UNITARITY CONSTRAINTS FOR STRING PROPAGATION IN THE PRESENCE OF BACKGROUND FIELDS

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We discuss the constraints imposed by the unitarity of the S-matrix on the theory of a closed bosonic string propagating in a background consisting of a condensate of the string modes. The equations of motion for some of the low mass modes are thus obtained to all orders in α' but to second order in a weak field expansion about a flat background.

Recently, there have been several investigations of string propagation in a background consisting of a condensate of the string modes [1,2]. In most of these approaches, the Polyakov path integral formulation of a string theory [3] was employed and the equations of motion for the background spacetime fields followed from the requirement that the β functions of a two-dimensional nonlinear sigma model vanish. Though this method gives covariant results, higher-order "stringy" corrections are tedious to incorporate.

In this letter, we show how the equations of motion for the background fields may be obtained to all orders in α', the inverse string tension, but to a given order in a weak field expansion about a flat background. Consistent string propagation requires conformal invariance (closure of the Virasoro algebra) which can be shown to guarantee the unitarity of the S-matrix. In our approach the constraints on the background fields follow directly from the unitarity of the S-matrix, i.e., from the condition that physical states in the remote past map into physical states in the distant future. There is close connection with the string tree amplitude calculations and the operator formalism is used to obtain explicit forms of the equations of motion to second order in a weak field expansion for some of the low mass modes.

The theory of a closed bosonic string propagating in a background consisting of a condensate of its massless modes, the symmetric tensor mode $g_{\mu\nu}(X)$ and the antisymmetric tensor mode $B_{\mu\nu}(X)$, is described by the following action:

$$S=\frac{-1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,[\sqrt{-\gamma}\,\gamma^{\alpha\beta}g_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu+\epsilon^{\alpha\beta}B_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu]\ . \tag{1}$$

Here, $\gamma_{\alpha\beta}(\tau,\sigma)$ is a metric tensor on the two-dimensional world-sheet parametrized by σ,τ. Henceforth we will choose the conformal gauge, set $\alpha'=\frac{1}{2}$, and restrict ourselves to 26-dimensional spacetime and to the string tree level. Classically, the Virasoro generators $L_n,\bar{L}_n$ and the hamiltonian may be constructed and it is found that the Virasoro algebra closes without any constraints on the background fields [4]. In order to study this theory at the first-quantized level, we introduce a weak field expansion around flat spacetime, i.e., we set $g_{\mu\nu}(X)=\eta_{\mu\nu}+h_{\mu\nu}(X), B_{\mu\nu}(X)=b_{\mu\nu}(X)$, and regard $h_{\mu\nu}(X)$ and $b_{\mu\nu}(X)$ as weak fields. The hamiltonian and the Virasoro operators may be expanded with respect to these fields and the zeroth-order terms correspond to the usual flat space expressions. The higher-order terms with respect to the weak fields are treated as a perturbation. We may now go to the interaction representation where the operators are expressed in terms of the "in" operators $X^{\mu(\mathrm{in})}, \pi_\mu^{(\mathrm{in})}$, which satisfy free equations of motion. The normal mode expansion for $X^{\mu(\mathrm{in})}$ is

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$$X^{\mu(\mathrm{in})}(\tau,\sigma)=x^\mu+\tfrac{1}{2}p^\mu\tau+\frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\{\alpha_n^\mu\exp[-\mathrm{i}n(\tau-\sigma)]+\tilde{\alpha}_n^\mu\exp[-\mathrm{i}n(\tau+\sigma)]\}\,, \tag{2}$$

and the canonically conjugate momentum, $\pi_\mu^{(\mathrm{in})}=\pi^{-1}\eta_{\mu\nu}\,\partial X^\nu/\partial\tau$. The x^μ, p^ν and the oscillators satisfy the usual commutation relations [5]. From now on, we omit the superscript "in". As mentioned above, the Virasoro and hamiltonian operators can be expanded as $L_n=L_n^{(0)}+L_n^{(1)}+...,\bar{L}_n=\bar{L}_n^{(0)}+\bar{L}_n^{(1)}+...,H=H^{(0)}+H^{(1)}+...$, where the superscript denotes the corresponding order in the weak field expansion. Their explicit expressions are

