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UNCOUNTABLE GROUPS HAVE MANY NONCONJUGATE SUBGROUPS

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We prove that any uncountable group G of power λ has at least λ subgroups not conjugate in pairs. The paper is very self-contained, assuming no knowledge except cardinal arithmetic (and the definition of an (abelian) group).

Contents

0. Introduction

This article is dedicated to the proof of

0.1. Main Theorem. *If G is a group of cardinality λ, λ an uncountable cardinal, $\mu = \mathrm{Min}\{\mu : 2^\mu \geq \lambda\}$, then* $\mathrm{nc}_{\leq\mu}(G) \geq \lambda$.

0.2. Definition. $\mathrm{nc}_\kappa(G)$ is the number of pairwise nonconjugate subgroups of G of power κ. We define $\mathrm{nc}_{\leq\kappa}(G)$, $\mathrm{nc}_{<\kappa}(G)$ similarly.

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We note that

0.3. Conclusion. *If λ is an uncountable cardinal, G a group of cardinality λ, then G has at least λ pairwise nonconjugate subgroups of power $<\lambda$.*

Proof of the Conclusion. If $\mu = \mathrm{Min}\{\mu : 2^{\mu} \geq \lambda\}$ is $<\lambda$, we finish by 0.1, hence we have to deal with λ strong limit only. If λ is singular, we get the result by 1.2(3). If λ is regular, then necessarily $\lambda = \aleph_{\lambda}$, and for each $\alpha < \lambda$, G has a subgroup G_{α} of power $\aleph_{\alpha}$; clearly the G_{α}'s are pairwise nonisomorphic, hence nonconjugate.

This paper continues [5] where the result was conjectured and proved under GCH, and for many cases (on λ, for every G). The motivation was a question of Rips; he built a group of power $\aleph_0$ with exactly three subgroups up to conjugacy, and he asks whether we can do something similar for higher cardinals.

Note that by [6] if $\lambda = \kappa^{+} = 2^{\kappa}$, then there is a group of power λ with λ subgroups (hence $\leq\lambda$ subgroups up to conjugacy). Rips [4] improves this to: If there is an algebra with countably many operations of power λ with $\leq\lambda$ subalgebras, then there is such a group.

Almost no special knowledge is required to understand the paper. The facts we use from mathematical logic which algebraists may not know are explained in the Appendix.

During the proof we prove the Main Theorem under various hypotheses on λ and then add the hypothesis eliminating those cases.

Really, we prove the theorem by induction on λ.

Some readers were disappointed complaining that "after at last I got an intuition, the class of groups we discuss disappears." We may want to look at classes of groups which essentially are discussed (that is, the one satisfying some intermediate consequences of being in $\mathcal{P}^m$ or Ω^m). See 8.4.

In Section 10 we give a generalization of 0.1, 0.3.

Notation

Set Theory. Let λ, μ be fixed cardinals as in the Main Theorem. Let $|A|$ be the power of A. Let χ, κ, θ, σ denote cardinals (almost always infinite), α, β, γ, i, j denote ordinals, δ denote a limit ordinal and η, ν, ρ denote sequences of ordinals. Let ${}^{\beta}\alpha$ be the set of sequences of length β of ordinals $<\alpha$. ${}^{\beta>}\alpha = \bigcup_{\gamma<\beta} {}^{\gamma}\alpha$, ${}^{\beta\geq}\alpha = \bigcup_{\gamma\leq\beta} {}^{\gamma}\alpha$. Let χ^{κ} be cardinal exponentation, $\chi^{<\kappa} = \sum_{\theta<\kappa} \chi^{\theta}$.

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ be the integers, rationals and reals, respectively. Let m, n, r denote natural numbers or integers, so $n<\omega$ ($i<\omega$) means n (i) is a natural number, $n \in \mathbb{Z}$ means n is an integer. Let $\langle a_t : t \in T\rangle$ denote a T-indexed sequence. Let F, f, h denote functions.

Group Theory. Let G, H, I, J, K, L, M, N denote groups. For $A \subseteq G$ let $\langle A\rangle_G$ denote the subgroup of G generated by A; but $\langle A_1, \ldots, A_n\rangle_G = \langle \bigcup_{i=1}^{r} A_i\rangle_G$, if $A_i = \{a_i\}$ we write a_i instead of A_i, and let $\langle A_t : t \in T\rangle_G$ denote $\langle \bigcup_{t\in T} A_t\rangle_G$.

Let a, b, c, d, x, y denote elements of groups, e the unit (e_G of G, if confusion may arise) and A, B, C, D denote sets of elements of groups.

For $g \in G$, $\square^g : G \to G$ is the function $\square^g(x) = gxg^{-1}$, $\square^g$ is an automorphism of G, and such an automorphism is called inner. So a normal subgroup of G is one preserved by all inner automorphisms and a characteristic subgroup is one preserved by all automorphisms of G (so being a characteristic subgroup is a transitive relation, being a normal subgroup not necessarily).

If $B, A \subseteq G$, $x \in G$, then $xA = \{xy : y \in A\}$, $AB = \{xy : x \in A, y \in B\}$; if N is a normal subgroup of G, then $G/N = \{xN : x \in G\}$ is the quotient group, and for $A \subseteq G$, $A/N = \{xN : x \in A\}$.

We say x, y commute in G if $xy = yx$; we say $A, B \subseteq G$ commute if every $x \in A$, $y \in B$ commute.

Let $\mathrm{Cent}\, G = \{x \in G : x \text{ commutes with } G\}$. $\mathrm{Cent}^\alpha(G)$ is defined by induction on α:

$$\mathrm{Cent}^0(G) = \{e\},$$

$$\mathrm{Cent}^{\alpha+1}(G) = \{x \in G : x\, \mathrm{Cent}^\alpha(G) \in \mathrm{Cent}(G/\mathrm{Cent}^\alpha(G))\},$$

$$\mathrm{Cent}^\delta(G) = \bigcup_{\alpha<\delta} \mathrm{Cent}^\alpha(G).$$

We can prove by induction on α that $\mathrm{Cent}^\alpha(G)$ is a normal (even characteristic) subgroup of G.

Let $\mathrm{Cent}^\infty(G) = \bigcup_\alpha \mathrm{Cent}^\alpha(G)$. Let $\mathrm{Cm}_G(A) = \{x \in G : x \text{ commutes with } A\}$, this is a subgroup.

Now $G^{(1)} = \langle xyx^{-1}y^{-1} : x, y \in G \rangle_G$ is called the commutator subgroup of G. We define $G^{(\alpha)}$ by induction on α: $G^{(0)} = G$, $G^{(\alpha+1)} = (G^{(\alpha)})^{(1)}$, $G^{(\delta)} = \bigcap_{\alpha<\delta} G^\alpha$, $G^{(\infty)} = \bigcap_\alpha G^\alpha$. We can prove by induction that they are all characteristic subgroups of G, and $G/G^{(1)}$ is commutative.

Let $(G:H)$ be the index of H in G, i.e., $|\{xH : x \in G\}|$. Let $\mathrm{Ker}(h)$ be the kernel of the homomorphism h.

0.4. Fact. *For* $A \subseteq G$,

$$\mathrm{Cent}(\langle A \rangle_G) = \langle A \rangle_G \cap \mathrm{Cm}_G(A) \subseteq \mathrm{Cent}(\mathrm{Cm}_G(A)).$$

Proof. Direct checking.

We say that $\{a_t : t \in S\}$ forms a basis of a commutative [free] group G if $G = \langle a_t : t \in S \rangle_G$, and $e = \prod_{l=1}^n (a_{t_l})^{n(l)}$ ($t_1, \ldots, t_n$ distinct, $n(l) \in \mathbb{Z}$) implies $(a_{t_l})^{n(l)} = e$ for each l [implies $n(l) = 0$ for each l].

1. The easy facts and the case $2^\mu > \lambda$

Remember that λ, μ are always as in the Main Theorem. We shall start to investigate counterexamples and

1.1. Definition. **(1)** Let $\mathcal{P}_\lambda = \mathcal{P}^0_\lambda = \{G : G \text{ has power } \lambda \text{ and } \mathrm{nc}_{\leq\mu}(G) < \lambda\}$.

(2) $\mathcal{P}^1_\lambda = \{G$: for some $L \subseteq \mathrm{Cent}(G)$, $|L| < \mu$ and $G/L \in \mathcal{P}_\lambda\}$.

(3) For A, B, $C \subseteq G$ we say B, C are conjugate over A in G (or B conjugate to C over A in G) if some inner automorphism of G maps C onto B and is the identity over A.

1.2. Fact. (1) *If $G \in \mathcal{P}_\lambda$, then G has at most λ subgroups of power $\leq\mu$.*

(2) *If $\mathcal{P}_\lambda \neq \emptyset$, then for no κ, $2^\kappa < \lambda < \lambda^\kappa$.*

(3) *If $2^\kappa < |G| < |G|^\kappa$, then* $\mathrm{nc}_\kappa(G) = |G|^\kappa$.

Proof. (1) Let $\{G_i : i < \alpha\}$ be a maximal family of pairwise nonconjugate subgroups of G each of power $\leq\mu$. AS $G \in \mathcal{P}_\lambda$ necessarily $\alpha < \lambda$. Now the family $\{\square^g G_i : g \in G,\ i < \alpha\}$ contains all subgroups of G of power $\leq\mu$ and has power $\leq|G| \cdot |\alpha| = \lambda$.

(2) Let $\{a_i : i < \lambda\}$ be a list of distinct elements of G; as $\lambda < \lambda^\kappa$, there is a list $\{u_\alpha : \alpha < \lambda^\kappa\}$ of distinct subsets of λ. Let $G_\alpha = \langle a_i : i \in u_\alpha \rangle_G$, so G_α is a subgroup of G of power κ, and $\kappa < \mu$ (as $2^\kappa < \lambda \leq 2^\mu$). Define an equivalence relation E on λ^κ:

$$\alpha E \beta \quad \text{iff} \quad G_\alpha = G_\beta.$$

What is the power of $\{\beta : \alpha E \beta\}$? It is at most the number of subsets of $\{i < \lambda : a_i \in G_\alpha\}$, but this set has power $\leq|G_\alpha| = \kappa$, hence the number of subsets of it is $\leq 2^\kappa$. Hence each E-equivalence class has power $\leq 2^\kappa$. As $2^\kappa < \lambda^\kappa$, the number of groups in $\{G_i : i < \lambda^\kappa\}$ is λ^κ. So G has λ^κ subgroups each of power $\leq\kappa < \mu$, hence by (1) we get a contradiction to $G \in \mathcal{P}_\lambda$.

(3) By the proofs of (1) and (2).

1.3. Fact. *For a commutative uncountable group G, and $\kappa \leq |G|$,* $\mathrm{nc}_\kappa(G) = |G|^\kappa$.

Proof. Easy (or see [5]): Choose by induction on $\alpha < |G|$, a_α, n_α such that $0 \leq n_\alpha < \omega$, $a_\alpha \in G$ and for every $m \in \mathbb{Z}$, $(a_\alpha)^m \in \langle a_\beta : \beta < \alpha \rangle_G$ iff $(a_\alpha)^m = e$ iff m is a multiple of n_α (and $a_\alpha \neq e$ of course). This can be done as G is uncountable. Now let for $S \subseteq |G|$, $|S| \leq \kappa$, $G_S = \langle a_\alpha : \alpha \in S \rangle_G$, so we have $|G|^\kappa$ distinct subgroups of G. But the only inner automorphism of G is the identity, so we finish.

1.4. Fact. (1) *If N is a normal subgroup of G and θ a cardinal $\geq\aleph_0$, then* $\mathrm{nc}_{\leq\theta}(G/N) \leq \mathrm{nc}_{\leq\theta}(G)$.

(2) $\mathrm{nc}_{\leq\theta}(G/N)$ *is the number of $\{H : N \subseteq H \subseteq G,\ (H : G) \leq \theta\}$ up to conjugacy in G.*

(3) *If $\theta < \mu$, $G \in \mathcal{P}^0_\lambda$, then* $\mathrm{nc}_{\leq\theta}(G) < \lambda$.

Proof. (1) Let $\kappa = \mathrm{nc}_{\leq\theta}(G/N)$, and let H_i $(i < \kappa)$ be pairwise nonconjugate subgroups of G/N each of power $\leqslant\theta$. Choose for each member of H_i a representative, so for some $x^i_\alpha \in G$ $(\alpha<|H_i|)$, $H_i=\{x^i_\alpha N:\alpha<|H_i|\}$, and let $K_i=\langle x^i_\alpha:\alpha<|H_i|\rangle_G$. So K_i is a subgroup of G of power $\leqslant|H_i|+\aleph_0\leqslant\theta$, and if $g\in G$, $i\neq j$, $\Box^g$ maps K_i onto K_j, then $\Box^{gN}$ maps H_i onto H_j, contradiction. So K_i $(i<\kappa)$ exemplify $\kappa\leqslant \mathrm{nc}_{\leqslant\theta}(G)$ as required.

(2), (3) should be clear.

1.5. Fact. *For a cardinal θ, and N a subgroup of G, $\mathrm{nc}_{\leqslant\theta}(N)\leqslant \mathrm{nc}_{\leqslant\theta}(G)\times(G:N)$.*

Proof. Let $\kappa=\mathrm{nc}_{\leqslant\theta}(N)$ and $\{H_i:i<\kappa\}$ be a maximal family of subgroups of N, nonconjugatge in N, each of power $\leqslant\theta$. Define an equivalence relation E on κ: $i\,E\,j$ if H_i, H_j are conjugate in G.

Clearly the number of E-equivalence classes is at most $\mathrm{nc}_{\leqslant\theta}(G)$, so it is enough to prove that each equivalence class has power $\leqslant(G:N)$. If $S=\{j:i\,E\,j\}$, then for every $j\in S$ for some $g_j\in G$, $\Box^{g_j}$ maps H_j onto H_i. If $|S|>(G:N)$ for some $\alpha\neq\beta\in S$, $g_\alpha N=g_\beta N$, hence $g_\beta^{-1}g_\alpha\in N$; now

$$\Box^{(g_\beta^{-1}g_\alpha)}(H_\alpha)=\Box^{g_\beta^{-1}}(\Box^{g_\alpha}H_\alpha)=\Box^{g_\beta^{-1}}(H_i)=H_\beta;$$

so H_α, H_β are conjugate in N, contradiction.

1.6. Fact. *If N is a normal subgroup of G, $G\in\mathcal{P}^m_\lambda$, $(G:N)=\lambda$, then $G/N\in\mathcal{P}^m_\lambda$ (for $m=0,1$).*

Proof. It suffices to prove the fact for $m=0$. As $(G:N)=\lambda$, G/N has power λ and by 1.4, $\mathrm{nc}_{\leqslant\mu}(G/N)\leqslant\mathrm{nc}_{\leqslant\mu}(G)<\lambda$.

1.7. Fact. *If N is a subgroup of G, $G\in\mathcal{P}^m_\lambda$, $(G:N)<\lambda$, then $N\in\mathcal{P}^m_\lambda$ (for $m=0,1$).*

Proof. If $m=1$, let L exemplify $G\in\mathcal{P}^1_\lambda$ (see 1.1(2)). We know $N/(L\cap N)\cong NL/L$ which is a normal subgroup of G/L, and $(G/L:NL/L)\leqslant(G:N)<\lambda$, so we reduce this to the case $m=0$ (remembering $L\subseteq\mathrm{Cent}(G)$, hence $N\cap L\subseteq\mathrm{Cent}(N)$).

It is known that $|G|=(G:N)\times|N|$, hence $|N|=\lambda$. By 1.5,

$$\mathrm{nc}_{\leqslant\mu}(N)\leqslant\mathrm{nc}_{\leqslant\mu}(G)\times(G:N)<\lambda.$$

1.8. Fact. *If $G\in\mathcal{P}^1_\lambda$, then*

(1) $\mathrm{Cent}(G)$ *has power* $<\mu+\aleph_1$.

(2) $\mathrm{Cent}(G)$ *has power* $\leqslant\aleph_0+\mathrm{nc}_{\leqslant\aleph_0}(G)$.

(3) $\mathrm{Cent}^\infty(G)$ *has power* $\leqslant\aleph_0+\mathrm{nc}_{\leqslant\aleph_0}(G)$.

(4) $(G:G^{(1)})$ *is* $<\mu+\aleph_1$.

Proof. Let L exemplify $G \in \mathcal{P}^1_\lambda$.

(1) If $|\mathrm{Cent}(G)| \geq \mu + \aleph_1$, then $|\mathrm{Cent}(G)/L| \geq \mu + \aleph_1$, hence by 1.3, $\mathrm{Cent}(G)/L$ has at least 2^μ distinct subgroups of power $\leq \mu$. By the definition of the center, they are pairwise nonconjugate in G/L. So $\mathrm{nc}_{\leq\mu}(G/L) \geq 2^\mu \geq \lambda$, contradiction.

