof. We begin by introducting the Jordan block operators. For a Hilbert space $\mathscr{K}$ and $k \geqslant 2$ we define on $\mathscr{K}^{(k)}$ the *Jordan block* operator $J_k[\mathscr{K}]$ of order k by

$$J_k[\mathscr{K}](x_1 \oplus \cdots \oplus x_k) = x_2 \oplus \cdots \oplus x_k \oplus 0, \qquad x_1 \oplus \cdots \oplus x_k \in \mathscr{K}^{(k)}.$$

It is easy to see that $J_k[\mathscr{K}]$ is unitarily equivalent to a direct sum of s copies of J_k, where s is the dimension of $\mathscr{K}$. By definition, the zero operator on $\mathscr{K}$ is a Jordan operator of order one. It is known (cf. [1, 14]) that the given nilpotent operator T is quasisimilar to a Jordan operator $J = J_{n_1}[\mathscr{K}_1] \oplus \cdots \oplus J_{n_p}[\mathscr{K}_p]$ acting on $\mathscr{K} = \mathscr{K}_1^{(n_1)} \oplus \cdots \oplus \mathscr{K}_p^{(n_p)}$, where $n_1,..., n_p$ are distinct positive integers, and $\mathscr{K}_1,..., \mathscr{K}_p$ are Hilbert spaces. In particular, there exists a bounded linear transformation X: $\mathscr{K} \to \mathscr{H}$ such that $TX = XJ$ and $\ker X = \ker X^* = (0)$. Since $T^jX = XJ^j$ for any nonnegative integer j, J is a nilpotent operator of order n. Hence $n = \max\{n_i: 1 \leqslant i \leqslant p\}$, and by reordering the Jordan blocks in J we may assume that $n = n_1$. Then

$$J^{n-1} = (J_n[\mathscr{K}_1])^{n-1} \oplus 0 \oplus \cdots \oplus 0$$

and

$$(J_n[\mathscr{K}_1])^{n-1}(\mathscr{K}_1^{(n)}) = \mathscr{K}_1 \oplus \cdots \oplus (0).$$

Therefore

$$\begin{aligned} \dim \mathscr{K}_1 &= \text{rank } J^{n-1} = \text{rank } XJ^{n-1} \\ &= \text{rank } T^{n-1}X = \text{rank } T^{n-1} \geqslant m, \end{aligned}$$

and consequently there exists a subspace $\mathscr{M} \subset \mathscr{K}_1^{(n)}$, which is invariant for $J_n[\mathscr{K}_1]$ and satisfies

$$J_n[\mathscr{K}_1] \mid \mathscr{M} = J_n \oplus \cdots \oplus J_n \qquad (m \text{ copies of } J_n).$$

Since $\mathscr{M}$ has finite dimension, there exists a bounded invertible linear transformation S: $\mathscr{K} \to \mathscr{H}$ such that $Sx = Xx$ for all $x \in \mathscr{M}$. Since $\mathscr{M}$ is invariant for J, for $x \in \mathscr{M}$ we have $TSx = TXx = XJx = SJx = SJ_n^{(m)}x$. Hence $S^{-1}TS \mid \mathscr{M} = J_n^{(m)}$. ∎

THEOREM 4. *Let $T \in \mathscr{L}(\mathscr{H})$ be an algebraic operator with minimal polynomial*

$$m_T(z) = (z-\lambda_1)^{n_1}(z-\lambda_2)^{n_2} \cdots (z-\lambda_k)^{n_k},$$

and set $\mathscr{H}_i = \ker(T-\lambda_i)^{n_i}$, $1 \leqslant i \leqslant k$. If m is any positive integer, then the following are equivalent:

(1) $\operatorname{rank}[(T-\lambda_i)^{n_i-1} \mid \mathscr{H}_i] \geqslant m$, $1 \leqslant i \leqslant k$,

(2) $\mathscr{A}_T$ has property $(\mathbf{B}_{m,\aleph_0})$,

(3) $\mathscr{A}_T$ has property $(\mathbf{B}_{m,m})$.