$$L_n^{(0)}=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\sigma\exp(-\mathrm{i}n\,\sigma)\,{:}P^\mu(\tau,\sigma)^2{:}\,,\quad \bar{L}_n^{(0)}=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\sigma\exp(+\mathrm{i}n\sigma)\,{:}\tilde{P}^\mu(\tau,\sigma)^2{:}\,, \tag{3,4}$$

$$L_n^{(1)}=\bar{L}_{-n}^{(1)}=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\sigma\exp(-\mathrm{i}n\,\sigma)\,{:}\rho_{\mu\nu}(X)\,P^\mu(\tau,\sigma)\tilde{P}^\nu(\tau,\sigma){:}\,, \tag{5}$$

$$H^{(0)}=L_0^{(0)}+\bar{L}_0^{(0)}-2\,,\quad H^{(1)}=2L_0^{(1)}\,,\quad \text{etc}\,. \tag{6}$$

In the above, $\rho_{\mu\nu}(X)\equiv-h_{\mu\nu}(X)+b_{\mu\nu}(X)$ and

$$P^\mu(\tau,\sigma)=(\partial/\partial\tau-\partial/\partial\sigma)X^\mu(\tau,\sigma)\,,\quad \tilde{P}^\mu(\tau,\sigma)=(\partial/\partial\tau+\partial/\partial\sigma)X^\mu(\tau,\sigma)\,. \tag{7}$$

In (3)–(6) a normal ordering prescription has been introduced. For string propagation in a flat background, normal ordering of the oscillator parts in $L_n^{(0)}$ and $\bar{L}_n^{(0)}$ is necessary in order to get the correct central charge terms in the Virasoro algebra [5]. We adopt the same normal ordering for the oscillators in $L_n^{(1)}$ and $\bar{L}_n^{(1)}$, however, because of the presence of $\rho_{\mu\nu}(X)$, we still have an ordering ambiguity for the zero mode part (i.e., x^μ and p^μ). This ambiguity can be removed by demanding $\partial^\mu\rho_{\mu\nu}(X)=\partial^\nu\rho_{\mu\nu}(x)=0$. As we shall see later, these conditions correspond to gauge conditions for the background fields.

Until now we have only considered condensates of the string massless modes. Tachyon and other massive tensor fields condensates can also be incorporated in our formalism. To do so within a weak field expansion, we simply include the corresponding terms in the hamiltonian. For example, if $\phi(X)$ represents a background tachyon field, then its contribution is included by adding an interaction hamiltonian, $\Delta H_{\mathrm{tach}}=(1/2\pi)\int\mathrm{d}\sigma \times{:}\phi(X){:}$. In general, the total hamiltonian takes the form

$$H=H^{(0)}+H^{(1)}+H^{(2)}+...+\Delta H_{\mathrm{tach}}+\Delta H_{\mathrm{1st\ massive}}+...\equiv H^{(0)}+H_{\mathrm{I}} \tag{8}$$

Thus, in a weak field expansion, the form of the interaction hamiltonian for each mass level is essentially given by the corresponding vertex operator.

Let us now discuss what constraints are imposed on the background fields due to the unitarity of the S-matrix. Consistent string propagation requires the closure of the Virasoro algebra which guarantees the decoupling of the ghost states. For string propagation in a flat background, if we specify the physical state conditions in the usual manner, then, using the Virasoro algebra with the correct central change term, one can show the decoupling of the ghost states from the physical spectrum [5]. Consider next the analogous problem for a string propagating in a background consisting of a condensate of some of its modes. In this case, even though the classical Virasoro algebra is satisfied, quantum mechanically, it is valid only if the background fields satisfy some constraint [4]. Order by order in a weak field expansion, it can be shown that if the Virasoro algebra is satisfied, then the S-matrix is unitary [4,6]. In order to make a precise statement of the unitary requirement within perturbation theory, we assume that the physical state conditions for the "in" states are specified as in the flat case, i.e.,