(2) Clearly $\{\langle a \rangle_G : a \in \mathrm{Cent}(G)\}$ is a family of pairwise nonconjugate countable subgroups of G, and if $|\mathrm{Cent}(G)| > \aleph_0$, then the family has power $|\mathrm{Cent}(G)|$. On the other hand the family has power $\leq \mathrm{nc}_{\leq\aleph_0}(G)$. Together we get the conclusion.

(3) Left to the reader.

(4) We know that $G^{(1)}$ is a normal subgroup of G, hence by 1.4, $\mathrm{nc}_{\leq\mu}(G/G^{(1)}) \leq \mathrm{nc}_{\leq\mu}(G)$. But $G/G^{(1)}$ is trivially commutative, so we can apply 1.3 (if $m=1$ we should divide by some L, $|L| < \mu$, so it does not matter).

1.9. Fact. *Suppose $A \subseteq G$, then on the set $\{H : A \subseteq H \subseteq G,\ |H| \leq \kappa\}$ the equivalence relation "being conjugate over A" has at most $\mathrm{nc}_{\leq\kappa}(G) + \kappa^{|A|}$ equivalence classes (and this number is $< \lambda$ if $G \in \mathcal{P}_\lambda$, $\kappa^{|A|} < \lambda$, $\kappa \leq \mu$).*

Proof. Let $\theta = \mathrm{nc}_{\leq\kappa}(G) + \kappa^{|A|}$, and suppose $A \subseteq H_i \subseteq G$, $|H_i| \leq \kappa$ for $i < \theta^+$, and the H_i's are pairwise nonconjugate over A in G. As $\theta \geq \mathrm{nc}_{\leq\kappa}(G)$, w.l.o.g. the H_i's are pairwise conjugate in G, so let $g_i \in G$, $\square^{g_i}$ maps H_i onto H_0. The number of possible functions $\square^{g_i} \restriction A$ is at most the number of functions from A into H_0, i.e., $|H_0|^{|A|} \leq \kappa^{|A|} \leq \theta$, hence w.l.o.g. $\square^{g_i} \restriction A$ is constant. So $\square^{(g_2^{-1} g_1)} = \square^{g_2^{-1}} \square^{g_1}$ is the identity on A and maps H_1 onto H_2, contradiction.

1.10. Fact. (1) *If $2^{|A|} < \lambda$, $A \subseteq G \in \mathcal{P}_\lambda$, then $\mathrm{Cm}_G(A)$ has power λ.*

(2) *If $A \subseteq G$, $|A| \leq \mu$, then $\mathrm{nc}_{\leq\mu}(\mathrm{Cm}_G(A)/\mathrm{Cent}(\langle A \rangle_G)) \leq \mathrm{nc}_{\leq\mu}(G) + \mu^{|A|}$* (remember $\mathrm{Cent}(\langle A \rangle_G) \subseteq \mathrm{Cent}(\mathrm{Cm}_G(A))$ by 0.4).

(3) *If $\aleph_0 < \mu$, $\mu^{|A|} < \lambda$, $A \subseteq G \in \mathcal{P}_\lambda$, then $\mathrm{Cm}_G(A) \in \mathcal{P}^1_\lambda$ (in fact $\mathrm{Cm}_G(A)/\mathrm{Cent}(\langle A \rangle_G) \in \mathcal{P}_\lambda$.)*

(4) *Parts* (1) *and* (3) *are true for $G \in \mathcal{P}^1_\lambda$ too.*

Proof. (1) Let $a_i \in G$ $(i < \lambda)$ be distinct members of G and let θ be any (infinite) cardinal such that $\aleph_0 + |A|^{|A|} + \mathrm{nc}_{\leq\mu}(G) \leq \theta < \lambda$. W.l.o.g. $\langle A, a_i \rangle_G$ $(i < \theta^+)$ are distinct, and by 1.9 w.l.o.g. $\langle A, a_i \rangle_G$ are pairwise conjugate over A. So let $g_i \in G$, $\square^{g_i}$ be the identity over A and maps $\langle A, a_0 \rangle_G$ onto $\langle A, a_i \rangle_G$. W.l.o.g. for some $b \in \langle A, a_0 \rangle$ for every $i > 0$, $\square^{g_i}(b) = a_i$. So g_i commutes with A, hence $g_i \in \mathrm{Cm}_G(A)$, and $g_i b g_i^{-1} = a_i$. As the a_i's are distinct, the g_i are distinct, hence $\mathrm{Cm}_G(A)$ has power $\geq \theta^+$. As θ was any cardinal $\aleph_0 + \mathrm{nc}_{\leq\mu}(G) + |A|^{|A|} \leq \theta < \lambda$, we finish.

(2) Use 1.9 and the proof of 1.4.

(3) Use (2).

(4) Easy. For (1) if $g_i L$ $(i < \lambda)$ are distinct members of $\mathrm{Cm}_{G/L}(A/L)$, then $\square^{g_i}$ maps each $a \in A$ into aL. As $2^{|A|} < \lambda$, $|A| < \mu$, hence $|AL| < \mu$, hence for each

$\theta<\lambda$, w.l.o.g. $\square^{g_i}\restriction A$ is the same for $i<\theta^+$, hence $|\mathrm{Cm}(A)|\geqslant|\{g_0^{-1}g_i:i<\theta^+\}|>\theta$. Hence $|\mathrm{Cm}(A)\geqslant\lambda$.

1.11. Theorem. *The main theorem holds if* $\lambda<2^\mu$.

Proof. Let $G\in\mathcal{P}_\lambda$. We choose by induction on $\alpha<\mu$, for every $\eta\in{}^\alpha 2$ an element $a_\eta\in G$ such that

(a) a_η commutes with $a_{\eta\restriction\beta}$ for every $\beta<l(\eta)$,

(b) $a_{\eta\hat{}\langle 0\rangle}$ and $a_{\eta\hat{}\langle 1\rangle}$ do not commute.

For α limit or zero, $\eta\in{}^\alpha 2$: choose $a_\eta=e$. For $\alpha=\beta+1$, $\eta\in{}^\beta 2$ we have to define $a_{\eta\hat{}\langle 0\rangle}$, $a_{\eta\hat{}\langle 1\rangle}$.

By 1.10(1), $\mathrm{Cm}_G\{a_{\eta\restriction\gamma}:\gamma\leqslant\beta\}$ has power λ (as $\beta<\mu$, so $2^{|\beta|}<\lambda$). Hence it is enough to find there two noncommuting elements. If we cannot find them, $\mathrm{Cm}_G\{a_{\eta\restriction\gamma}:\gamma\leqslant\beta\}$ is a commutative subgroup of G of power λ, so by 1.3 it has 2^μ subgroups of power μ, hence G has 2^μ subgroups of power μ, contradiction to 1.2(1).

So the a_η are defined, and let for $\eta\in{}^\mu 2$, $H_\eta=\langle a_{\eta\restriction\alpha}:\alpha<\mu\rangle_G$. Clearly H_η is a commutative subgroup of G of power $\leqslant\mu$. Also $\eta\neq\nu\Rightarrow H_\eta\neq H_\nu$; otherwise let $\beta=\mathrm{Min}\{\beta:\eta(\beta)\neq\nu(\beta)\}$, then $a_{\nu\restriction(\beta+1)}$ does not commute with $a_{\eta\restriction(\beta+1)}$ but $a_{\eta\restriction(\beta+1)}\in H_\eta$, $a_{\nu\restriction(\beta+1)}\in H_\nu$ and both are commutative. So G has $\geqslant 2^\mu>\lambda$ subgroups of power λ, contradicting 1.2(1). So there is no $G\in\mathcal{P}_\lambda$.

1.12. Fact. *If* $A\subseteq G\in\mathcal{P}^1_\lambda$, $|A|<\mu$, *and* N *is a normal subgroup of* G *which includes* $\mathrm{Cm}_G(A)$, *then* $(G:N)<\lambda$.

Proof. Cent $G\subseteq N$ (as Cent $G\subseteq\mathrm{Cm}_G(A)$). Suppose $(G:N)=\lambda$, so by induction one chooses $a_i\in G-\langle N,A,a_j:j<i\rangle_G$. As in the proof of 1.10(1) for some $i<j<\lambda$ and $g\in G$, $\square^g$ maps $\langle A,a_i\rangle_G$ onto $\langle A,a_j\rangle_G$ and is the identity on A, so $g\in\mathrm{Cm}(A)\subseteq N$, and for some $b\in\langle A,a_j\rangle$, $a_i=gbg^{-1}\in\langle A,a_j,g\rangle_G\subseteq\langle N,A,a_\alpha:\alpha<i\rangle$, contradiction.

1.13. Fact. *If* $G\in\mathcal{P}^1_\lambda$ *then*:

(1) *The number of* H, Cent $G\subseteq H\subseteq G$, $|H|\leqslant\mu$ *up to conjugacy in* G *is* $<\lambda$.

(2) *The number of* $H\subseteq G$, $H^{(1)}=H$, *up to conjugacy in* G *is* $<\lambda$.

Proof. (1) We know $G/\mathrm{Cent}\,G\in\mathcal{P}_\lambda$, and use 1.4(2).

(2) This is because for such H, $(\langle H,\mathrm{Cent}\,G\rangle_G)^{(1)}=H$.

2. The case $\mu=\aleph_0$

In fact this was the original question (i.e., $\lambda=\aleph_1$) and in [5] we have proved $\mathrm{nc}(G)\geqslant\lambda$ when $\aleph_0<|G|\leqslant 2^{\aleph_0}$, however here we want to prove $\mathrm{nc}_{\leqslant\aleph_0}(G)\geqslant|G|$.

To this end we eventually build many non-isomorphic finitely generated subgroups (after analyzing a possible counterexample).

During this section we assume $\mu = \aleph_0$, and later assume $\mu > \aleph_0$.

2.1. Fact. *If* $\mu = \aleph_0$, *then every* $G \in \mathcal{P}^1_\lambda$ *has an element of order* ∞ (*i.e.*, $(\forall n > 0)\, g^n \neq e$).

Proof. Let $G \in \mathcal{P}^{(1)}_\lambda$. By 1.8, $\mathrm{Cent}^\infty(G)$ has power $< \lambda$, hence by 1.6, $G/\mathrm{Cent}^\infty(G) \in \mathcal{P}_\lambda$. If G was a counterexample to the fact, then so is $G/\mathrm{Cent}^\infty(G)$, hence w.l.o.g. G is with a trivial center.

Clearly in such a G (i.e., a counterexample with trivial center):

(∗) every finitely generated commutative subgroup of G is finite.

We shall prove later:

2.2. Subfact. *For* G *as above for each finite commutative* A, *some* $g \in G$ *commutes with* A, $\langle g \rangle_G \cap \langle A \rangle = \{e\}$ *and* $g \neq e$, *of course.*

So we can choose by induction on $n < \omega$, $a_n \in G$, $a_n \neq e$, so that $\{a_n : n < \omega\}$ is a basis of a commutative subgroup of G. (Note that $\langle a_m : m \leqslant n \rangle_G$ is finite by (∗)). Let $\{n : n < \omega\} = \{n_t : t \in \mathbb{Q}\}$, and for every real r, let $H_r = \{a_{n_t} : t \in \mathbb{Q}, t < r\}$. So $\{H_r : r \in \mathbb{R}\}$ is a family of $2^{\aleph_0} \geqslant \lambda$ subgroups of G. As $G \in \mathcal{P}_\lambda$ for some $r(1) < r(2)$, $H_{r(2)}$ is conjugate to $H_{r(1)}$ in G. Hence for some $g \in G$, $\Box^g$ maps $H_{r(2)}$ into $H_{r(1)}$ which is a proper subgroup of $H_{r(2)}$ (obviously $H_{r(1)} \subseteq H_{r(2)}$, but for some rational t, $r(1) < t < r(2)$, hence $a_{n_t} \in H_{r(2)} - H_{r(1)}$). So necessarily $g^n \neq e$ for $0 < n < \omega$, hence we prove 2.1 except that we have to prove 2.2.

Proof of Subfact 2.2. As G has a trivial center choose a finite B, $A \subseteq B \subseteq G$, such that each $a \in \langle A \rangle_G$ (except e) does not commute with some $b \in B$ (possible as $\langle A \rangle_G$ is finite, by (∗)). Now by 1.9, $\mathrm{Cm}_G(B)$ has power λ, but by the choice of B, $\mathrm{Cm}_G(B) \cap \langle A \rangle_G = \{e\}$, so any $g \in \mathrm{Cm}_G(B)$, $g \neq e$ is as required.

2.3. Conclusion. *Let* $\mu = \aleph_0$. *If* $G \in \mathcal{P}^1_\lambda$, $A \subseteq G$ *is finite, then there is a* $g \in \mathrm{Cm}_G(A)$ *such that* $g^n \notin \langle A \rangle_G$ *for every* $0 < n < \omega$.

Proof. As A is finite, by 1.10(2), letting $G_1 \overset{\mathrm{df}}{=} \mathrm{Cm}_G(A)$, $G_1/\mathrm{Cent}\, G_1$ belongs to $\mathcal{P}_\lambda$. By 2.1 there is a $g \in G_1$ such that $g\, \mathrm{Cent}\, G_1$ has infinite order. So $g^n \notin \mathrm{Cent}\, G_1$ for $0 < n < \omega$, also $g \in \mathrm{Cm}_G(A)$ and

$$\langle A \rangle_G \cap \langle g \rangle_G \subseteq \langle A \rangle_G \cap \mathrm{Cm}_G(A) \subseteq \mathrm{Cent}\, \mathrm{Cm}_G(A) = \mathrm{Cent}\, G_1.$$

2.4. Fact. *Suppose* $G \in \mathcal{P}_\lambda$, $\mu = \aleph_0$. *There are* $b_n \in G$ (*for* $n \in \mathbb{Z}$) *and* $g \in G$ *such that*:

(1) $\{b_n : n \in \mathbb{Z}\}$ *forms a basis of a free commutative group.*

(2) $\Box^g b_n = b_{n+1}$.

Proof. By 2.3 there are $a_n \in G$ $(n<\omega)$ which form a free basis of a commutative subgroup of G. Let N_t $(t \in \mathbb{Q})$, H_r $(r \in \mathbb{R})$ be as in the proof of 2.1 and so again for some $r(1)<r(2)$ *in* $\mathbb{R}$ *and* $g \in G$, $\Box^g$ maps $H_{r(2)}$ into $H_{r(1)}$. Let $t \in \mathbb{Q}$, $r(1)<t<r(2)$, and $b_m = \Box^{g^m}(a_{n_t})$ for $m \in \mathbb{Z}$. The checking is easy.

2.5. Proof of the Main Theorem for $\mu=\aleph_0$. We use g, b_n $(n \in \mathbb{Z})$ from 2.4. Denote $H=\langle\{b_n : n<\omega\}\rangle_G$, this group has $2^{\aleph_0}$ subgroups. Let $\chi = \mathrm{nc}_{\leqslant\aleph_0}(G)+\aleph_0$, so H has χ^+ distinct subgroups which are conjugate in pairs (by elements of G). For $i<\chi^+$ let H_i be a subgroup of H and $g_i \in G$ such that $i \neq j \Rightarrow H_i \neq H_j$ and $g_i H_i g_i^{-1} = H_0$. Now for every $i<\chi^+$ define $K_i = \langle b_0, g_i, g\rangle_G$, a subgroup of G. If we shall find $S \subseteq \chi^+$, $|S|=\chi^+$ such that for i, $j \in S$, $i \neq j \Rightarrow K_i \not\cong K_j$, we clearly obtain a contradiction to the choice of χ (non-isomorphic groups cannot be conjugate).

How shall we do it? View K_i as models of group theory with three additional constants b, g, h (where in K_i, b is interpreted by b_0, g by g_i, h by g). Those models cannot be isomorphic since for $i \neq j<\chi^+$, g_i behaves on $\langle b_0, g\rangle$ differently than g_j.

Now we use the following observation: Given a countable group it can be expanded by 3 new constants in only $\aleph_0$ ways ($\aleph_0^3=\aleph_0$) (this is a special case of Lemma VIII 1.3 from [7]). So there is $S \subseteq \chi^+$, $|S|=\chi^+$, such that $i \neq j \Rightarrow \bar{K}_i \not\cong \bar{K}_i$ when $\bar{K}_i$ is the reduct to the language of group theory (containing only e and multiplication) of K_i.

3. Eliminating the normal subgroups with small index

In this section we shall show that any $G \in \mathcal{P}_\lambda$ has a normal subgroup N with index $<\lambda$, which has no proper normal subgroup with index $<\lambda$. We then prove that for any such N, $N=N^{(1)}$ and any $x \in N - \mathrm{Cent}(N)$ has $\geqslant \mu$ conjugates in N.