Proof. From the Riesz functional calculus it follows that $(\bigvee_{i \neq j} \mathscr{H}_i) \cap \mathscr{H}_j = (0)$ for $1 \leqslant j \leqslant k$ and $\mathscr{H}_1 \dotplus \cdots \dotplus \mathscr{H}_k = \mathscr{H}$. Furthermore, if E_i is the projection on $\mathscr{H}_i$ along $\sum_{j \neq i} \mathscr{H}_j$, then $E_i \in \mathscr{A}_T$.

Now we begin the proof that (1) implies (2). Replacing T by an operator similar to it, we may assume that the subspaces $\mathscr{H}_1, \ldots, \mathscr{H}_k$ are pairwise orthogonal. Let $T_i = T \mid \mathscr{H}_i$. Then $T_i - \lambda_i$ is nilpotent of order n_i, and by assumption $\operatorname{rank}(T_i - \lambda_i)^{n_i - 1} \geqslant m$. Thus by Lemma 3, there exist an invertible operator $S_i \in \mathscr{L}(\mathscr{H}_i)$ and a subspace $\mathscr{M}_i \subset \mathscr{H}_i$ such that $\mathscr{M}_i$ is invariant under $S_i^{-1}(T_i - \lambda_i)S_i$ and

$$S_i^{-1}(T_i - \lambda_i)S_i \mid \mathscr{M}_i = J_{n_i} \oplus \cdots \oplus J_{n_i} \qquad (m \text{ copies of } J_{n_i}).$$

Let $S = S_1 \oplus \cdots \oplus S_k \in \mathscr{L}(\mathscr{H})$ and $\mathscr{M} = \mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_k$. Then $S^{-1}TS \mid \mathscr{M}_i = J_{n_i}^{(m)} + \lambda_i$. Therefore, $S^{-1}TS \mid \mathscr{M} = A^{(m)}$, where $A = (J_{n_1} + \lambda_1) \oplus \cdots \oplus (J_{n_k} + \lambda_k) \in \mathscr{L}(\mathscr{M}_0)$ and $\mathscr{M}_0 \subset \mathscr{M}$. Clearly the minimal polynomial of A is equal to m_T, and its degree $N = n_1 + \cdots + n_k$ is equal to the dimension of $\mathscr{M}_0$. Therefore, A has a cyclic vector, which is also necessarily separating for $\mathscr{A}_A$. Thus, from Lemma 2, we conclude that $\mathscr{A}_{A^{(m)}}$ has property $(\mathbf{B}_{m,\aleph_0})$. Let $T_0 = S^{-1}TS$ and let $\{[L_{ij}]: 1 \leqslant i \leqslant m, 1 \leqslant j < \aleph_0\}$ be a system in Q_{T_0}. Since $\mathscr{A}_{A^{(m)}}$ and $\mathscr{A}_{T_0}$ have dimension N, there exist $[L'_{ij}] \in Q_{A^{(m)}}$ such that

$$\langle (A^m)^{(s)}, [L'_{ij}] \rangle = \langle T_0^s, [L_{ij}] \rangle, \qquad 1 \leqslant s \leqslant N, 1 \leqslant i \leqslant m, 1 \leqslant j < \aleph_0.$$

Since $\mathscr{A}_{A^{(m)}}$ has property $(\mathbf{B}_{m,\aleph_0})$, there exist sequences $\{x'_i\}_{i=1}^m$, $\{y'_j\}_{j=1}^\infty$ from $\mathscr{M}_0^{(m)}$ satisfying $[L'_{ij}] = [x'_i \otimes y'_j]$, $1 \leqslant i \leqslant m$, $1 \leqslant j < \aleph_0$. Now the two sequences in $\mathscr{H}$ defined by $x_i = x'_i \oplus 0$, $y_j = y'_j \oplus 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j < \aleph_0$, will satisfy $[L_{ij}] = [x_i \otimes y_j]$, $1 \leqslant i \leqslant m$, $1 \leqslant j < \aleph_0$. We have proved that $T_0 = S^{-1}TS$ has property $(\mathbf{B}_{m,\aleph_0})$, and hence the same is true for T.