$$[L_0^{(0)}(\phi_{\mathrm{in}})+\bar{L}_0^{(0)}(\phi_{\mathrm{in}})-2]|\mathrm{phys}\rangle=0\,,\quad [L_0^{(0)}(\phi_{\mathrm{in}})-\bar{L}_0^{(0)}(\phi_{\mathrm{in}})]|\mathrm{phys}\rangle=0\,,$$

$$L_n^{(0)}(\phi_{\mathrm{in}})|\mathrm{phys}\rangle=0\quad(n>0)\,,\qquad \bar{L}_n^{(0)}(\phi_{\mathrm{in}})|\mathrm{phys}\rangle=0\quad(n>0)\,. \tag{9}$$

Here we have used the adiabatic hypothesis such that $L_n, \bar{L}_n$ reduce to the flat space expressions $L_n^{(0)}, \bar{L}_n^{(0)}$ at $\tau \to -\infty$. At intermediate times the string can interact with the background fields and in order to maintain unitarity, we have to require that at $\tau \to +\infty$ the physical state conditions are

$$[L_0^{(0)}(\phi_{\rm in}) + \bar{L}_0^{(0)}(\phi_{\rm in}) - 2] S|{\rm phys}\rangle = 0\,, \quad [L_0^{(0)}(\phi_{\rm in}) - \bar{L}_0^{(0)}(\phi_{\rm in})] S|{\rm phys}\rangle = 0\,,$$

$$L_n^{(0)}(\phi_{\rm in}) S|{\rm phys}\rangle = 0 \quad (n>0)\,, \qquad \bar{L}_n^{(0)}(\phi_{\rm in}) S|{\rm phys}\rangle = 0 \quad (n>0)\,, \tag{10}$$

where $S = T\exp[-\mathrm{i}\int_{-\infty}^{\infty} H_{\rm I}(\phi_{\rm in})\,\mathrm{d}\tau]$ is the S-matrix. Therefore, the unitarity requirement can be stated thus: Take a physical state $|B\rangle$ satisfying (9) and consider the matrix element

$$T = \langle A|L_n^{(0)} S|B\rangle\,, \tag{11}$$

with $\langle A|L_n^{(0)}$ an on-shell state, i.e., $\langle A|(H^{(0)}+n) = 0$, but not necessarily a physical state. Unitarity then demands that T vanish for any choice of the states $|A\rangle$ and $|B\rangle$. An explicit connection between the above requirement and the closure of the Virasoro algebra can be established to each order in a weak field expansion [4,6].

Our approach in determining the equations of motion for the background fields consists in expanding the matrix element T order by order in a weak field expansion and demanding that it vanish. The simplest choice of T corresponds to taking $n = 1$ and the following states for $\langle A|$ and $|B\rangle$)

$$\langle A| = \langle 0, p'|\tilde{\alpha}_1^{\lambda}\,, \quad p'^2 = 0\,, \qquad |B\rangle = |0,p\rangle\,, \quad p^2 = 8\,, \tag{12}$$

where $\langle 0,p'|$ and $|0,p\rangle$ are ground states with momenta p' and p respectively. The first-order contribution in the weak field expansion of T (denoted by $T^{(1)}$) is readily evaluated and the results for the massless and tachyon mode contributions are [omitting an overall $2\pi\delta(0)$ factor]

$$T^{(1)}_{\rm massless} = (-\mathrm{i})\langle A|L_1^{(0)} H^{(1)}|B\rangle$$

$$= (-\mathrm{i})\int \frac{\mathrm{d}^D k}{(2\pi)^D}(2\pi)^D\delta^D(p'-p-k)\,\tfrac{1}{8}k^2\rho_{\mu\nu}(k)(Y_{\rm massless})^{\mu\nu,\lambda}\,, \tag{13}$$

$$T^{(1)}_{\rm tach} = (-\mathrm{i})\langle A|L_1^{(0)}\Delta H_{\rm tach}|B\rangle$$

$$= (-\mathrm{i})\int \frac{\mathrm{d}^D k}{(2\pi)^D}(2\pi)^D\delta^D(p'-p-k)(\tfrac{1}{8}k^2-1)\,\phi(k)(Y_{\rm tach})^{\lambda}\,. \tag{14}$$