Those subgroups N (and variants) play an important role in the sequel.

3.1. Claim. *Suppose that θ is an uncountable cardinal, and N_α $(\alpha<\theta)$ is a strictly decreasing sequence of normal subgroups of G. Then* $\mathrm{nc}_{\leqslant\theta}(G) \geqslant 2^\theta$.

Proof. We shall define by induction on $\alpha<\theta$, an element a_α and an ordinal β_α such that:

(a) $a_\alpha \in N_{\beta_\alpha} - N_{\beta_\alpha+1}$,
(b) $\beta_\alpha < |\omega+\alpha|^+$,
(c) for every $\gamma<\alpha$, $\beta_\gamma<\beta_\alpha$,
(d) $a_\alpha \notin \langle N_{\beta_\alpha+1} \cup \{a_\gamma : \gamma<\alpha\}\rangle_G$.

Suppose we have defined a_γ, β_γ for every $\gamma<\alpha$ and we shall define a_α, β_α. Clearly the subgroup $H_\alpha = \langle a_\gamma : \gamma<\alpha\rangle_G$ has power $<|\omega+\alpha|^+$, hence for some ordinal β_α, $\bigcup_{\gamma<\alpha}\beta_\gamma < \beta_\alpha < |\omega+\alpha|^+$, and $H_\alpha \cap (N_{\beta_\alpha} - N_{\beta_\alpha+1}) = \emptyset$. Choose $a_\alpha \in N_{\beta_\alpha} - N_{\beta_\alpha+1}$; now (a), (b), (c) hold trivially. As for (d), if it fails, then $a_\alpha N_{\beta_\alpha+1}$

belongs to $\{gN_{\beta_\alpha+1}: g \in H_\alpha\}$ (i.e., the homomorphic image of H_α in $G/N_{\beta_\alpha+1}$ by the canonical homomorphism). Hence for some $g \in H_\alpha$, $gN_{\beta_\alpha+1} = a_\alpha N_{\beta_\alpha+1}$ hence $a_\alpha^{-1}g \in N_{\beta_\alpha+1}$. But $a_\alpha \in N_{\beta_\alpha}$, $N_{\beta_\alpha+1} \subseteq N_{\beta_\alpha}$, so necessarily $g = a_\alpha(a_\alpha^{-1}g) \in N_{\beta_\alpha}$, however $g \in H$, $H \cap N_{\beta_\alpha} = H \cap N_{\beta_\alpha+1}$ hence $g \in N_{\beta_\alpha+1}$, so $a_\alpha = g(g^{-1}a_\alpha) = g(a_\alpha^{-1}g)^{-1}$ belongs to $N_{\beta_\alpha+1}$, contradicting the choice of a_α. So (d) holds too, so we have carried successfully the definition by induction of a_α, β_α. Now for any $S \subseteq \theta$ we define

$$H_S = \langle a_\alpha : \alpha \in S \rangle_G.$$

Clearly it suffices to prove that for any distinct subsets S, T of θ, H_S is not conjugate to H_T. Now as $S \neq T$ w.l.o.g. for some α, $\alpha \in S$, $\alpha \notin T$. As $a_\alpha \in N_{\beta_\alpha} - N_{\beta_\alpha+1}$, and $\alpha \in S$, clearly $H_S \cap (N_{\beta_\alpha} - N_{\beta_\alpha+1}) \neq \emptyset$. On the other hand as $\alpha \notin T$,

$$\begin{aligned} H_T = \langle a_\gamma : \gamma \in T \rangle &\subseteq \langle N_{\beta_\alpha+1} \cup \{a_\gamma : \in T\} \rangle_G \\ &\subseteq \langle N_{\beta_\alpha+1} \cup \{a_\gamma : \gamma \in T, \gamma \leqslant \alpha\} \rangle_G \\ &\subseteq \langle N_{\beta_\alpha+1} \cup \{a_\gamma : \gamma < \alpha\} \rangle_G \end{aligned}$$

Hence $H_T \cap (N_{\beta_\alpha} - N_{\beta_\alpha+1}) = \emptyset$ by the proof of (d).

As the set $N_{\beta_\alpha} - N_{\beta_\alpha+1}$ is preserved by inner automorphisms of G, H_T is disjoint to it whereas H_S is not disjoint to it, clearly H_S, H_T are not conjugates.

3.2. Claim. *Suppose that N is a normal subgroup of G, $A \subseteq G$, $|A| \leqslant \kappa$, $(G:N) < \sigma = |G|$, σ is an uncountable cardinal, $\mathcal{P}_\sigma = \emptyset$, $\kappa^{|A|} < \sigma$ and $\sigma \leqslant 2^\kappa$.*

Then N has subsets B_i (for $i < \sigma$) such that $|B_i| \leqslant \kappa$ and the subgroups $\langle A, B_i \rangle_G$ (for $i < \sigma$) are pairwise nonconjugates in G.

Proof. Suppose not and there are only $\theta_0 < \sigma$ nonconjugate such subgroups. Let $\theta = \theta_0 + \kappa^{|A|} + (G:N) + \aleph_0$, so clearly $\theta < \sigma$. We first prove:

(∗) $K = N \cap \mathrm{Cm}_G(A)$ has power σ.

For let $\theta_1 = \theta + |K|$ and assume $\theta_1 < \sigma$. Let b_i $(i < \theta_1^+)$ be distinct members of N (N has power σ as $|G| = \sigma > (G:N)$, σ infinite). As we have assumed that the claim fails and as $\theta_0 \leqslant \theta \leqslant \theta_1$, among the subgroups $\langle A, b_i \rangle_G$ $(i < \theta_1^+)$ there are θ_1^+ which are pairwise conjugates in G. So w.l.o.g. all $\langle A, b_i \rangle$ $(i < \theta_1^+)$ are conjugates in G. So let $\Box^{g_i}$ be a conjugation which maps $\langle A, b_i \rangle_G$ onto $\langle A, b_0 \rangle_G$. As $\kappa^{|A|} \leqslant \theta \leqslant \theta_1$, $|A| \leqslant \kappa$, and $|\langle A, b_i \rangle_G| \leqslant \kappa$ the number of functions from A to $\langle A, b_0 \rangle_G$ is $\leqslant \theta_1$, hence w.l.o.g. $\Box^{g_i} \restriction A$ is constant, hence (for i, $j < \theta_1^+$), $(\Box^{g_j})^{-1}\Box^{g_i} = \Box^{(g_j^{-1}g_i)}$ is the identity on A, which means $g_j^{-1}g_i \in \mathrm{Cm}_G(A)$.

As $\Box^{g_i}(b_i)$ has $\leqslant |\langle A, b_0 \rangle_G| \leqslant \kappa \leqslant \theta_1$ possible values, w.l.o.g. for $i > 0$, it is constant, hence $(\Box^{g_j})^{-1}\Box^{g_i}(b_i) = b_j$. Also the number of possible cosets g_iN is at most $(G:N) \leqslant \theta \leqslant \theta_1$, hence w.l.o.g. for every $i > 0$, $g_iN = g_1N$, so $g_i^{-1}g_1 \in N$.

So we have gotten that $g_i^{-1}g_1$ $(1<i<\theta_1^+)$ are θ_1^+ distinct members of K, contradiction to "$(*)$ fails." So we have proved $(*)$

So $|K|=\sigma$. Now

$(**)$ there are subgroups H_i of K (for $i<\sigma$) such that $H_i=K\cap\langle H_i,A\rangle_G$, $|H_i|\leqslant\kappa$, and the H_i's are pairwise nonconjugate in K.

$(**)$ *suffices*: suppose 3.2 fails. By the proof of 1.9, for some $i<\sigma$ for θ^+ j's, $\langle A,H_j\rangle_G$ is conjugate to $\langle A,H_i\rangle_G$ say by $\Box^{g_{i,j}}$, and w.l.o.g. $\Box^{g_{i,j}}\restriction A=$ the identity, hence $g_{i,j}\in \mathrm{Cm}_G(A)$.

Now $\theta\geqslant(G:N)$, hence for some such $j(1)\neq j(2)$, $g_{i,j(1)}N=g_{i,j(2)}N$, so $g\stackrel{\mathrm{def}}{=}g_{i,j(2)}^{-1}g_{i,j(1)}\in N\cap\mathrm{Cm}_G(A)=K$, and $\Box^g$ maps $\langle A,H_{j(1)}\rangle_G$ onto $\langle A,H_{j(2)}\rangle_G$, and as $g\in K$, $\Box^g$ maps $\langle A,H_{j(1)}\rangle_G\cap K$ onto $\langle A,H_{j(2)}\rangle_G\cap K$, but for every j, $\langle A,H_j\rangle_G\cap K=H_j$. So $j(1)\neq j(2)$ but $H_{j(1)}$, $H_{j(2)}$ were assumed to be nonconjugate in K, contradiction, hence $(**)$ really suffices.

Proof of $(**)$. Now if $\mathrm{Cent}(K)/(\mathrm{Cent}(K)\cap\langle A\rangle_G)$ has power $\geqslant\kappa+\aleph_1$, by 1.3, $\mathrm{Cent}(K)$ has at least $2^\kappa\geqslant\sigma$ subgroups of power κ extending $\mathrm{Cent}(K)\cap\langle A\rangle_G$, trivially nonconjugates in K (being in the center), and for each such H easily $H=\langle A,H\rangle_H\cap K$.

So $\mathrm{Cent}(K)/(\mathrm{Cent}(K)\cap\langle A\rangle_G)$ has power $<\kappa+\aleph_1$ and as $\langle A\rangle_G$ has power $<\sigma$ (as $\kappa^{|A|}<\sigma$, σ uncountable) easily $\mathrm{Cent}(K)$ has power $<\sigma$.

So $K/\mathrm{Cent}(K)$ has power σ, and as "$\mathscr{P}_\sigma=\emptyset$" is a hypothesis and as $2^\kappa\geqslant\sigma$, clearly $\mathrm{nc}_{\leqslant\kappa}(K/\mathrm{Cent}\,K)$ is $\geqslant\sigma$.

So let K_i $(i<\sigma)$ be subgroups of K of power $\leqslant\kappa$, such that $K_i/\mathrm{Cent}\,K$ $(i<\sigma)$ are pairwise nonconjugate subgroups of $K/\mathrm{Cent}\,K$. Let $H_i=\langle K_i,A\rangle_G\cap K$; as $\langle A\rangle_G\cap\mathrm{Cm}_G(A)\subseteq\mathrm{Cent}\,\mathrm{Cm}_G(A)$ (see 0.4) it is easy to check that the H_i $(i<\sigma)$ are as required in $(**)$. (Note that K_i, A commute, $\langle K_i,A\rangle_G=\{xy:x\in K_i, y\in A\}$, and $H_i=\{xy:x\in K_i, y\in A\cap K\}$.)

3.3. Fact. *For any subgroups N_α $(\alpha<\beta)$ of N,*

$$(G:\bigcap_{\alpha<\beta}N_\alpha)\leqslant\prod_{\alpha<\beta}(G:N_\alpha).$$

Proof. Trivial: Define a function F from G to $\prod_{\alpha<\beta}G/N_\alpha$ by $F(x)=\langle xN_\alpha:\alpha<\beta\rangle$. The power of the range of F is $\leqslant\prod_{\alpha<\beta}|G/N_\alpha|=\prod_{\alpha<\beta}(G:N_\alpha)$. Also

$$F(x)=F(y)\Leftrightarrow(\forall\alpha<\beta)(xN_\alpha=yN_\alpha)\Leftrightarrow(\forall\alpha<\beta)(y^{-1}x\in N_\alpha)$$

$$\Leftrightarrow y^{-1}x\in\bigcap_{\alpha<\beta}N_\alpha\Leftrightarrow x\Big(\bigcap_\alpha N_\alpha\Big)=y\Big(\bigcap_\alpha N_\alpha\Big).$$

So $\prod_{\alpha<\beta}(G:N_\alpha)\geqslant|\mathrm{Rang}(F)|\geqslant(G:\bigcap_{\alpha<\beta}N_\alpha)$, and so the conclusion is clear. From now on we assume

3.4. Hypothesis. $\mathscr{P}_\sigma=\emptyset$ for every uncountable $\sigma<\lambda$, and $\mu>\aleph_0$, $2^\mu=\lambda$.

And for this section sometimes we assume

3.4A. Statement. μ is not strong limit singular (hence $|A|<\mu \Rightarrow \mu^{|A|}<\lambda$, see 1.10(3), (4)).

3.5. Lemma. *If $G \in \mathscr{P}_\lambda$, then*

$$\operatorname{Min} G \stackrel{\text{def}}{=} \bigcap \{N : N \textit{ a normal subgroup of } G,\ (G:N)<\lambda\}$$

is a characteristic (hence normal) subgroup of G and has index $<\lambda$.

Proof. Being characteristic is trivial, so we shall prove the "index $<\lambda$."

We choose by induction on $\alpha<\mu$ a normal subgroup N_α of G, such that $N_0=G$, N_α is a proper subgroup of N_β for every $\beta<\alpha$ and $(G:N_\alpha)<\lambda$.

If we succeed we shall get by 3.1 that $\mathrm{nc}_{\leqslant\mu}(G) \geqslant 2^\mu$ but $2^\mu \geqslant \lambda$, hence this contradicts $G \in \mathscr{P}_\lambda$. So for some $\alpha<\mu$ we cannot find N_α as required. If α is a successor ordinal, i.e., $\alpha=\beta+1$ note that for any normal subgroup N of G with index $<\lambda$, $N \cap N_\beta$ is a normal subgroup of G with index $<\lambda$. As $N \cap N_\beta$ cannot serve as N_α, necessarily $N \cap N_\beta = N_\beta$. So N_β is equal to $\operatorname{Min} G$, hence $(G:\operatorname{Min} G)=(N:N_\beta)<\lambda$, and we finish the proof.

So we assume α is a limit ordinal. Then necessarily $N \stackrel{\text{def}}{=} \bigcap_{\beta<\alpha} N_\beta$ has index λ (in G). By 3.3, $(G:N) \leqslant \prod_{\beta<\alpha}(G:N_\beta)$, let $\sigma_\beta=(G:N_\beta)$, clearly $\sigma_\beta<\lambda$ for $\beta<\alpha$ and by α's choice $\prod_{\beta<\alpha}\sigma_\beta \geqslant \lambda$ and $\beta<\gamma \Rightarrow \sigma_\beta \leqslant \sigma_\gamma$.

Let $\mu_\beta=\operatorname{Min}\{\theta : 2^\theta \geqslant \sigma_\beta\}$, as $\sigma_\beta<\lambda$ clearly $\mu_\beta \leqslant \mu$, and obviously $\beta<\gamma<\alpha \Rightarrow \mu_\beta \leqslant \mu_\gamma$.

Case (a): $\operatorname{Sup}\{\mu_\beta : \beta<\alpha\}$ *is* $<\mu$. Then we can find $\kappa<\mu$ such that $|\alpha| \leqslant \kappa$ and $\mu_\beta \leqslant \kappa$ for every $\beta<\alpha$. So for each $\beta<\alpha$, $\sigma_\beta \leqslant 2^{\mu_\beta} \leqslant 2^\kappa$, hence $\prod_{\beta<\sigma}\sigma_\beta \leqslant (2^\kappa)^{|\alpha|}=2^\kappa$, contradicting $\lambda \leqslant \prod_{\beta<\alpha}\sigma_\beta$.

Case (b): *Not case* (a) *and* σ_β ($\beta<\alpha$) *is eventually constant.* So by renaming w.l.o.g. $\sigma_\beta=\sigma$ for every $\beta<\alpha$ and $\lambda \leqslant \prod_{\beta<\alpha}\sigma_\beta=\sigma^{|\alpha|}$. As $\alpha<\mu$, by μ's definition $2^{|\alpha|}<\lambda$, hence by cardinal arithmetic $2^{|\alpha|}<\sigma$, and as α is a limit ordinal, $|\alpha|$ is infinite. So G/N_1 has power σ, $|\alpha|$ is infinite, $2^{|\alpha|}<\sigma<\lambda \leqslant \sigma^{|\alpha|}$, hence by 1.2(3), $\mathrm{nc}_{\leqslant|\alpha|}(G/N_1) \geqslant \sigma^{|\alpha|}$, but (see above) $\sigma^{|\alpha|} \geqslant \lambda$ and $|\alpha|<\mu$ hence $\mathrm{nc}_{\leqslant\mu}(G/N_1) \geqslant \lambda$. But by 1.4, $\mathrm{nc}_{\leqslant\mu}(G) \geqslant \mathrm{nc}_{\leqslant\mu}(G/N_1) \geqslant \lambda$, contradiction.