For the proof that (3) implies (1) we recall from the first paragraph that $E_i \in \mathscr{A}_T$, $1 \leqslant i \leqslant k$. Let p be a fixed positive integer such that $1 \leqslant p \leqslant k$. We have $(T - \lambda_p)^{n_p - 1} E_p \in \mathscr{A}_T$. Let $[L] \in Q_T$ such that $\langle (T-\lambda_p)^{n_p - 1} E_p, [L] \rangle = 1$. We define a system $\{[L_{ij}]: 1 \leqslant i, j \leqslant m\}$ in Q_T

by setting $[L_{ii}]=[L]$, $1\leqslant i\leqslant m$, and $[L_{ij}]=0$ for $i\neq j$. Since $\mathscr{A}_T$ has property $(\mathbf{B}_{m,m})$, there exist sequences $\{x_i\}_{i=1}^m$, $\{y_j\}_{j=1}^m$ in $\mathscr{H}$ satisfying $[L_{ij}]=[x_i\otimes y_j]$, $1\leqslant i, j\leqslant m$. Let $u_i=(T-\lambda_p)^{n_p-1}E_p x_i$, $1\leqslant i\leqslant m$, and assume that $\sum_{i=1}^m \alpha_i u_i=0$ for some complex numbers $\alpha_1,..., \alpha_m$. Then for $1\leqslant j\leqslant m$,

$$0=\left(\sum_{i=1}^m \alpha_i u_i, y_j\right)=\alpha_j\langle (T-\lambda_p)^{n_p-1}E_p, [L]\rangle=\alpha_j.$$

Hence the vectors $u_1,..., u_m$ are linearly independent, and since they belong to the range of $(T-\lambda_p)^{n_p-1}E_p$, rank $[(T-\lambda_p)^{n_p-1} \mid \mathscr{H}_p]\geqslant m$, as was required. ∎

Corollary 5. *If in Theorem* 4 *we further assume that the Hilbert space* $\mathscr{H}$ *has finite dimension, then* (2) *and* (3) *are equivalent to the following statement*:

(1′) *In the Jordan canonical form of* T, *corresponding to each eigenvalue* λ_i, $1\leqslant i\leqslant k$, *there are at least* m *Jordan blocks of size* n_i.

Proof. By consideration of the Jordan canonical form of T, it is easy to see that $\operatorname{rank}[(T-\lambda_i)^{n_i-1} \mid \mathscr{H}_i]$ is equal to the number of Jordan blocks of size n_i corresponding to the eigenvalue λ_i. Hence the corollary follows immediately from the theorem. ∎

Corollary 6. *If* $T\in\mathscr{L}(\mathscr{H})$ *is an algebraic operator and if* $\mathscr{A}_T$ *has property* $(\mathbf{B}_{2,2})$ *then* T *is reflexive.*

Proof. This is an immediate consequence of Theorem 4 and the description from [12, Theorem 5.11] of the algebraic operators that are reflexive. ∎

Remark 7. We note that the converse of Corollary 6 is false. For instance, the operator

$$T=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\oplus\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is reflexive, but by Corollary 5, it does not have property $(\mathbf{B}_{2,2})$. On the other hand we know that property $(\mathbf{B}_{1,1})$ is not sufficient for reflexivity since, by Theorem 4 or [12], every algebraic operator has property $(\mathbf{B}_{1,1})$.

Proposition 8. *Let* $T=\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ *be acting on* $\mathbf{C}^2$. *Then* $\mathscr{A}_T$ *has property* $(\mathbf{B}_{1,\aleph_0})$, *but* $\mathscr{A}_T$ *does not have property* $(\tilde{\mathbf{B}}_{1,2})$.