Here, we have introduced the Fourier transformation of $\rho_{\mu\nu}(X)$ and $\phi(X)$:

$$:\rho_{\mu\nu}(X): = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\rho_{\mu\nu}(k)\,:\exp(\mathrm{i}kX):, \qquad :\phi(X): = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\phi(k)\,:\exp(\mathrm{i}kX):, \tag{15}$$

and the tensor structures $(Y_{\rm massless})^{\mu\nu,\lambda}, (Y_{\rm tach})^{\lambda}$ are given by

$$(Y_{\rm massless})^{\mu\nu,\lambda} = \tfrac{1}{2}q^{\mu}[\eta^{\nu\lambda} + \tfrac{1}{2}q^{\nu}\,\tfrac{1}{2}(p'-p)^{\lambda}]\,, \quad (Y_{\rm tach})^{\lambda} = \tfrac{1}{2}(p'-p)^{\lambda}\,, \tag{16,17}$$

with $q = \frac{1}{2}(p+p')$. If we demand that (13) and (14) vanish irrespective of p and p' then we obtain the free part of the equations of motion for the corresponding background fields. In x-space these are

$$\partial^2\rho_{\mu\nu}(x) = 0\,, \quad (\partial^2+8)\,\phi(x) = 0\,. \tag{18,19}$$

(18) is the free equation of motion for the massless modes, i.e., the graviton, antisymmetric tensor and dilaton modes in the gauge $\partial^{\mu}\rho_{\mu\nu}(x) = \partial^{\nu}\rho_{\mu\nu}(x) = 0$. In this gauge the trace of $\rho_{\mu\nu}(x)$ represents the dilaton. If we require these on-shell conditions on $H^{(1)}$ and $\Delta H_{\rm tach}$ then they reduce to the corresponding vertex operators, however, it is important to notice that (18) and (19) arise in our formalism as a result of consistency requirements.

The first-order results may be easily extended to the massive modes. For example, the requirement that the contribution to $T^{(1)}$ from the term in the interaction hamiltonian, $\Delta H_{1\mathrm{st\,massive}}$, vanish, gives the free equation of motion $(\partial^2 - 8)M_{\mu\alpha,\nu\beta}(x) = 0$, and so on.

We next consider the second-order term in T (denoted by $T^{(2)}$). Upon combining with the first-order result, this will correspond to the interaction terms in the equations of motion. In this letter, we will only consider the contribution to $T^{(2)}$ corresponding to second order in $\rho_{\mu\nu}(X)$ and $\phi(X)$ (denoted by $T^{(2)}_{\rho^2}$ and $T^{(2)}_{\phi^2}$ respectively). $T^{(2)}_{\rho^2}$ is given by

$$T^{(2)}_{\rho^2} = (-\mathrm{i})^3 (\langle A|L_1^{(0)} H^{(1)} \Delta^{-1} H^{(1)}|B\rangle - \langle A|L_1^{(0)} H^{(2)}|B\rangle) \tag{20}$$

where, $\Delta = H^{(0)}$ is the inverse propagator. Consider the second term on the right-hand side of (20). In theories with derivative interactions, one should be careful to distinguish between T and T^* products. In the hamiltonian formulation we must use the T product, whereas the standard string operator formalism corresponds to using a T^* product. A careful examination along the lines of ref. [7] reveals that the difference between these two products is cancelled by the second term in (20), in accordance with the Mathews–Nambu theorem [8]. Thus, dropping this term and using the T^* product, we get

$$\langle A|L_1^{(0)} H(1)\,\Delta^{-1} H(1)|B\rangle = -\frac{1}{16}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\,\rho_{\mu\nu}(k_1)\,\rho_{\alpha\beta}(k_2)(2\pi)^D \delta^D(p' - k_1 - k_2 - p)$$

$$\times \frac{\Gamma(-1-\frac{1}{8}s)\,\Gamma(-1-\frac{1}{8}u)\,\Gamma(-1-\frac{1}{8}\tilde{t})}{\Gamma(2+\frac{1}{8}s)\,\Gamma(2+\frac{1}{8}u)\,\Gamma(2+\frac{1}{8}\tilde{t})}\,(\tfrac{1}{2}p')_\sigma K^{\sigma\mu\alpha} K^{\lambda\nu\beta}\,, \tag{21}$$