Case (c): *Not case* (a) *and* $\langle \mu_\beta : \beta<\alpha \rangle$ *is not eventually constant.*

Subcase (c1): 3.4A *holds*. So, w.l.o.g. $2^{\mu_0} \geqslant \mu$, $\mu_0>\aleph_0$ and $\langle \mu_\beta : \beta<\alpha \rangle$ is strictly increasing. We now define by induction on $\beta<\alpha$ for every $\eta \in \prod_{\gamma<\beta}\sigma_{\gamma+1}$ a subgroup H_η of G such that:

(i) $H_{\eta \restriction \gamma} \subseteq H_\eta \subseteq \langle H_{\eta \restriction \gamma}, N_{\gamma+1} \rangle_G$ for $\gamma<l(\eta)$.

(ii) $|H_\eta| \leqslant \mu_{l(\eta)}$.

(iii) If $\beta=l(\eta)$ is a limit ordinal, then $H_\eta=\bigcup_{\gamma<\beta} H_{\eta \restriction \gamma}$.

(iv) The subgroups $\langle H_{\eta^\frown\langle i \rangle}, N_{l(\eta)+2} \rangle_G$ for all $i<\sigma_{l(\eta)+1}$ (for a fix η) are pairwise nonconjugate in G.

The induction step is done by Lemma 3.2 (possible as $\mu_\beta \leqslant \mu$ hence $\mu_\beta<\mu$ for every $\beta<\alpha$). With $G/N_{l(\eta)+2}$, $N_{l(\eta)+1}/N_{l(\eta)+2}$, $\sigma_{l(\eta)+1}$, H_η, $\mu_{l(\eta)+1}$ here standing for G, N, σ, A, κ there respectively (note that $(\mu_{l(\eta)+1})^{|H_\eta|} \leq (2^{\mu_0})^{\mu_{l(\eta)}} = 2^{\mu_{l(\eta)}} < \sigma_{l(\eta)+1}$ as necessarily 2^{μ_β} $(\beta < \alpha)$ is strictly increasing too). Now for each $\eta \in \prod_{\beta<\alpha} \sigma_{\beta+1}$, $H_\eta = \bigcup_{\gamma<\beta} H_{\eta \restriction \gamma}$; clearly by (ii) $|H_\eta| \leq \sum_{\beta<\alpha} \mu_\beta \leq \mu\, |\alpha| = \mu$, by (i) + (iv) the H_η's are pairwise nonconjugate. So $\{H_\eta : \eta \in \prod_{\beta<\alpha} \mu_{\beta+1}\}$ exemplifies $\mathrm{nc}_{\leq\mu}(G) \geq \prod_{\beta<\alpha} \sigma_{\beta+1} = \prod_{\beta<\alpha} \sigma_\beta \geq \lambda$, contradicting $G \in \mathcal{P}_\lambda$.

Subcase (c2): 3.4A *fails* (i.e., μ is strong limit singular). By cardinal arithmetic, μ_i, $\sigma_i < \mu$ for each i and $\mathrm{cf}\, \alpha = \mathrm{cf}\, \mu$. Let $\mu = \sum_{i<\mathrm{cf}\,\mu} \chi(i)$, $|\alpha| + \mathrm{cf}\, \mu < \chi(i)$, $\chi(i+1) = \chi(i+1)^{\chi(i)}$. Let $\beta(i) = \mathrm{Min}\{\beta < \alpha : \mu_\beta > \chi(i+1)\}$, then $\beta(i) = \gamma(i) + 1$ (as $|\alpha| < \chi(i)$). We can now imitate the proof of (c1).

Case (d): *For some* $\beta < \gamma$, $\mu_\beta = \mu$, $\sigma_\beta < \sigma_\gamma$ *and for every* $A \subseteq N_\beta/N_\gamma$, $|A| < \mu$, *the set* $\mathrm{Cm}_{n_\beta/N_\gamma}(A)$ *has power* $\geq \mu$. Clearly in this case N_β/N_γ has a commutative subgroup of power μ, hence by 1.3 has $2^\mu = \lambda$ subgroups H of power μ hence by 1.2(1) (apply to σ_β, μ standing for λ, μ) $\mathrm{nc}_{\leq\mu}(N_\beta/N_\gamma) \geq \lambda$, hence by 1.4, $\mathrm{nc}_{\leq\mu}(N_\beta) \geq \lambda$, contradicting $G \in \mathcal{P}_\lambda$ by 1.5 (as $(G : N_\beta) = \sigma_\beta < \lambda$).

Case (e): *No previous cases.* As not case (a) w.l.o.g. $\mathrm{Sup}\{\mu_\beta : \beta < \alpha\} = \mu$, hence w.l.o.g. $\sigma_\beta > \aleph_0$. As not case (c) w.l.o.g. $\langle \mu_\beta : \beta < \alpha \rangle$ is eventually constant, so necessarily $\mu_\beta = \mu$ for every β large enough, and w.l.o.g. $\mu_\beta = \mu$ for every $\beta < \alpha$. As not case (b) w.l.o.g. $\langle \sigma_\beta : \beta < \alpha \rangle$ is strictly increasing, $\alpha = \mathrm{cf}\, \alpha$, and let $\alpha(*) = \mathrm{Min}\{\alpha, \mathrm{cf}\, \mu\}$. Note that $\prod_{\beta<\alpha(*)} \sigma_\beta \geq \lambda$: *if* $\alpha(*) = \alpha$ obviously, and *if* $\alpha(*) = \mathrm{cf}\, \mu \neq \alpha$, then necessarily $\mathrm{cf}\, \mu < \mu$, hence $\lambda = 2^\mu = (2^{<\mu})^{\mathrm{cf}\,\mu} \leq \prod_{\beta<\mathrm{cf}\,\mu} \sigma_{\beta+1}$ (note that $2^{<\mu} \leq \sigma_{\beta+1}$, as $\mu_{\beta+1} = \mu$).

We now define by induction on $\beta < \alpha(*)$ for every $\eta \in \prod_{\gamma<\beta} \sigma_{\gamma+1}$ a subgroup H_η of G such that:

(i) $H_{\eta \restriction \gamma} \subseteq H_\eta \subseteq \langle H_{\eta \restriction \gamma}, N_{\gamma+2} \rangle_G$ for $\gamma < l(\eta)$.

(ii) $|H_\eta|$ is strictly smaller than μ.

(iii) If $\beta = l(\eta)$ is a limit ordinal, then $H_\eta = \bigcup_{\gamma<\beta} H_{\eta \restriction \gamma}$.

(iv) The subgroups $\langle H_{\eta \wedge \langle i \rangle}, N_{l(\eta)+2} \rangle_G$ for all $i < \sigma_{l(\eta)+1}$ (for a fix η) are pairwise nonconjugate in G.

The problem is the induction step. Suppose H_η is defined, $l(\eta) = \beta$, and we shall define $H_{\eta \wedge \langle i \rangle}$ $(i < \sigma_{\beta+1})$. Note that as $(G : N_\gamma) = \sigma_\gamma$, σ_γ strictly increasing, clearly $(N_\beta : N_\gamma) = \sigma_\gamma$ for $\beta < \gamma < \alpha(*)$. As not case (d), there is a set $A_\eta \subseteq N_{\beta+1}$ such that $|A_\eta| < \mu$ and $\mathrm{Cm}_{N_{\beta+1}/N_{\beta+2}}(A_\eta/N_{\beta+2})$ has power $< \mu$. So (as in the proof of 1.10) there are $\sigma_{\beta+2}$ elements of $N_{\beta+1}/N_{\beta+2}$ which are pairwise nonconjugate over $A_\eta/N_{\beta+2}$. As $(G : N_{\beta+1}) = \sigma_{\beta+1} < \sigma_{\beta+2}$, there are $(\sigma_{\beta+1})^+$ members of $N_{\beta+1}/N_{\beta+2}$ which are pairwise nonconjugate over $H_\eta \cup A_\eta$ in $G/N_{\beta+2}$. As $2^{|H_\eta \cup A_\eta|} < \sigma_{\beta+1}$, as in 1.10, we can find $a_i \in N_{\beta+1}$ $(i < \sigma_{\beta+1})$ such that the subgroups $\langle H_\eta/N_{\beta+2} \cup A_\eta/N_{\beta+2} \cup \{a_i N_{\beta+2}\} \rangle_{G/N_{\beta+2}}$ are pairwise nonconjugate. Now the subgroups $H_{\eta \wedge \langle i \rangle} \stackrel{\mathrm{def}}{=} \langle H_\eta \cup A_\eta \cup \{a_i\} \rangle_G$ (for $i < \sigma_{\beta+1}$) are as required.

At last the subgroups $\{\bigcup_{\gamma<\alpha(*)} H_{\eta \restriction \gamma} : \eta \in \prod_{\beta<\alpha(*)} \sigma_{\beta+1}\}$ are pairwise nonconjugate subgroups of G, each of power $\leq \mu$, and their number is $\prod_{\beta<\alpha(*)} \sigma_{\beta+1} \geq \lambda$ (by the choice of $\alpha(*)$). This contradicts $G \in \mathcal{P}_\lambda$, hence we finish case (e).

3.6. Lemma. *Suppose* $G \in \mathcal{P}^1_\lambda$, *then* $(G : \mathrm{Min}\, G) < \lambda$.

Proof. Let $L \subseteq \text{Cent}\, G$, $|L| < \mu$, $G/L \in \mathcal{P}_\lambda$. and let $K = \{x \in G : xL \in \text{Min}(G/L)\}$. We know that $G/L \in \mathcal{P}_\lambda$, hence by 3.5, $(G/L : \text{Min}(G/L)) < \lambda$, hence $(G:K) < \lambda$, and clearly K is a normal subgroup of G. Clearly

$$\text{Min}\, G = \bigcap \{N : N \subseteq K,\ N \text{ a normal subgroup of } G \text{ and } (K:N) < \lambda\}.$$

However for any such N, clearly $\langle N, L \cap K \rangle_G = K$ hence $(K:N) \leq |L|$. So, repeating the beginning of the proof of 3.5, $(K : \text{Min}\, G) \leq 2^{|L|} < 2^\mu = \lambda$ but $(G:K) < \lambda$, hence $(G : \text{Min}\, G) < \lambda$.

3.7. Definition. Let $\Omega^m = \{\text{Min}\, G : G \in \mathcal{P}^m_\lambda\}$ (for $m = 0, 1$, if $m = 0$ we may omit it).

3.8. Lemma. *For every* $G \in \Omega^m_\lambda$ ($m = 0, 1$):

(1) $G \in \mathcal{P}^m_\lambda$.

(2) $\text{Min}\, G = G$ (*hence* $\Omega^m_\lambda = \{G \in \mathcal{P}^m_\lambda : G = \text{Min}\, G\}$).

(3) $G = G^{(1)}$.

(4) *Every* $x \in G - \text{Cent}\, G$ *has at least* μ *conjugates* (*in* G).

Proof. As $G \in \Omega^m_\lambda$, let $G = \text{Min}\, G^*$, $G^* \in \mathcal{P}^m_\lambda$.

(1) Immediate: $|G| = \lambda$ as $|G^*| = \lambda$, and by 3.5 and 3.6, $\lambda > (G^* : \text{Min}\, G^*) = (G^* : G)$ and use 1.7.

(2) The problem is that "being a normal subgroup" is not a transitive relation. However being a characteristic subgroup is a transitive relation. Now G is a characteristic subgroup of G^* (by the definition of Min in 3.5) and $\text{Min}\, G$ is a characteristic subgroup of G (similarly), so: $\text{Min}\, G$ is a characteristic subgroup of G^*, hence $\text{Min}\, G$ is a normal subgroup of G^*. Now we know $(G^* : \text{Min}\, G) = (G^* : G)(G : \text{Min}\, G)$, $G^* \in \mathcal{P}^m_\lambda$ (by its choice), $G \in \mathcal{P}^m_\lambda$ (by (1)), $(G^* : G) < \lambda$ (by 3.5 or 3.6), $(G : \text{Min}\, G) < \lambda$ (by 3.5 or 3.6), hence $(G^* : \text{Min}\, G) < \lambda$. So by the definition of $\text{Min}\, G^*$, $\text{Min}\, G^* \subseteq \text{Min}\, G$, but $\text{Min}\, G \subseteq G = \text{Min}\, G^*$, hence $G = \text{Min}\, G^* = \text{Min}\, G$, as required.

(3) We know that $G^{(1)}$ is a normal subgroup of G, and by 1.8(4), $(G : G^{(1)}) < \mu + \aleph_1 < \lambda$. As $G = \text{Min}\, G$ (by (2)) necessarily $G^{(1)} = G$.

(4) Suppose $x \in G - \text{Cent}\, G$ and $A = \{gxg^{-1} : g \in G\}$ has power $< \mu$. Then $K \overset{\text{def}}{=} \text{Cm}_G(A)$ is a normal subgroup of G (as any inner automorphisms of G maps A onto itself, hence $\text{Cm}_G(A)$ onto itself). Also for $a, b \in G$, $aK = bK$ iff $\square^a \restriction A = \square^b \restriction A$ (as both are permutations of A and they are equal iff $\square^{b^{-1}a} \restriction A =$ the identity, i.e., $b^{-1}a \in K$). So $(G:K) \leq |\{\square^g \restriction A : g \in G\}| \leq |\{h : h$ is a permutation of $A\}| \leq 2^{|A|} < 2^\mu = \lambda$ ($2^{|A|} < 2^\mu$ as $\mu = \text{Min}\{\sigma : 2^\sigma \geq \lambda\}$). So $(G:K) < \lambda$, K a normal subgroup of G, hence $\text{Min}\, G \subseteq K$, but $\text{Min}\, G = G$, so $K = G$, but then (as $x \in A$) $x \in \text{Cent}(G)$, contradiction.

3.9 Claim. (1) *If* 3.4A *holds*, $G \in \mathcal{P}^1_\lambda$, $A \subseteq B$ *are subsets of* G *of power* $< \mu$, *then* $\text{Min}\, \text{Cm}_G(B) \subseteq \text{Min}\, \text{Cm}_G(A)$.

(2) *For* $G \in \mathcal{P}^1_\lambda$, $\text{Min}\, G$ *is the maximal subgroup of* G *with no proper normal subgroup of index* $< \lambda$.

(3) *If* 3.4A *holds,* $B \subseteq H \overset{\text{def}}{=} \text{Min}\, \text{Cm}_G(A)$, $A \subseteq G$, $G \in \mathcal{P}^1_\lambda$, *A and B of power* $<\mu$, *then* $\text{Min}\, \text{Cm}_G(A \cup B) = \text{Min}\, \text{Cm}_H(B)$.

Proof. (1) Trivially $\text{Cm}_G(B) \subseteq \text{Cm}_G(A)$. As $\text{Min}\, \text{Cm}_G(A)$ is a normal subgroup of $\text{Cm}_G(A)$ of index $<\lambda$, $\text{Cm}_G(B) \cap \text{Min}\, \text{Cm}_G(A)$ is a subgroup of $\text{Cm}_G(B)$, is a normal subgroup of $\text{Cm}_G(B)$ and

$$(\text{Cm}_G(B) : (\text{Cm}_G(B) \cap \text{Min}\, \text{Cm}_G(A)) \leq (\text{Cm}_G(A) : \text{Min}\, \text{Cm}_G(B)) < \lambda.$$

Hence $\text{Cm}_G(B) \cap \text{Min}\, \text{Cm}_G(A)$ includes $\text{Min}\, \text{Cm}_G(B)$, which gives the desired conclusion.

(2) Trivial: If N is such a subgroup, then $N \cap \text{Min}\, G$ is a normal subgroup of N of index $<\lambda$, hence $N \cap \text{Min}\, G = N$, i.e., $N \subseteq \text{Min}\, G$.

(3) By 3.4A, 1.10, clearly all subgroups mentioned are in $\mathcal{P}^1_\lambda$, $\text{Min}\, \text{Cm}_H(B)$ is a normal subgroup of $\text{Cm}_G(A \cup B)$ and has no proper normal subgroup of index $<\lambda$, hence by 3.9(2),

$$\text{Min}\, \text{Cm}_H(B) \subseteq \text{Min}\, \text{Cm}_G(A \cup B).$$

By 3.9(1), $\text{Min}\, \text{Cm}_G(A \cup B) \subseteq \text{Min}\, \text{Cm}_G(A) = H$, hence trivially $\text{Min}\, \text{Cm}_G(A \cup B) \subseteq \text{Cm}_H(B)$. As $\text{Min}\, \text{Cm}_G(A \cup B)$ has no proper normal subgroup of index $<\lambda$, clearly by 3.9(2),

$$\text{Min}\, \text{Cm}_G(A \cup B) \subseteq \text{Min}\, \text{Cm}_H(B).$$

Together they complete the proof.