Proof. That $\mathscr{A}_T$ has property $(\mathbf{B}_{1,\aleph_0})$ follows from Corollary 5. Checking against the basis $\{I, T\}$ of $\mathscr{A}_T$, we see that $[\begin{pmatrix} x & y \\ z & w \end{pmatrix}] = [\begin{pmatrix} x+w & 0 \\ z & 0 \end{pmatrix}]$ in $\mathscr{L}(\mathbf{C}^2) / {}^{\perp}\mathscr{A}_T$. Let $L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $L_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. For $0 < \delta < 1$ we define $u' = v' = 1 \oplus \delta \in \mathbf{C}^2$. Then $u' \otimes v' = \begin{pmatrix} 1 & \delta \\ \delta & \delta^2 \end{pmatrix}$ and therefore

$$\| [L_1] - [u' \otimes v'] \| = \left\| \left[\begin{pmatrix} 0 & \delta \\ \delta & \delta^2 \end{pmatrix} \right] \right\| = \left\| \left[\begin{pmatrix} \delta^2 & 0 \\ \delta & 0 \end{pmatrix} \right] \right\|$$

$$\leqslant \left\| \begin{pmatrix} \delta^2 & 0 \\ \delta & 0 \end{pmatrix} \right\|_1 < \sqrt{2}\, \delta.$$

Since u' is cyclic for $\mathscr{A}_T$, from Lemma 2 there exists $v'' \in \mathbf{C}^2$ such that $[L_2] = [u' \otimes v'']$. Hence the system $\{[u' \otimes v'], [u' \otimes v'']\}$ is an approximate solution of the system $\{[L_1], [L_2]\}$. Since $\mathscr{A}_T$ has property $(\mathbf{B}_{1,\aleph_0})$, there exist u, v_1, $v_2 \in \mathbf{C}^2$ such that $[L_1] = [u \otimes v_1]$ and $[L_2] = [u \otimes v_2]$. Let $\{e_1, e_2\}$ be the canonical basis for $\mathbf{C}^2$. Then

$$(Tu, v_2) = \langle T, [u \otimes v_2] \rangle = \langle T, [L_2] \rangle = \mathrm{tr}(TL_2) = 1$$

imply that $(u, e_2) \neq 0$. This, and

$$(Tu, v_1) = \langle T, [u \otimes v_1] \rangle = \langle T, [L_1] \rangle = \mathrm{tr}(TL_1) = 0$$

imply that $(v_1, e_1) = 0$. Hence $\| v' - v_1 \| \geqslant |(v' - v_1, e_1)| = 1$. This completes the proof that $\mathscr{A}_T$ does not have property $(\tilde{\mathbf{B}}_{1,2})$. ∎

Concluding Remarks. Recall that the class C_0 is defined to be the set of all completely nonunitary contractions T in $\mathscr{L}(\mathscr{H})$ (with $\dim \mathscr{H} = \aleph_0$) such that there exists some nonzero H^∞-function f satisfying $f(T) = 0$. (Here the H^∞-funtional calculus $f \to f(T)$ is that discussed by Sz.-Nagy and Foias in [13].) It is obvious that (up to a scalar multiple) all algebraic operators belong to C_0, and there is a (fairly satisfactory) necessary and sufficient condition known [4] in order that an operator in C_0 be reflexive. Therefore, it would be interesting to know exactly which operators in C_0 have which properties $(\mathbf{A}_n)$ (cf. [8] for the definition of the properties), since this would then generalize Theorem 4 above, and, moreover, likely give necessary and sufficient conditions in terms of the properties $(\mathbf{A}_n)$ that an operator in C_0 be reflexive. The authors conjecture that a necessary and sufficient condition that the dual algebra $\mathscr{A}_T$ generated by an operator T in C_0 have property $(\mathbf{A}_n)$ is that the Jordan model $\bigoplus_{j=0}^{\infty} S(\theta_j)$ of T (cf. [13]) satisfy $\theta_1 = \cdots = \theta_n$. Consequently we conjecture that an operator T in C_0 such that $\mathscr{A}_T$ has property $(\mathbf{A}_2)$ is reflexive.

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