where

$$s = -(p+k_2)^2\,, \quad \tilde{t} = -(p'-p)^2 + k_1^2 + k_2^2\,, \quad u = -(p+k_1)^2\,, \tag{22}$$

and $K^{\sigma\mu\alpha}, K^{\lambda\mu\beta}$ are tensors composed of the invariants $s, \tilde{t}, u$ and the momenta [6]. It can be shown that if we use the first-order result $k_1^2\rho_{\mu\nu}(k_1) = k_2^2\rho_{\mu\nu}(k_2) = 0$, first, then the above expression vanishes. This is due to the gauge invariance of the massless modes and corresponds to the spacetime Ward identity. However, nontrivial contributions arise when we set t near one of the on-mass shell values. Then, one picks up the corresponding t-channel poles and the above expression reduces to the indeterminate 0/0 form. By using a proper infrared regularization, these contributions precisely correspond to the interaction terms in the equations of motion. For t near the on-shell values of the tachyon, massless and massive modes, we can thus pick up the corresponding contributions to the equations of motion. The infrared regularization we adopt here corresponds to shifting the mass shell condition for $\rho_{\mu\nu}(k_1), \rho_{\mu\nu}(k_2)$, i.e., we let $(k_1^2 + m^2)\,\rho_{\mu\nu}(k_1) = (k_2^2 + m^2)\,\rho_{\mu\nu}(k_2) = 0$, then we take t near one of the on-shell values and at the end consider the limit $m^2 \to 0$. $T^{(2)}_{\phi^2}$ can be treated in analogous manner and below we list our results for $T^{(2)}_{\rho^2}$ and $T^{(2)}_{\phi^2}$ when t is near on-mass shell for the massless and the tachyon modes ($k = k_1 + k_2$):

$$(T^{(2)}_{\rho^2})_{t\sim 0} = -(-\mathrm{i})^3\,\frac{1}{4}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}(2\pi)^D\delta^D(p' - k_1 - k_2 - p)$$

$$\times\, \rho_{\mu'\nu'}(k_1)\,\rho_{\alpha'\beta'}(k_2)\, t^{\mu'\alpha'\mu} t^{\nu'\beta'\nu} (Y_{\mathrm{massless}})_{\mu\nu}{}^{\lambda}, \tag{23}$$

with $t^{\mu'\alpha'\mu} = [\eta^{\alpha'\mu}\,\frac{1}{2}k^{\mu'} - \eta^{\mu\mu'}\,\frac{1}{2}k^{\alpha'} + (\eta^{\alpha'\mu'} - \frac{1}{2}k^{\mu'}\,\frac{1}{2}k^{\alpha'})\,\frac{1}{4}(k_1 - k_2)^{\mu}]$,

$$(T^{(2)}_{\rho^2})_{t\sim -8} = -(-\mathrm{i})^3\,\frac{1}{4}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}(2\pi)^D\delta^D(p' - k_1 - k_2 - p)$$

$$\times\, \rho_{\mu\nu}(k_1)\,\rho_{\alpha\beta}(k_2)(\eta^{\alpha\mu} - \tfrac{1}{2}k^{\alpha}\,\tfrac{1}{2}k^{\mu})(\eta^{\beta\nu} - \tfrac{1}{2}k^{\beta}\tfrac{1}{2}\,k^{\nu})(Y_{\mathrm{tach}})^{\lambda}\,, \tag{24}$$

$$(T^{(2)}_{\phi^2})_{t\sim 0} = (-\mathrm{i})^3 \frac{1}{32} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} (2\pi)^D \delta^D(p' - k_1 - k_2 - p)$$

$$\times\, \phi(k_1)\, \phi(k_2)(k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu})(Y_{\mathrm{massless}})^{\mu\nu,\lambda}\,, \tag{25}$$

$$(T^{(2)}_{\phi^2})_{t\sim -8} = -(-\mathrm{i})^3 \frac{1}{4} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} (2\pi)^D \delta^D(p' - k_1 - k_2 - p)\, \phi(k_1)\, \phi(k_2)(Y_{\mathrm{tach}})^\lambda\,. \tag{26}$$