3.10. Fact. *For every G and every cardinal* θ:

(1) *There is a (unique) subgroup* $N = \text{Min}_\theta\, G$ *such that N is a maximal subgroup of G satisfying*: (α) $N^{(1)} = N$, *and* (β) *for every* $x \in N - \text{Cent}\, N$, $|\{gxg^{-1} : g \in N\}| \geq \theta$.

(2) $\text{Min}_\theta\, G$ *is a characteristic (hence normal) subgroup of G, and* $\theta \leq \kappa \Rightarrow \text{Min}_\kappa\, G \subseteq \text{Min}_\theta\, G$.

(3) *If* $G \in \mathcal{P}^m_\lambda$, *then* $\text{Min}\, G \subseteq \text{Min}_\theta\, G$, *hence* $(G : \text{Min}_\theta\, G) < \lambda$, *also* $\text{Min}_\theta\, G \in \mathcal{P}^m_\lambda$ *(provided that* $\theta \leq \mu$ *of course).*

(4) *There are an ordinal* α, *a non-decreasing continuous function* $h: \alpha \to \alpha$ *such that* $h(0) = 0$, $h(i) \leq i$, $h(h(i)) = h(i)$, $[h(i) < h(j) \Rightarrow i < h(j)]$ *and a strictly decreasing continuous sequence* $\langle N_i : i \leq \alpha \rangle$ *of subgroups of G such that* $N_0 = G$, $N_\alpha = \text{Min}_\theta\, G$, *and for each i*, $N_{h(i)}$ *is a characteristic subgroup of G and even of* $N_{h(j)}$ *for* $j < i$ *and* $N_{i+1} = N_i^{(1)}$ *or* $N_i = N_i^{(1)}$ *and* $N_{i+1} = \text{Cm}_{N_i}(A)$ *for some set* $A \subseteq N_{h(i)}$, $|A| < \theta$, *where A is the set of conjugates in* $N_{h(i)}$ *of some* $x \in N_{h(i)} - \text{Cent}\, N_{h(i)}$ *and* N_i *is a normal subgroup of* $N_{h(i)}$.

Proof. Easy: Let $\langle N_i : i \leq \alpha \rangle$ be a maximal sequence as required in (4) (except $N_\alpha = \text{Min}_\theta(G)$). By the maximality, N_α satisfies (1)(α), (1)(β): Also if N satisfies (1)(α) and (1)(β), then we can prove by induction on i that $N \subseteq N_i$, hence $N \subseteq N_\alpha$. So N_α is the maximal subgroup of G satisfying (1)(α), (1)(β). So we have proved (1) and (4). Parts (2) and (3) also cause no problem.

3.11. Fact. *If $G \in \Omega^1_\lambda$, then $G/\mathrm{Cent}\, G \in \Omega_\lambda$.*

3.12. Fact. *If* 3.4A *holds, $A \subseteq G \in \Omega^1_\lambda$, $|A| < \mu$, N a normal subgroup of G and* $\mathrm{Min}\, \mathrm{Cm}_G(A) \subseteq N$, *then $N = G$.*

Proof. *Similar to that of* 1.12. *Suppose $N \neq G$.* As $G \in \Omega^1_\lambda$, necessarily $\mathrm{Cent}\, G$ has power $< \mu$, hence w.l.o.g. $\mathrm{Cent}\, G \subseteq A$. As $N \neq G$, N a normal subgroup of G, $G \in \Omega^1_\lambda$ necessarily $(G:N) = \lambda$. Denote $M \stackrel{\text{def}}{=} \mathrm{Min}\, \mathrm{Cm}_G(A)$. We know that $\kappa \stackrel{\text{def}}{=} (\mathrm{Cm}_G(A):M) < \lambda$, hence there is $B \subseteq \mathrm{Cm}_G(A)$, $|B| < \lambda$ such that for every $x \in \mathrm{Cm}_G(A)$ for some $b \in B$, $xb^{-1} \in M$. Now we can define by induction on $i < \lambda$, $a_i \in G$, $a_i \notin \langle N, B, A, a_j : j < i\rangle_G$ (this is possible since otherwise G/N is generated by $\{bN : b \in B\} \cup \{aN : a \in A\} \cup \{a_jN : j < i\}$ which has power $< \lambda$, hence the group G/N has power $< \lambda$, contradicting an assumption).

Now $\langle a_i, A\rangle_G/\mathrm{Cent}\, G$ (for $i < \lambda$) are subgroups of $G/\mathrm{Cent}\, G$, which belong to Ω_λ, hence letting $\chi = 2^{|A| + \aleph_0} + (\mathrm{Cm}_G(A):M)$, w.l.o.g. $\langle a_i, A\rangle_G/\mathrm{Cent}\, G$ $(i < \chi^+)$ are pairwise conjugate in $G/\mathrm{Cent}\, G$. As $\mathrm{Cent}\, G \subseteq A \subseteq \langle a_i. A\rangle_G$ the subgroups $\langle a_i, A\rangle_G$ $(i < \chi^+)$ are pairwise conjugate in G, and let $\Box^{g_i}$ map $\langle a_i, A\rangle_G$ onto $\langle a_0, A\rangle_G$. W.l.o.g. $\Box^{g_i} \restriction A$ is the same as well as $\Box^{g_i}(a_i)$ (for $0 < i < \chi^+$). So for $0 < i, j < \chi^+$, $g_i^{-1} g_j \in \mathrm{Cm}_G(A)$ and $\Box^{g_i^{-1} g_j}$ maps a_j to a_i, so $a_2 \in \langle g_2^{-1} g_1, a_1\rangle_G \subseteq \langle \mathrm{Cm}(A), a_1\rangle_G \subseteq \langle M, B, a_i : i < 2\rangle_G \subseteq \langle N, B, a_i : i < 2\rangle$, contradiction.

4. Direct decomposition and semi-decomposition

As we know that $A \subseteq G$, $G \in \mathcal{P}_\lambda$, $\mu^{|A|} < \lambda$ implies $\mathrm{Cm}_G(A) \in \mathcal{P}_\lambda$, we are able to build groups, which are generated by pairwise commuting subgroups. If we start with G with no center (such G's exist in $\mathcal{P}_\lambda$) we can get direct decomposition (see 4.1). This leads naturally to problems of the uniqueness of a decomposition and a common refinement of two decompositions, and for suitable G's, to the Boolean algebra which the direct summands form. However in our later proofs it seems necessary to demand only that the subgroups are commuting, thus forming a semi-decomposition, semi-summands, etc. We may want to divide by the center, but we are interested in the inner automorphisms of a larger group.

At last we consider problems of the form: When do the groups $H \subseteq K$ have essentially the same decompositions; the natural function is

$$K = \sum_{t \in T} K_t \quad \Rightarrow \quad H = \sum_{t \in T} (H \cap K_t).$$

We complicate this by considering semi-decomposition and decompositions to normal subgroups of some extensions H', K', respectively.

4.1. Definition. (1) $G = \sum_{t \in T} G_t$ (a direct decomposition) if the G_t are pairwise commuting subgroups of G, $G = \langle G_t : t \in T\rangle_G$ and $G_t \cap \langle G_s : s \in T,\ s \neq t\rangle = \{e\}$

(so every $g \in G_t$ has a unique representation $\prod_{t\in T} g_t$ where $g_t = e$ for all but finitely many $t \in T$).

The function $g \to g_t$ is denoted by End_{G,G_t} (more exactly $\mathrm{End}^t_{\langle G_t: t\in T\rangle}$) and is a homomorphism from G onto G_t, which is the identity on G_t.

(2) $G = \sum'_{t\in T} G_t$ (a semi-decomposition) if the G_t are pairwise commuting subgroups of G, and $G = \langle G_t : t \in T\rangle_G$ (so each $g \in G_t$ has a representation $\prod_{t\in T} g_t$ where $g_t = e$ for all but finitely many $t \in T$, but the representation is not necessarily unique). We define $\mathrm{End}^s_{\langle G_t: t\in T\rangle}(A) = \{a_s$: for some $a \in A$, $a = \prod_{t\in T} a_t$, $a_t \in G_t$, and $s \in T\}$. Each G_t is called a semi-summand.

(3) A semi-decomposition $\sum'_{t\in T} G_t$ is called nice if $G_t^{(1)} = G_t$.

4.2. Fact. (1) *If* $G = \sum_{t\in T} G_t$, *then* $G = \sum'_{t\in T} G_t$.

(2) *If* $G = \sum'_{t\in T} G_t$, *then* $\mathrm{Cent}\, G = \sum'_{t\in T} \mathrm{Cent}\, G_t$, $\mathrm{Cent}^\alpha G_t = \sum'_{t\in T} \mathrm{Cent}^\alpha G$ (*for* α *an ordinal or* ∞).

(3) *If* $G = \sum'_{t\in T} G_t$, *then* $G^{(\alpha)} = \sum'_{t\in T} G_t^{(\alpha)}$.

(4) *If* $a \in G = \sum'_{t\in T} G_t$, $a = \prod_{t\in T} a_t$ (*see* 4.1(2)), *then*:

(i) $\mathrm{Cm}_G(a) = \sum' \{\mathrm{Cm}_{G_t}(a_t) : t \in T\}$.

(ii) $\mathrm{Cent}\,\mathrm{Cm}_G(a) = \sum' \{\mathrm{Cent}\,\mathrm{Cm}_{G_t}(a_t) : t \in T\}$.

(iii) $a_t \in \mathrm{Cent}\,\mathrm{Cm}_G(a)$.

(iv) *If* $a \in G_t$, *then* $\mathrm{Cent}\,\mathrm{Cm}_G(a) = \langle \mathrm{Cent}\,\mathrm{Cm}_{G_t}(a), \mathrm{Cent}\, G\rangle_G$.

(5) *If* $G = \sum'_{t\in T} G_t$, *then* $G_t \cap \langle G_s : s \in T, s \neq t\rangle_G \subseteq \mathrm{Cent}\, G$.

(6) *If* N_t *is a normal subgroup of* G_t, *then* $N = \sum'_{t\in T} N_t$ *is a normal subgroup of* G. *If in addition*, $\mathrm{Cent}(G_t) \subseteq N_t$, *then* $G/N = \sum_{t\in T} G_t/N_t$ (*more exactly* $G/N = \sum_{t\in T} \langle G_t, N\rangle_G/N$ *and* $\langle G_t, N\rangle/N$ *is canonically isomorphic to* G_t/N_t).

Proof. Left to the reader.

4.3. Fact. *Suppose* $G = \sum_{t\in T} H_t = \sum_{s\in S} K_s$. *Then*

(1) $G/\mathrm{Cent}\, G = \sum_{t\in T, s\in S} (H_t/\mathrm{Cent}\, G \cap K_s/\mathrm{Cent}\, G)$,

(2) $G^{(1)} = \sum_{t\in T, s\in S} H_t^{(1)} \cap K_s^{(1)}$.

Proof. (2) Let $\bar{H} = \langle H_t : t \in T\rangle$, $\bar{K} = \langle K_s : s \in S\rangle$ (i.e., the sequences, not the subgroups they generate), and $f_t^0 = \mathrm{End}^t_{\bar{H}}$, $f_s^1 = \mathrm{End}^s_{\bar{K}}$. Let $t(*) \in T$, s.t. for $s \in S$, $t \in T$, $f_t^0 f_s^1 \restriction H_{t(*)}$ is a homomorphism from $H_{t(*)}$ into H_t. By 4.2(4)(ii) (applied to f_t^0, then to f_s^1), for all $x \in H_{t(*)}$

$$\mathrm{Cent}\,\mathrm{Cm}_G[f_t^0 f_s^1(x)] \subseteq \mathrm{Cent}\,\mathrm{Cm}_G[f_s^1(x)] \subseteq \mathrm{Cent}\,\mathrm{Cm}_G(x).$$

So if $t \neq t(*)$, by 4.2(4)(iii) and (iv), (i)

$$f_t^0 f_s^1(x) \in \mathrm{Cent}\,\mathrm{Cm}_G(x) = \langle \mathrm{Cent}\,\mathrm{Cm}_{H_{t(*)}}(x), \mathrm{Cent}\, G_q : q \in T\rangle_G.$$

But

$$\langle \mathrm{Cent}\,\mathrm{Cm}_{G_{t(*)}}(x), \mathrm{Cent}\, H_q : q \in T\rangle_G \cap H_t = \mathrm{Cent}\, H_t,$$

hence $f_t^0 f_s^1(x) \in \mathrm{Cent}\, H_t \subseteq \mathrm{Cent}\, G$. Now for $x, y \in H_{t(*)}$, $f_t^0 f_s^1(xyx^{-1}y^{-1}) = (f_t^0 f_s^1(x))(f_t^0 f_s^1(y))(f_t^0 f_s^1(x)^{-1}(f_t^0 f_s^1(y))^{-1}$ which is e by the last sentence; so

$f_t^0 f_s^1 \restriction H_{t(*)}$ is trivial on $H_{t(*)}^{(1)}$ and it induces a trivial homomorphism from $H_{t(*)}/\text{Cent}\, H_{t(*)}$ into $G_t/\text{Cent}\, G_t$. The rest should be clear (or see 4.8's proof).

4.4. Fact. *If $G = G^{(1)}$ or* Cent $G = \{e\}$ *and $G = \sum_{t\in T} H_t = \sum_{s\in S} K_s$, then*

$$G = \sum_{t\in T, s\in S} H_t \cap K_s.$$

Proof. By 4.3(2), if $G = G^{(1)}$ and by 4.3(1), if Cent $G = \{e\}$.

4.5. Fact. *If $G = G^{(1)}$ and $G = \sum'_{t\in T} H_t = \sum'_{s\in T} K_s$, then $G = \sum'_{t\in T, s\in S} H_t \cap K_s$.*

Proof. The only nontrivial point is why $G \subseteq G' \stackrel{\text{def}}{=} \langle H_t \cap K_s : t \in T,\ s \in S\rangle_G$.

4.5A. Subfact. $G = \sum'_{t\in T} H_t^{(\infty)} = \sum'_{t\in T} K_s^{(\infty)}$.

This is so because $G = G^{(\infty)} = \sum'_{t\in T} H_t^{(\infty)}$. So w.l.o.g. $H_t^{(\infty)} = H_t$, $K_s^{(\infty)} = K_s$ (as we just need that they generate G).

It is enough to prove that every $a \in H_t$ belongs to G'. As $G = \sum'_{s\in S} K_s$, clearly $a = \prod_{s\in S} a_s$ for some $a_s \in K_s$, hence it is enough to prove that w.l.o.g. for each $s \in S$, $a_s \in H_t$ (remember the a_s are not uniquely defined). But we have assumed $H_t = H_t^{(1)}$. First suppose a is a commutator $a = xyx^{-1}y^{-1}$ for some $x, y \in H_t$, and let $x = \prod_{s\in S} x_s$, $y = \prod_{s\in S} y_s$, where x_s, $y_s \in K_s$. Easily $a = xyx^{-1}y^{-1} = \prod_{s\in S} x_s y_s x_s^{-1} y_s^{-1}$. As $x = \prod_{s\in S} x_s$, $x_s \in K_s$, clearly for each $s \in S$, $\text{Cent}\, \text{Cm}_G(x_s) \subseteq \text{Cent}\, \text{Cm}_G(x)$, hence for some $b_s \in H_t$, $b_s^{-1} x_s \in \text{Cent}\, G$.

Similarly, for some $c_s \in H_t$, $c_s^{-1} y_s \in \text{Cent}\, G$. Now

$$\begin{aligned} x_s y_s x_s^{-1} y_s^{-1} &= b_s (b_s^{-1} x_s) c_s (c_s^{-1} y_s)(b_s^{-1} x_s)^{-1} b_s^{-1} (c_s^{-1} y_s)^{-1} c_s^{-1} \\ &= b_s c_s b_s^{-1} c_s^{-1} \in H_t \end{aligned}$$

and it also belongs to K_s; hence it belongs to $H_t \cap K_s$.

So $a = xyx^{-1}y^{-1} = \prod_s x_s y_s x_s^{-1} y_s^{-1} = \prod b_s c_s b_s^{-1} c_s^{-1}$ is as required. As the commutators in H_t generate H_t, the proof is complete.

4.6. Definition. (1) For any group G, $G = G^{(1)}$ or Cent $G = \{e\}$ we define the structure BA(G): its elements are the direct summands of G, i.e., $\{I$: for some J, $G = I + J\}$; its operations are union and intersection:

$$I \cup J \stackrel{\text{def}}{=} \langle I, J\rangle_G,$$

$$I \cap J = \text{the usual intersection.}$$

(2) If $G = G^{(1)}$, BA$'(G)$ is the following structure: its elements are the semi-summands I of G satisfying $I = I^{(1)}$. The operations are as in (1). Note $I \in \text{BA}'(G)$ is commutative iff it is trivial.

4.7. Fact. (1) *$\mathrm{BA}(G)$ is a Boolean algebra with zero $\{e\}$, one G, and if $G = I + J$, I is the complement of J.*

(2) *For $I \in \mathrm{BA}(G)$ there is a unique endomorphism $\mathrm{End}_G^I\, G$ from G onto I which is the identity on I such that $\mathrm{End}_G^I = \mathrm{End}^0_{\langle I, I_1\rangle}$ where $G = I_0 + I_1$, $I_0 = I$.*

(3) *If $G = G^{(1)}$, $\mathrm{BA}'(G)$ is a Boolean algebra with zero $\{e\}$, one G; and for every $I \in \mathrm{BA}'(G)$ there is a unique $J = J^{(1)}$, $G = I +' J$.*

(4) *For $I \in \mathrm{BA}'(G)$ we define End_G^I as in* 4.7(2), 4.1(2).

Proof. (1) Apply 4.4.

(2) Follows.

(3) Apply 4.5.

4.8. Fact. (1) *If $G \in \Omega^1_\lambda$, then G has no nontrivial direct summand of power $< \lambda$, nor such a noncommutative semi-summand.*

(2) *If $G = \sum_{t \in T} G_t$, then for any $A \subseteq G$*

$$\mathrm{Cm}_G(A) = \sum_{t \in T} \mathrm{Cm}_{G_t}(\mathrm{End}_G^{G_t} A).$$

Similarly for $G = \sum'_{t \in T} G_t$.

(3) *If $G = \sum_{t \in T} G_t \in \mathcal{P}_\lambda$, then for each $t \in T$, $[|G_t| = \lambda \Rightarrow G_t \in \mathcal{P}_\lambda]$ and* $\mathrm{Min}\, G = \sum \{\mathrm{Min}\, G_t : t \in T, |G_t| = \lambda\}$. *Similarly for $G = \sum'_{t \in T} G^0_t$, (no G^0_t is commutative by* 1.3) *and/or for $\mathcal{P}^1_\lambda$.*

(4) *For no $G \in \Omega_\lambda$ there are an infinite T and G_t $(t \in T)$ each of power λ such that $G = \sum'_{t \in T} G_t$. This holds for $G \in \Omega^1_\lambda$ too.*

(5) *There is no $G \in \mathcal{P}_\lambda$, $G = G^{(1)}$ or* $\mathrm{Cent}\, G = \{e\}$, *and $a_i \in G\text{-}\{e\}$ (for $i < \mu$) such that for $j < i$ there is a direct decomposition $I + J$ of G such that $a_i \in I$, $a_j \in J$.*

(6) *If in* (5) *$a_i \in G$-*$\mathrm{Cent}\, G$, *we can use semi-decomposition even for $G \in \mathcal{P}^1_\lambda$.*

Proof. (1) If I is a direct summand of G, then for some J, $G = I + J$, so J is a normal subgroup of G, $(G : J) = |I|$. As $G = \mathrm{Min}\, G$, $1 < |I| < \lambda$ is impossible. The proof for semi-summand is similar.

(2) Note that $a, b \in G$ commute iff for every $s \in T$, $\mathrm{End}^s_{\langle G_t : t \in T\rangle}(a)$ commutes with $\mathrm{End}^s_{\langle G_t : t \in T\rangle}(b)$.

(3) Note that subgroups of G_t are conjugate in G iff they are conjugate in G_t, hence $(*)$ for every σ, $\mathrm{nc}_{\leq\sigma}(G_t) \leq \mathrm{nc}_{\leq\sigma}(G)$.

So if $G \in \mathcal{P}_\lambda$, $|G_t| = \lambda$, clearly $G_t \in \mathcal{P}_\lambda$. For any t, $\mathrm{Min}\, G_t + \sum_{s \neq t} G_s$ is a normal subgroup of G of index $(G_t : \mathrm{Min}\, G_t) < \lambda$, hence it includes $\mathrm{Min}\, G$. We can conclude that:

$$\mathrm{Min}\, G \subseteq \sum \{\mathrm{Min}\ G_t : |G_t| = \lambda\}.$$

Suppose equality fails, and $x \in \sum \{\mathrm{Min}\, G_t : t \in T\}$ but $x \notin \mathrm{Min}\, G$, so $x = \prod_{t \in T} x_t$, $x_t \in \mathrm{Min}\, G_t$. Hence we can assume $x \in G_t$ for some t. Necessarily $|G_t| = \lambda$, and

$x \notin (\text{Min}\, G) \cap G_t$, but this is a normal subgroup of G_t of index $< \lambda$, hence should include Min G_t, but x does not belong to it, contradiction.

(4) W.l.o.g., T is the set of natural numbers. For each $n < \omega$, $|G_n| = \lambda$, we know $|\text{Cent}\, G_n| < \mu$.

We now choose for $n < \omega$, $i < \lambda$, $a_{n,i} \in G_n$ such that $\langle a_{n,i}, \text{Cent}\, G_n \rangle_{G_n}$ are pairwise distinct, and define for $i < \lambda$, $H_i = \langle a_{n,i} : n < \omega \rangle_G$.

It is clear that $a_{n,i} \in H_i \cap G_n \subseteq \langle a_{n,i}, \text{Cent}\, G_n \rangle_G$. Now suppose that $i \neq j < \lambda$, $g \in G$ and $\square^g$ maps H_i onto H_j. Clearly for some n, $g \in \sum_{m<n} G_m$, apply $\text{End}^n_{\langle G_m : m < \omega \rangle}$ to H_i, H_j, g and we see that $\square^g$ maps $\langle a_{n,i}, \text{Cent}\, G_t \rangle_{G_t}$ onto $\langle a_{n,j}, \text{Cent}\, G_t \rangle_{G_t}$, contradiction. For $G \in \Omega^1_\lambda$, the same proof works.

4.8A. Remark. Really, the proof shows that e.g., if $|T| \leqslant \sigma$, e.g., $|G_t| \geqslant 2^\sigma$,

$$\text{nc}_\sigma\Big(\sum_{t \in T} G_t\Big) \geqslant \prod_{t \in T} |G_t|^\sigma / (\text{the ideal of finite subsets of } T).$$

(5) For any set $S \subseteq \mu$ let

$$H_S = \langle a_i : i \in S \rangle_G.$$

By the hypothesis for $j < i$, a_i commutes with a_j, hence H_S is commutative, so suppose S, T are distinct subsets of μ, $g \in G$, but $\square^g$ maps H_S onto H_T. As $S \neq T$ w.l.o.g., there is $\alpha \in S - T$, so as $\square^g(a_\alpha) \in H_T$ there are $\beta_1, \ldots, \beta_n \in T$, and $m(1), \ldots, m(n) \in \mathbb{Z}$ such that $\square^g(a_\alpha) = \prod_{k=1}^n (a_{\beta_k})^{m(k)}$ (remember H_S is commutative.) Note that $\beta_k \neq \alpha$ for $k = 1, n$.

For each k there is a direct decomposition $G = I_k + J_k$, $a_\alpha \in I_k$, $a_{\beta_k} \in J_k$. So $a_\alpha \in I \overset{\text{def}}{=} \bigcap_{k=1}^n I_k$, I is a direct summand of G, $G = I + J$, and $J_k \subseteq J$ for $k = 1, n$. Hence $a_{\beta_k} \in J_k \subseteq J$. Now $\square^g a_\alpha = \prod_k (a_{\beta_k})^{m(k)}$ is trivially contradictory (as $a_\alpha \neq e$).

(6) Similar, or use $G/\text{Cent}\, G$.

4.9. Definition. (1) For a group G and $A \subseteq G$ let $\langle A \rangle^{\text{cg}}_G$ be $\langle g a g^{-1} : a \in A, g \in G \rangle_G$, or equivalently the smallest normal subgroup of G which includes A.

(2) Let $\text{cg}(G) = \text{Min}\{|A| : G = \langle A \rangle^{\text{cg}}_G\}$.

(3) For a group K and a normal subgroup H let

$$\text{cg}_K(H) = \text{cg}(H, K) = \text{Min}\{|A| : H = \langle A \rangle^{\text{cg}}_K\}.$$

4.10. Definition. (1) We say $\sum_{s \in S} H_s$ is a direct decomposition of H inside K if $H = \sum_{s \in S} H_s$, and each H_s is a normal subgroup of K. Similarly for semi-direct decompositions.

(2) $\text{BA}(H, K) = \text{BA}_K(H) = \{I \in \text{BA}(H) : I$ is a normal subgroup of $K\}$ where H is a normal subgroup of K.

(3) $\text{BA}'_K(H) = \{I \in \text{BA}'(H) : I$ is a normal subgroup of $K\}$.

4.11. Fact. (1) *If H is a normal subgroup of K, then* $\mathrm{BA}_K(H)$ *is a Boolean subalgebra of* $\mathrm{BA}(H)$.

(2) *If H is a normal subgroup of K, $H = H^{(1)}$, then* $\mathrm{BA}'_K(H)$ *is a Boolean subalgebra of* $\mathrm{BA}'(H)$.

Proof. (1) Clearly $\mathrm{BA}_K(H)$ is closed under the operations of union and intersection. Obviously, $\{e\}$, $H \in \mathrm{BA}_K(H)$. As for complementation if $I + J = H$, $I \in \mathrm{BA}_K(H)$, then I is a normal subgroup of K. So for any $a \in H$, $\square^a$ maps I onto itself, and $H = \square^a(H) = \square^a(I) + \square^a(J) = I + \square^a(J)$, hence necessarily $\square^a(J) = J$, so J is a normal subgroup of K hence $J \in \mathrm{BA}_K(H)$.

(2) Similarly.

4.12. Fact. *Suppose $H \subseteq K$, and for every $x \in H$*

$(*)$ Cent $\mathrm{Cm}_K(x) \subseteq H$ *or even* $y \in K \wedge$ Cent $\mathrm{Cm}_K(y) \subseteq$ Cent $\mathrm{Cm}_K(x) \Rightarrow y \in H$.

Then (1) *If $K = \sum_{t \in T} K_t$, then $H = \sum_{t \in T}(H \cap K_t)$.*

(2) *If $K = \sum'_{t \in T} K_t$, then $H = \sum'_{t \in T}(H \cap K_t)$.*

Remark. In $(*)$ the second condition is weaker than the first.

Proof. In both cases the least trivial point is $H = \langle H \cap K_t : t \in T \rangle_K$. For this it is enough to prove that if $x \in H$, then for some $x_t \in H \cap K_t$, $x = \prod_{t \in T} x_t$. By the hypothesis, $x = \prod_{t \in T} x_t$ for some $x_t \in K_t$. But by $(*)$ $x \in H \Rightarrow x_t \in H$ for each $t \in T$.

4.13. Claim. *Suppose H is a normal subgroup of K, and $\langle A \rangle_K^{\mathrm{cg}} = H$.*

If (K^1, H^1) is an elementary submodel (see Ap 1) *of (K, H), $A \subseteq H^1$, then*

$(*)$ *For any direct decomposition $\sum_{t \in T} H_t^1$ of H^1 inside K^1 there is a unique direct decomposition $\sum_{t \in T} H_t$ of H inside K such that $H_t \cap H^1 = H_t^1$.*

$(**)$ *For any nice semi-decomposition $\sum'_{t \in T} H_t^1$ of H^1 inside K^1 there is a unique nice semi-decomposition $\sum_{t \in T} H_t$ of H inside K such that $H_t \cap H^1 = H_t^1$* (see 4.1(3)).

Proof. As the proofs are similar we give them together; only (b) is for $(**)$ only, (e) for $(*)$ only.

We define $H_t = \langle H_t^1 \rangle_K^{\mathrm{cg}}$, and let $A_t = \{aba^{-1} : a \in K,\ b \in H_t^1\}$.

(a) H_t is a normal subgroup of K, $H_t^1 \subseteq H_t$. (This is obvious.)

(b) $H_t = H_t^{(1)}$ (for $(**)$ only).

If $c \in A_t$, then $c = aba^{-1}$, $b \in H_t^1$, $a \in K$; and as $(H_t^1)^{(1)} = H_t^1$, $b = \prod_{m=1}^n x_m y_m x_m^{-1} y_m^{-1}$, $x_m, y_m \in H_t^1$(i.e., b is the product of commutators). Hence

$$aba^{-1} = \prod_{m=1}^{n} (ax_m a^{-1})(ay_m a^{-1})(ax_m a^{-1})^{-1}(ay_m a^{-1})^{-1} \in H_t^{(1)};$$

hence $A_t \subseteq H_t^{(1)}$, so $H_t = H_t^{(1)}$.

(c) For $t \neq s$ (in T), H_t and H_s commutes.

For suppose $x \in H_t$, $y \in H_s$ do not commute. As $x \in H_t$ there are $n < \omega$, $x_1, \ldots, x_n \in H_t^1$ and $a_1, \ldots, a_n \in K$ such that $x = \prod_{i-1}^n (a_i x_i a_i^{-1})$. Similarly, there are $m < \omega$, $y_1, \ldots, y_m \in H_s^1$ and $b_1, \ldots, b_m \in K$ such that $y = \prod_{i=1}^m (b_i y_i b_i^{-1})$. So

$$(K, H) \models (\exists z_1, \ldots, z_n)(\exists u_1, \ldots, u_m)$$
$$\left[\prod_{i=1}^n (z_i x_i z_i^{-1}) \prod_{j=1}^m u_j y_j u_j^{-1} \neq \prod_{j=1}^m (u_j y_j u_j^{-1}) \prod_{i=1}^n (z_i x_i z_i^{-1})\right]$$

(so $x_i, \ldots, x_n, y_1, \ldots, y_m$ are here parameters; the formula is satisfied as the $a_1, \ldots, a_n, b_1, \ldots, b_m$ are witnesses for the existence).

As (K^1, H^1) is an elementary submodel of (K, H) and as the parameters $x_1, \ldots, x_n, y_1, \ldots, y_n$ are in $H^1 \subseteq K^1$, also (K^1, H^1) satisfies this formula, hence there are $a_1', \ldots, a_n', b_1', \ldots, b_m' \in K^1$ such that $x' = \prod_{i=1}^n a_i' x_i (a_i')^{-1}$ and $y' = \prod_{j=1}^m b_j' y_j (b_j')^{-1}$ do not commute. As $x_1, \ldots, x_n \in H_t^1$, $a_1', \ldots a_n' \in K^1$, and H^1 is a normal subgroup of K^1, clearly $x' \in H_t^1$. Similarly, $y' \in H_s^1$. But H_t^1, H_s^1 commute, contradiction.

(d) $H' \stackrel{\text{def}}{=} \langle \bigcup_{t \in T} H_t \rangle_H$ is a normal subgroup of K, is included in H, it includes H^1, hence A. So $H = \langle A \rangle_K^{\text{cg}} \subseteq \langle H' \rangle_K^{\text{cg}} = \langle H' \rangle = H' \subseteq H$ as required.

(e) $H_t \cap \langle \bigcup_{s \neq t} H_s \rangle_K = \{e\}$ (for $(*)$ only). The proof is like that of (c).

(f) Uniqueness of H_t. If H_t' are other candidates, first prove $H_t \subseteq H_t'$, then an inequality contradicts $H = \sum_{t \in T} H_t$ (or $H = \sum_{t \in T}' H_t$ using niceness).

5. A kind of derivative and required subgroups

When we are dealing with $G \in \mathcal{P}_\lambda$, we have found that for $A \subseteq G$, $|A| < \mu$, $\mathrm{Cm}_G(A) \in \mathcal{P}_\lambda$. We want to exploit this to prove that every G has subgroups of many isomorphism types, this being proved by induction of some notion of depth of groups of those isomorphic types. So in stage α, we shall try to build conjugate subgroups $H \subset K$ of G such that $\mathrm{Cm}_G(H) \cap K$ includes a direct sum L of many subgroups of smaller depth. If $\square^g$ maps K onto H, then $\langle L, g \rangle_G$ has quite a clear structure. Note that we do not have much control on the center (hence we shall divide by it in 5.1).

In this section we deal with a suitable notion of depth.

5.1. Definition. (1) For any group G let

$$\mathcal{D}v(G) = \langle \{x\colon \text{in } G/\mathrm{Cent}^\infty G,\ x\,\mathrm{Cent}^\infty G \text{ belongs to a normal countable abelian subgroup}\} \rangle_G.$$

(2) For any group G and ordinal α we define $\mathcal{D}v^\alpha(G)$ by induction on α:

(α) $\mathcal{D}v^0(G) = \{e\}$,

(β) $\mathcal{D}v^{\alpha+1}(G) = \{x : x\mathcal{D}v^\alpha(G) \in \mathcal{D}v(G/\mathcal{D}v^\alpha(G)),$

(γ) for $\alpha = \delta$ a limit ordinal $\mathcal{D}v^\delta(G) = \bigcup_{\beta < \gamma} \mathcal{D}v^\beta(G)$.

(3) For any group G, $\mathcal{D}v^\infty(G) = \bigcup_\alpha \mathcal{D}v^\alpha(G)$.