From these we note that the $t \sim 0$ contributions for both $T^{(2)}_{\rho^2}$ and $T^{(2)}_{\phi^2}$ have the same factorized form as $T^{(1)}_{\mathrm{massless}}$ and in particular the same tensor structure $(Y_{\mathrm{massless}})^{\mu\nu,\lambda}$. This enables us to identify the corresponding equations of motion consistently. Similar remarks apply to the $t \sim -8$ contributions which give the tachyon equation of motion. To second order these equations in x-space are

$$\partial^2 \rho_{\mu\nu}(x) = \tfrac{1}{2}\,\partial_\mu\phi\partial_\nu\phi - [\partial^\lambda\partial^\sigma(\rho_{\lambda\sigma}\rho_{\mu\nu} - \rho_{\mu\lambda}\rho_{\sigma\nu}) + \tfrac{1}{2}(\eta^{\lambda\sigma} + \tfrac{1}{2}\partial^\lambda\tfrac{1}{2}\partial^\sigma)\,\partial^\gamma(\partial_\nu\rho_{\gamma\delta}\,\rho_{\mu\lambda} - \rho_{\gamma\sigma}\partial_\nu\rho_{\mu\lambda})$$
$$+ \tfrac{1}{2}(\eta^{\lambda\sigma} + \tfrac{1}{2}\partial^\lambda\tfrac{1}{2}\partial^\sigma)\,\partial^\gamma(\partial_\mu\rho_{\sigma\gamma}\rho_{\lambda\nu} - \rho_{\sigma\gamma}\partial_\mu\rho_{\lambda\nu})$$
$$+ \tfrac{1}{4}(\eta^{\lambda\sigma} + \tfrac{1}{2}\partial^\lambda\tfrac{1}{2}\partial^\sigma)(\eta^{\gamma\delta} + \tfrac{1}{2}\partial^\gamma\tfrac{1}{2}\partial^\delta)(\partial_\mu\partial_\nu\rho_{\sigma\delta}\,\rho_{\lambda\gamma} - \partial_\mu\rho_{\sigma\delta}\,\partial_\nu\rho_{\lambda\gamma})]\,, \tag{27}$$

$$(\partial^2 + 8)\,\phi(x) = 2\phi^2 + 2(\eta^{\alpha\mu} + \tfrac{1}{2}\partial^\alpha\tfrac{1}{2}\partial^\mu)(\eta^{\beta\nu} + \tfrac{1}{2}\partial^\beta\tfrac{1}{2}\partial^\nu)(\rho_{\mu\nu}\rho_{\alpha\beta})\,. \tag{28}$$

The various couplings in the above equations are consistent with the string tree amplitude calculations [5].

The equations of motion for the massive modes can be similarly determined by taking the corresponding t-channel pole in $T^{(2)}$. For example the $\rho\rho$ and $\phi\phi$ contributions to the equation of motion for the first massive mode can be obtained by taking the $t \sim 8$ pole in $T^{(2)}_{\rho^2}$, $T^{(2)}_{\phi^2}$. These are

$$(T^{(2)}_{\rho^2})_{t\sim 8} = -(-\mathrm{i})^3 \frac{1}{16} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} (2\pi)^D \delta^D(p' - k_1 - k_2 - p)$$

$$\times\, \rho_{\mu'\nu'}(k_1)\, \rho_{\alpha'\beta'}(k_2)\, M^{\mu'\alpha'}{}_{\mu\alpha} M^{\nu'\beta'}{}_{\nu\beta}(Y_{\mathrm{massive}})^{\mu\alpha\nu\beta,\lambda}\,, \tag{29}$$

$$(T^{(2)}_{\phi^2})_{t\sim 8} = -(-\mathrm{i})^3 \frac{1}{16} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} (2\pi)^D \delta^D(p' - k_1 - k_2 - p)$$