5.2. Claim. (1) $\mathscr{D}v^1(G) = \mathscr{D}v(G)$.

(2) $\mathscr{D}v^{\alpha+\beta}(G) = \{x \in G : x\mathscr{D}v^{\alpha}(G) \in \mathscr{D}v^{\beta}(G/\mathscr{D}v^{\alpha}(G))\}$.

(3) *$\mathscr{D}v^{\alpha}(G)$ is a normal, and even characteristic subgroup of G.*

(4) $\mathscr{D}v^{\alpha}(G) \subseteq \mathscr{D}v^{\beta}(G) \subseteq \mathscr{D}v^{\infty}(G)$ *if* $\alpha \leqslant \beta$.

(5) *If* $\mathscr{D}v^{\alpha}(G) = \mathscr{D}v^{\alpha+1}(G)$, *then* $\mathscr{D}v^{\alpha}(G) = \mathscr{D}v^{\beta}(G)$ *for every* $\beta \geqslant \alpha$, *hence* $\mathscr{D}v^{\alpha}(G) = \mathscr{D}v^{\infty}(G)$.

(6) *For some* $\alpha < |G|^+$, $\mathscr{D}v^{\alpha}(G) = \mathscr{D}v^{\infty}(G)$.

(7) $\mathscr{D}v^{\alpha}(G/\mathrm{Cent}^{\infty}(G)) = \mathscr{D}v^{\alpha}(G)/\mathrm{Cent}^{\infty}(G)$.

(8) *For any homomorphism h from G onto K, h maps $\mathscr{D}v^{\alpha}(G)$ into $\mathscr{D}v^{\alpha}(K)$.*

Proof. Immediate.

5.3. Claim. *For any pairwise commuting subgroups G_i $(i < \alpha)$ of G, and for any γ*

$$\mathscr{D}v^{\gamma}\Big(\sum_{i<\alpha}{}' G_i\Big) = \sum_{i<\alpha}{}' \mathscr{D}v^{\gamma}(G_i),$$

also

$$\mathscr{D}v^{\infty}\Big(\sum_{i<\alpha}{}' G_i\Big) = \sum_{i<\alpha}{}' \mathscr{D}v^{\infty}(G_i).$$

Proof. Easy.

5.4. Definition. We call H a γ-required group if $\mathscr{D}v^{\gamma+1}(H) = H \neq \mathscr{D}v^{\gamma}(G)$, H has power $\leqslant |\gamma| + \aleph_0$ and $H/\mathrm{Cent}^{\infty} H$ is indecomposable when $\gamma > 0$.

5.5. Definition. For any group G let $\gamma(G)$ be the first ordinal γ such that G has no β-required subgroup for $\gamma \leqslant \beta < (\aleph_0 + |\gamma|)^+$.

5.6. Claim. (1) *If* $L \subseteq K$, *then* $\gamma(L) \leqslant \gamma(K)$.

(2) $\gamma(G) = \gamma(G/\mathrm{Cent}^{\infty} G)$.

(3) *Any abelian nontrivial group is a γ-required subgroup for $\gamma = 0$.*

(4) *For any nontrivial G, $\gamma(G) \geqslant 0$ and $\gamma(G) \leqslant |G|^+$, even $\gamma(G) < \aleph_0 + |G|^+$.*

Proof. Trivial.

5.7. Lemma. *For every $G \in \mathscr{P}_{\lambda}$, for some $A \subseteq G$, $|A| < \mu$ and $\gamma(\mathrm{Cm}_G(A)) < \mu$.*

Proof. Suppose that there is no such A. We define by induction on $\alpha < \mu$ a subgroup H_{α} of G such that:

(i) H_{α} has power $\leqslant |\alpha| + \aleph_0$,

(ii) $H_{\alpha} \subseteq \mathrm{Cm}_G(\bigcup_{\beta<\alpha} H_{\beta})$,

(iii) H_{α} is an γ_{α}-required group for some $\gamma_{\alpha} < \aleph_0 + |\alpha|^+$, $\gamma_{\alpha} > \alpha$.

In stage α, $|\bigcup_{\beta<\alpha} H_{\beta}| \leqslant \sum_{\beta<\alpha} |H_{\beta}| \leqslant |\alpha| \cdot (|\alpha| + \aleph_0) < \mu$, so as we have assumed that there is no A as mentioned in the lemma, necessarily there is γ_α, $\alpha \leqslant \gamma_\alpha < \aleph_0 + |\alpha|^+$ and a γ_α-required subgroup H_α of $\mathrm{Cm}_G(\bigcup_{\beta<\alpha} H_\beta)$, w.l.o.g. $\gamma_\alpha \neq \gamma_\beta$ for $\alpha \neq \beta$.

Now for any set $S \subseteq \{\gamma : \omega \leqslant \gamma < \mu\}$ let $K_S = \langle H_\gamma : \gamma \in S\rangle_G$, so it is enough to prove that K_S, K_T are nonconjugate subgroups of G for $S \neq T$. We shall prove more: that K_S, K_T are not isomorphic. If h is an isomorphism from K_S onto K_T, then it induces an isomorphism h' from $K_S/\mathrm{Cent}^\infty K_S$ onto $K_T/\mathrm{Cent}^\infty K_T$. As by (ii) the H_α's are pairwise commuting $K_S = \sum'_{\alpha\in S} H_\alpha$, so by 4.2(2), $\mathrm{Cent}^\infty K_S = \sum'_{\alpha\in S} \mathrm{Cent}^\infty H_\alpha$, and $K_S/\mathrm{Cent}^\infty K_S = \sum_{\alpha \in S} H_\alpha/\mathrm{Cent}^\infty H_\alpha$. The same holds for K_T; so as $H_\alpha/\mathrm{Cent}^\infty H_\alpha$ is indecomposable (remember Definition 5.4) by 4.4 for some one-to-one function f from S onto T, h' maps $H_\alpha/\mathrm{Cent}^\infty K_S$ onto $H_{f(\alpha)}/\mathrm{Cent}^\infty K_T$ (for $\alpha \in S$). But by 5.2(7) this easily implies $f(\alpha) = \alpha$ (for $\alpha \in S$), hence $S = T$.

5.8. Lemma. *Suppose H_m ($m \in \mathbb{Z}$) are pairwise commuting subgroups of G, F^m_n is isomorphism from H_m onto H_n (for n, $m \in \mathbb{Z}$) $F^m_m =$ the identity, $F^m_n F^k_m = F^k_n$, $H^* = \langle H_m : m \in \mathbb{Z}\rangle_G$, K is a subgroup of $\bigcap_{m\in\mathbb{Z}} \mathrm{Cent}\, H_m$, $K \neq H_0$, $K = H_m \cap \langle H_k : k \in \mathbb{Z},\ k \neq m\rangle_G$ for each $m \in \mathbb{Z}$ and F^n_m maps K onto K for every n, m.*

Suppose further $g \in G$, $F^m_{m+1} \subseteq \Box^g$ for every m and let $H \stackrel{\mathrm{def}}{=} \langle H^*, g\rangle_G$ and assume $H_n \neq K$. Then

(a) *$L \stackrel{\mathrm{def}}{=} \mathrm{Cent}^\infty H$ is a subgroup of K.*

(b) *K is a normal subgroup of H.*

(c) *$H^*/K = \sum'_{m\in\mathbb{Z}} H_m/K$.*

(d) *$K \subseteq \mathscr{D}v(H)$.*

(e) (i) *$\mathscr{D}v^\gamma(H_0) \neq H_0$ for every $\gamma < \beta$ implies $\mathscr{D}v^\beta(H) \neq H$,* (ii) *$\mathscr{D}v^\infty(H_m) = H_m$ implies $\mathscr{D}v^\infty(H) = H$, and* (iii) *$\mathscr{D}v^\infty(H_m) \neq H_m$ implies $\mathscr{D}v^\infty(H) = \sum'_{m\in\mathbb{Z}} \mathscr{D}v^\infty(H_m)$.*

(f) *$H/\mathrm{Cent}^\infty H$ is indecomposable.*

Proof. First note that (b), (c) are trivial.

(a) Suppose $(yK) \in H/K - \{eK\}$ is in the center of H/K. In H/K, any yK has a unique representation $(gK)^n \prod_{m\in\mathbb{Z}} (y_m K)$, $y_m \in H_m$, $\{m : y_m K \neq K\}$ is finite. So, if $y_m K \neq K$, then for some r, $y_{m-r} \in K$; as $yK \in \mathrm{Cent}(H/K)$, $yK = \Box^{(gK)^r}(yK)$, but the latter is an element of $\langle H_i, K : i \neq m\rangle_G$, hence $y_m \in K$, contradiction. Hence $yK \in \{g^r K : r \in \mathbb{Z}\}$, but for $r \neq 0$ trivially $g^r K \notin \mathrm{Cent}(H/K)$, so $\mathrm{Cent}(H/K) = \{K\}$, hence (a) holds.

(d) For every $a \in K$, $A_a = \{g^r a g^{-r} : r \in \mathbb{Z}\}$ is a subset of K (as F^n_m maps K onto K), and is closed under conjugation in G [by $\Box^g$ by its definition, under $\Box^b$, $b \in H^*$, as $A_a \subseteq K \subseteq \mathrm{Cent}\, H^*$, and those elements generate H]. So $\langle A_a\rangle_H$ is a normal subgroup; as A_a is countable abelian, $\langle A_a\rangle_H$ is countable abelian, and clearly $\langle A_a\rangle_H \subseteq K$ (as $A_a \subseteq K$). Hence $\langle A_\alpha\rangle_H \subseteq \mathscr{D}v(H)$. As $a \in K$ was arbitrary, $K \subseteq \mathscr{D}v(H)$.

(e) (i) By 5.2(8) w.l.o.g. $K = \{e\}$. We now prove by induction on β that

(*) if for $\gamma < \beta$, $\mathscr{D}v^\gamma(H_0) \neq H_0$ then

$$\mathscr{D}v^\beta(H) \cap H_m \subseteq \mathscr{D}v^\beta(H_m), \qquad \mathscr{D}v^\beta(H) = \sum_{m\in\mathbb{Z}} (\mathscr{D}v^\beta(H) \cap H_m).$$

For $\beta = 0$ or β limit, this is trivial. So let $\beta = \alpha + 1$, so $\mathscr{D}v^{\alpha}(H_m) \neq H_m$, and it is easy to compute for $H/\mathscr{D}v^{\alpha}(H)$: it has a trivial center and $\mathscr{D}v(H/\mathscr{D}v^{\alpha}(H))$ is generated by $\{x\,\mathscr{D}v^{\alpha}(H)\colon m \in \mathbb{Z},\ x \in H_m,\ x(\mathscr{D}v^{\alpha}(H) \cap H_m) \in \mathscr{D}v(H_m/(\mathscr{D}v^{\alpha}(H) \cap H_m))\}$. Now everything is easily checked.

Before we continue note

5.8A. Fact. *If h is a homomorphism from G onto K, and the kernel of h is included in $\mathscr{D}v^{\infty}(G)$, then h maps $\mathscr{D}v^{\infty}(G)$ onto $\mathscr{D}v^{\infty}(K)$.*

This follows by

5.8B. Fact. *$\mathscr{D}v^{\infty}(G)$ is the minimal normal subgroup K of G such that G/K has trivial center and $\mathscr{D}v(G/K) = \{e_{G/K}\}$ (equivalently, G/K has no nontrivial normal countable commutative subgroups) and $\mathscr{D}v^{\infty}(G) = \bigcap\{K\colon K$ a normal subgroup of G such that $\mathscr{D}v(G/K) = \{e_{G/K}\}\}$.* [Note that any countable subgroup of the center of a group is a countable normal commutative subgroup.]

(ii) By 5.8A w.l.o.g. $K = \{e\}$. As in the proof of (i) we can prove by induction on β that

(*) if for $\gamma < \beta$, $\mathscr{D}v^{\gamma}(H) \cap H_0 \neq H_0$ then

$$\mathscr{D}v^{\beta}(H) = \sum_{m\in\mathbb{Z}} (\mathscr{D}v^{\beta}(H) \cap H_m).$$

First assume $\mathscr{D}v^{\infty}(H) \cap H_0 \neq H_0$. Then (by 5.2(4), (6)) for every α, $\mathscr{D}v^{\alpha}(H) \cap H_0 \subseteq \mathscr{D}v^{\infty}(H) \cap H_0 \neq H_0$ hence $\mathscr{D}v^{\infty}(H) = \sum_{m\in\mathbb{Z}} (\mathscr{D}v^{\infty}(H) \cap H_m)$. As $\mathscr{D}v^{\infty}(H)$ is a characteristic subgroup of H, $\Box^{g}$ maps $\mathscr{D}v^{\infty}(H) \cap H_m$ onto $\mathscr{D}v^{\infty}(H) \cup H_{m+1}$.

Clearly $\mathscr{D}v^{\infty}(H) \cap H_0$ is a proper normal subgroup of H_0. But we have assumed $\mathscr{D}v^{\infty}(H_0) = H_0$, so by 5.8B $H_0/(\mathscr{D}v^{\infty}(H) \cap H_0)$ has a countable normal commutative subgroup, and let $x(\mathscr{D}v^{\infty}(H) \cap H_0)$ be a nontrivial element of such subgroup. Now the normal subgroup of $H/\mathscr{D}v^{\infty}(H)$ which $x\mathscr{D}v^{\infty}(H)$ generates, is countable normal and commutative, contradicting 5.8B.

So $H_0 \subseteq \mathscr{D}v^{\infty}(H)$ hence $H_m \subseteq \mathscr{D}v^{\infty}(H)$ for $m \in \mathbb{Z}$ hence $H^* \subseteq \mathscr{D}v^{\infty}(H)$. But $H/\mathscr{D}v^{\infty}(H)$, being a homomorphic image of H/H^*, is commutative and countable, so by 5.8A $\mathscr{D}v^{\infty}(H) = H$.

(iii) Simpler than the proof of (ii).

(f) Suppose $L \subseteq I_1$, $L \subseteq I_2$ and $H/L = I_1/L + I_2/L$ (and $I_1 \neq L$, $I_2 \neq L$), and let $gL = g_1L + g_2L$ where $g_1 \in I_1$, $g_2 \in I_2$.

First assume $I_1 \subseteq K$, and choose $b \in I_1 - L$. Then bL commutes with g_2L (as $b \in I_1$, $g_2 \in I_2$) and bL commutes with g_1L as b commutes with g_1 (as both are in K). Hence bL commutes with gL, but (as $b \in K$) it commutes with dL for $d \in H^*$, hence $bL \in \mathrm{Cent}(H/L)$; but $b \notin L$, $L = \mathrm{Cent}^{\infty} H$, contradiction.

So $I_1 \not\subseteq K$, and by the symmetry, $I_2 \not\subseteq K$. It is impossible that $g_1 \in H^*$, $g_2 \in H^*$, so w.l.o.g. $g_1 \notin H^*$. Let x be any member of $I_2 - L$, y any member of $I_1 - L$. Now g_1, $x \in H$ hence have represenatations

$$g_1 = g^n \prod_{m\in\mathbb{Z}} a^1_m, \qquad x = g^k \prod_{m\in\mathbb{Z}} a^2_m, \quad n \in \mathbb{Z} - \{0\},$$

$k \in \mathbb{Z}$, a^1_m, $a^2_m \in H_m$, $\{m : a^1_m, a^2_m$ are not both $e\}$ is finite. Remember that in H/L, every conjugate of x commutes with g_1 (and conversely). As $g_1 \notin H_1^*$, $n \neq 0$.

There is $r \in \mathbb{Z}$ s.t. $[a^l_m \notin e \Rightarrow 3\,|m| + 8 < r]$. Now $g_1 L$, $g^r x g^{-r} L$ commute (as they belong to I_1/L, I_2/L, respectively). This implies $a^2_m \in K$ for every m, so as $x \in I_2$ was arbitrary, $I_2 \subseteq \langle K, g\rangle_H$; but $I_2 \not\subseteq K$, hence there is $x_2 \in I_2 - K$. Working with x_2 and y instead of g_1, x we can prove $I_1 \subseteq \langle K, g\rangle_H$, so $H \subseteq \langle K, g\rangle_H$, contradiction to $K \neq H_0$.

5.9. Conclusion. *Suppose $J \subseteq L \subseteq G$, and in G, J and L are conjugates. Suppose further that H is a subgroup of* $\mathrm{Cm}_L(J) = L \cap \mathrm{Cm}_G(J)$, *of power* $\leqslant |\gamma| + \aleph_0$, *$H$ is not a subgroup of* $\mathrm{Cent}\,\mathrm{Cm}_L(J)$ *and γ is minimal such that $\mathscr{D}v^\gamma(H) = H$.*

Then G has a γ_1-required subgroup for some γ_1, $\gamma < \gamma_1 < (\aleph_0 + |\gamma|)^+$.