$$\times\, \tfrac{1}{4}(k_1 - k_2)_\mu\, \tfrac{1}{4}(k_1 - k_2)_\alpha\, \tfrac{1}{4}(k_1 - k_2)_\nu\, \tfrac{1}{4}(k_1 - k_2)_\beta\, \phi(k_1)\, \phi(k_2)(Y_{\mathrm{massive}})^{\mu\alpha\nu\beta,\lambda}\,, \tag{30}$$

with

$$M^{\mu'\alpha'}{}_{\mu\alpha} = [\eta^{\mu'\alpha'} - \tfrac{1}{2}k^{\mu'}\tfrac{1}{2}k^{\alpha'}\tfrac{1}{4}(k_1 - k_2)_\mu\tfrac{1}{4}(k_1 - k_2)_\alpha + \eta^{\alpha'}_\mu\tfrac{1}{2}k^{\mu'}\tfrac{1}{4}(k_1 - k_2)_\alpha$$
$$+ \eta^{\alpha'}_\alpha\tfrac{1}{2}k^{\mu'}\tfrac{1}{4}(k_1 - k_2)_\mu - \eta^{\mu'}_\mu\tfrac{1}{2}k^{\alpha'}\tfrac{1}{4}(k_1 - k_2)_\alpha - \eta^{\mu'}_\alpha\tfrac{1}{2}k^{\alpha'}\tfrac{1}{4}(k_1 - k_2)_\mu$$
$$+ \eta^{\alpha'}_\mu\eta^{\mu'}_\alpha + \eta^{\alpha'}_\alpha\eta^{\mu'}_\mu - \tfrac{1}{2}\cdot\tfrac{1}{2}k^{\mu'}\tfrac{1}{2}k^{\alpha'}\tfrac{1}{2}k_\mu\,\tfrac{1}{2}k_\alpha]\,, \tag{31}$$

$$(Y_{\mathrm{massive}})^{\mu\alpha\nu\beta,\lambda} = \tfrac{1}{2}q^\mu\,\tfrac{1}{2}q^\alpha\,\tfrac{1}{2}(p' - p)^\lambda\,\tfrac{1}{2}q^\nu\,\tfrac{1}{2}q^\beta + \eta^{\lambda\nu}\tfrac{1}{2}p^\beta + \eta^{\lambda\beta}\tfrac{1}{2}p^\nu\,. \tag{32}$$

The different couplings are again consistent with the string tree amplitude calculations. The interaction terms in the equations of motion for the massive modes correspond to taking $t > 0$ poles in our approach and would involve the introduction of nonrenormalizable terms in the action in the Polyakov path integral formulation [9].

In conclusion, we have obtained the equations of motion near on-shell for some of the low mass modes of a string in background fields by directly demanding unitarity of the S-matrix. These are consistent with the string tree level three-particle couplings for the modes to all orders in α'. Unitarity is formally guaranteed by conformal invariance [4] and the equations of motion for background fields have been obtained in refs. [1,2] by demanding the vanishing of the β-function in a σ-model approach. In the σ-model approach an explicit coupling of the dilaton background to the world-sheet curvature is included, ensuring that the equations of motion from the vanishing of the β-function are gauge covariant. On the other hand, we do not include such a coupling for the dilaton in (1) and the equations of motion derived here are in the gauge $\partial^\mu \rho_{\mu\nu} = \partial^\nu \rho_{\mu\nu} = 0$. As discussed in refs. [10,11] in the context of the string three point function calculation in the same gauge, the states of definite spin (i.e., the symmetric traceless graviton, the antisymmetric tensor and the dilaton) correspond to various projections of $\rho_{\mu\nu}$. The spin zero projection in fact just corresponds to the dilaton. A gauge invariant effective action for the three massless modes that reproduces the string three point function with massless external legs, to order α', has been obtained by Nepomechie [11]. The effective action for the graviton, antisymmetric tensor and the dilaton that will be obtained from our results will be the same as Nepomechie's to order α'. Callan, Klebanov and Perry [2] have shown that the effective action to order α' in the σ-model approach also agrees with that of refs. [10,11] up to field redefinitions.

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