Proof. Let $\Box^g$ map L onto J, $g \in G$, and let

$$H^* = \langle \Box^{g^m} H : m \in \mathbb{Z}\rangle_G, \qquad K_1 = \mathrm{Cent}\,\langle \Box^{g^m}(\mathrm{Cm}_L(J)) : m \in \mathbb{Z}\rangle_G,$$
$$K = K_1 \cap H^*, \qquad H_m = \langle \Box^{g^m} H, K\rangle_G.$$

The $\Box^{g^m} H$ $(m \in \mathbb{Z})$ are pairwise commuting (as for $m > 0$, $\Box^{g^m} H \subseteq J$, $H \subseteq \mathrm{Cm}_L(J)$, then use $\Box^{g^k}$ for other pairs). Similarly $\Box^{g^m}(\mathrm{Cm}_L(J))$ $(m \in \mathbb{Z})$ are pairwise commuting. Hence $K_1 = \sum'_{m\in\mathbb{Z}} \mathrm{Cent}\,\Box^{g^m}\mathrm{Cm}_L(J) = \sum'_{m\in\mathbb{Z}} \Box^{g^m}\,\mathrm{Cent}\,\mathrm{Cm}_L(J)$ and $H^* = \sum'_{m\in\mathbb{Z}} \Box^{g^m} H$.

If $a \in \mathrm{Cm}_L(J) - \mathrm{Cent}\,\mathrm{Cm}_L(J)$, then $a \notin K_1$, and $a \notin \langle \Box^{g^m}(\mathrm{Cm}_L(J)) : m > 0\rangle_G$.

Clearly, K_1 is a commutative group, hence so is K, and $K \subseteq \mathrm{Cent}\,H^* = \sum'_m \mathrm{Cent}\,\Box^{g^m} H = \sum'_m \Box^{g^m}\,\mathrm{Cent}\,H$ (by the definition of K), but $K \subseteq H_m \subseteq H^*$, hence $K \subseteq \mathrm{Cent}\,H_m$. As K is closed under $\Box^{g^m}$ $(m \in \mathbb{Z})$, K is a normal subgroup of $H^* \subseteq \langle H^*, g\rangle$. Now $H^*/K = \sum_m H_m/K$; for suppose $a_m \in H_m$ for $n(0) \geqslant m \geqslant n(1)$, $a_{n(1)} \notin K$, but $\prod_m a_m \in K$, then by applying $\Box^{g^{-n(1)}}$ we can assume $n(1) = 0$ and get a contradiction. Also $H_0 \neq K$, otherwise $H \subseteq H_0 \subseteq K \subseteq K_1$, but there is $a \in H - \mathrm{Cent}\,\mathrm{Cm}_J(L)$, and we have said such a is not in K_1. So we can apply 5.8, so $H^+ = \langle H^*, g\rangle_G$ is a subgroup of G, $H^+/\mathrm{Cent}^\infty H^+$ is indecomposable, $\mathscr{D}v^\gamma(H^+) \neq H^+$, but for some $\beta < (\aleph_0 + |H|)^+ \leqslant (|\gamma| + \aleph_0)^+$, $\mathscr{D}v^\beta(H^+) = H^+$. So by 5.8 we have completed the proof.

Remark. We could have defined $\mathscr{D}v$ in a finer way.

6. On limit μ — the easy cases

In this section we first show that for $G \in \mathscr{P}_\lambda$, $\mathrm{nc}_{\leqslant\mu}(G) \geqslant \prod_{\theta<\mu} \theta^+$ (thus proving the main theorem for a large class of λ's, e.g., the case μ is a strong limit and μ a limit regular cardinal). We use for this the previous section; by 5.9 (and (5.7) we can build for each $\theta < \mu$ an increasing sequence of subgroups of G of power θ, $\langle K_i^\theta : i < \theta^+\rangle$, no two of which are conjugate. We shall do it by induction on θ so that $K_i^\theta \subseteq \mathrm{Cm}_G(\bigcup\{K_j^\kappa : j < \kappa^+,\ \kappa < \theta)$. Now we want to show that for the subgroups

$$K_\eta = \langle K^\theta_{\eta(\theta)} : \theta < \kappa \rangle_G \quad \left(\text{for } \eta \in \prod_{\theta<\mu} \theta^+\right)$$

are pairwise nonconjugate. For this we want to be able to reconstruct the $K^\theta_{\eta(\theta)}$ from the K_η. So we restrict ourselves to K^θ_i such that this is easy (θ-groups); to get such K^θ_i we find them as subgroups of $\mathrm{Min}[\mathrm{Cm}_G(\bigcup \{K^\kappa_j : \kappa < \theta, j < \theta^+\})]$.

We then proceed to deduce something for any limit μ.

6.1. Theorem. *If μ is a strong limit cardinal of power $>\aleph_0$, then the main theorem holds.*

Proof. Suppose $\mathcal{P}_\lambda \neq \emptyset$. Then by Section 3 for some $G \in \mathcal{P}_\lambda$, $\mathrm{Min}_{\aleph_1} G = G$ and $\mathrm{Cent}\, G = \{e\}$ [Why? By 3.6-8, 3.11, there is $G \in \Omega_\lambda$, by 1.8(1) $|\mathrm{Cent}^\infty(G)| < \mu$, so every $x \in G - \mathrm{Cent}\, G$ has $\geq \mu$ conjugates in G; hence $x\, \mathrm{Cent}\, G$ has $\geq \mu$ conjugates in $G/\mathrm{Cent}\, G$, so $G/\mathrm{Cent}\, G$ has trivial center and by 3.11, it belongs to Ω_λ], and we shall deal with this G.

Let $\gamma^* = \mathrm{Min}\{\gamma(\mathrm{Cm}_G(A)) : A \subseteq G,\ |A| < \mu\}$ (see Definition 5.5). By 5.7, $\gamma^* < \mu$, and choose $A_0 \subseteq G$, $|A_0| < \mu$ such that $\gamma^* = \gamma(\mathrm{Cm}_G(A_0))$. We shall now define by induction on $i < \mu$, a group H_i, K_i such that

(a) $H_i \subseteq G$, $\bigcup_{j<i} H_j \cup A_0 \cup \bigcup_{j<i} K_j \subseteq K_i \subseteq G$.

(b) If $i = \gamma^* j_1 + j_2$, $j_2 < \gamma^*$, then H_i is a γ_i-required subgroup of $\mathrm{Cm}_G(K_i)$ for some γ_i, $j_2 \leq \gamma_i < (\aleph_0 + |j_2|)^+$ (hence $|H_i| \leq \aleph_0 + |\gamma^*|$).

(c) H_i commutes with K_i (follows from (b)).

(d) $|K_i| \leq |\gamma^*| + |A_0| + |i| + \aleph_0$ and $|\{gxg^{-1} : g \in K_i\}| > \aleph_0$ for $x \in K_i - \{e\}$, $K_i^{(1)} = K_i$. (Note that if $\gamma(G) = \gamma^*$ the A_0 would not be necessary.)

In the ith step, we know that $A_0 \cup \bigcup_{j<i} H_j \cup \bigcup_{j<i} K_j$ has power $\leq |A_0| + |\gamma^*|\, |i| + \aleph_0 < \mu$, hence there is K_i, $A_0 \cup \bigcup_{j<i} H_j \cup \bigcup_{j<i} K_j \subseteq K_i \subseteq G$, $|K_i| \leq |i| + |A_0| + |\gamma^*| + \aleph_0$ and $\mathrm{Cent}(K_i) = \{e\}$, $K_i^{(1)} = K_i$ and every $x \in K_i - \{e\}$ has $\geq \aleph_1$ conjugates by elements of K_i. (See AP 1.3, 4.)

By the definition of γ^*, there is an $H_i \subseteq \mathrm{Cm}_G(K_i)$ satisfying (b). Now (a), (c), (d) are immediate.

As μ is a strong limit, there are linear orders S, T, $|S| = \mu$, $|T| = 2^\mu = \lambda$, $S \subseteq T$, S dense in T (e.g., $S = {}^{\mathrm{cf}\,\mu >}\mu$, $T = {}^{\mathrm{cf}\,\mu \geq}\mu$, ordered lexicographically). Let $S = \{s_i : i < \mu\}$ and for every $t \in T$, let $M_t = \langle K_0, N_t \rangle_G$ where $N_t = \langle H_{\gamma^* i + j} : j < \gamma^*,\ s_i < t \rangle_G$. Clearly $M_t = K_0 + N_t$.

As $G \in \mathcal{P}_\lambda$, there are distinct $t_\alpha \in T - S$ (for $\alpha < \mu$) such that the M_{t_α} are conjugate. Let $\square^{g^\alpha}$ map M_{t_α} onto M_{t_0}.

Now by (d), $\mathcal{D}v^1(K_0) = \{e\}$, hence $\mathcal{D}v^{(\infty)}(K_0) = \{e\}$ whereas $\mathcal{D}v^\infty(N_t) = N_t$ (for every $t \in T$), this holds by 5.4. Hence

$$\mathcal{D}v^\infty(M_{t_\alpha}) = \mathcal{D}v^\infty(K_0 + N_{t_\alpha}) = \mathcal{D}v^\infty(K_0) + \mathcal{D}v^\infty(N_{t_\alpha}) = N_{t_\alpha}.$$

So necessarily $\square^{g_\alpha}$ maps N_{t_α} onto N_{t_0}, hence

$$M_{t_0} = K_0 + N_{t_0} = \square^{g_\alpha}(K_0) + N_{t_0}.$$

So by 4.3(2) remembering $K_0 = K_0^{(1)}$, hence $\square^{g_\alpha}(K_0) = (\square^{g_\alpha}(K_0))^{(1)}$, and that the intersection of each of them with N_{t_0} is $\{e\}$:

$$M_{t_0}^{(1)} = K_0 \cap (\square^{g_\alpha} K_0) + N_{t_0}^{(1)},$$

but also $M_{t_0}^{(1)} = K_0 + N_{t_0}^{(1)}$ and $K_0 \cap N_{t_0} = \{e\}$, so necessarily $K_0 \cap (\square^{g_\alpha} K_0)$ cannot be a proper subgroup of K_0, hence $K_0 = \square^{g_\alpha} K_0$. As $|K_0| < \mu$, μ strong limit necessarily for some $\alpha \neq \beta$, $\square^{g_\alpha} \upharpoonright K_0 = \square^{g_\beta} \upharpoonright K_0$, let $g = g_\beta^{-1} g_\alpha$, then $\square^g \upharpoonright K_0 =$ the identity, hence $g \in \mathrm{Cm}_G(K_0)$, and (see above) $\square^g$ maps N_{t_α} onto N_{t_β}. W.l.o.g., $t_\alpha < t_\beta$ and choose $i < \mu$ such that $t_\alpha < s_i < t_\beta$. Now we apply 5.9 and get a contradiction to the choice of γ^*, A_0.

6.2. Hypothesis. μ is not strong limit.

6.3. Fact. *If $\theta < \mu$, then $\mu^\theta < \lambda$.*

Proof. For some $\kappa < \mu$, $\mu \leq 2^\kappa$ (as μ is not strong limit), hence $\mu^\theta \leq 2^{\kappa+\theta}$, but $\kappa + \theta < \mu$ so by μ's choice $2^{\kappa+\theta} < \lambda$.

6.4. Conclusion. *If $A \subseteq G \in \mathcal{P}_\lambda^1$, $|A| < \mu$, then $\mathrm{Cm}_G(A) \in \mathcal{P}_\lambda^1$.*

Proof. By 1.10(3), (4).

6.5. Theorem. *If μ is a limit cardinal, $\mathcal{P}_\lambda \neq \emptyset$, then for some $\kappa < \mu$, $\prod_{\theta < \mu, \theta \geq \kappa} \theta^+ < \lambda$.*

The theorem follows from 6.7(1), 6.10. First we introduce a notion.

6.6. Definition. We say $G \in \mathcal{P}_\lambda^2$, ($G$ is a minimal member of $\mathcal{P}_\lambda^1$) if $G \in \mathcal{P}_\lambda^1$ and for every $A \subseteq G$ of power $< \mu$, $\gamma(G) \leq \gamma(\mathrm{Min}\,\mathrm{Cm}_G(A))$.

6.7. Claim. (1) *For every $G \in \mathcal{P}_\lambda^1$ for some $A \subseteq G$, $|A| < \mu$ and* $\mathrm{Min}\,\mathrm{Cm}_G(A)$ *belong to $\mathcal{P}_\lambda^2$.*

(2) *If $G \in \mathcal{P}_\lambda^2$, for every $A \subseteq G$, $|A| < \mu$, then* $\gamma(G) = \gamma(\mathrm{Cm}_G(A)) = \gamma(\mathrm{Min}\,\mathrm{Cm}_G(A))$ *and* $\mathrm{Cm}_G(A) \in \mathcal{P}_\lambda^2$, $\mathrm{Min}\,\mathrm{Cm}_G(A) \in \mathcal{P}_\lambda^2$.

Proof. (1) Define by induction on n, $G_n \in \mathcal{P}_\lambda^1$, A_n, such that $G_0 = G$, A_n a subset of G_n of power $< \mu$ such that $\gamma[\mathrm{Min}\,\mathrm{Cm}_{G_n}(A_n)] < \gamma(G_n)$ and let $G_{n+1} = \mathrm{Min}\,\mathrm{Cm}_{G_n}(A_n)$; by 3.6, 6.4 and 1.7, $G_{n+1} \in \mathcal{P}_\lambda^1$. For some n we cannot define A_n, so $G_n \in \mathcal{P}_\lambda^2$. But by 3.9(3), $G_m = \mathrm{Min}\,\mathrm{Cm}_G(\bigcup_{m<n} A_m)$, hence we finish.

(2) Left to the reader.

6.8. Definition. (1) G is a θ-group [explicit θ-group] if $|G| = \theta$, $G = G^{(1)}$ and G has no semi-direct summand of power $< \theta$ [and every $x \in G - \mathrm{Cent}(G)$ has θ conjugates (at least)].

(2) G is a $[\theta, \kappa)$-group [explicit $[\theta, \kappa)$-group] if $\theta \leqslant |G| < \kappa$, $G = G^{(1)}$ and G has no semi-direct summand of power $< \theta$ [and every $x \in G - \mathrm{Cent}(G)$ has at least θ conjugates].

6.9. Fact. (1) *If $G = \sum'_{t\in T} H_t$, H_t is a $[\theta_t, \kappa_t)$-group, and for no $t \neq s$, $\theta_t \leqslant \theta_s < \kappa_t$, then H_t is the maximal normal $[\theta_t, \kappa_t)$-subgroup of G which is a semi-direct summand. If we restrict ourselves to explicit $[\theta_t, \kappa_t)$-group the 'direct summand' is not necessary.*

(2) *G is a (explicit) θ-group iff G is a (explicit) $[\theta, \theta^+)$-group.*

(3) *If G is an explicit $[\theta, \kappa)$-group, then G is a $[\theta, \kappa)$-group.*

6.10. Lemma. *If $G \in \mathscr{P}^2_\lambda$, then*

(1) $$\mathrm{nc}_{\leqslant\mu}(G) \geqslant \prod_{\substack{\theta<\mu \\ \theta \geqslant |\gamma(G)|+\aleph_0}} (\theta^+).$$

(2) *Moreover, also $G/\mathrm{Cent}\, G$ satisfies this.*

Proof. (1) We shall define by induction on θ, $|\gamma(G)| + \aleph_0 \leqslant \theta < \mu$ subgroups K^θ_i $(i < \theta)$ such that

(i) K^θ_i is a subgroup of $G_\theta \overset{\text{def}}{=} \mathrm{Min}[\mathrm{Cm}_G(\bigcup \{K^\kappa_j : |\gamma(G)| + \aleph_0 \leqslant \kappa < \theta, j < \kappa^+\})]$.

(ii) K^θ_i has power θ.

(iii) K^θ_i is an explicit θ-group.

(iv) For $i \neq j$, K^θ_i, K^θ_j are not conjugates in G.

This is enough, as then for every $\eta \in \prod \{\theta^+ : |\gamma(G)| + \aleph_0 \leqslant \theta < \mu\}$, we define $L_\eta = \langle K^\theta_{\eta(\theta)} : |\gamma(G)| + \aleph_0 \leqslant \theta < \mu \rangle_G$. Now L_η is a subgroup of G of power μ, and, for each θ, the $K^\theta_{\eta(\theta)}$ are definable in L_η ($K^\theta_{\eta(\theta)}$ is the maximal normal explicit θ-subgroup of L_η); hence by (iv), $\eta \neq \nu$ implies L_η, L_ν are not conjugate in G, and since the number of L_η's is as required, we would have finished.

So let us carry out the induction. Clearly $G_\theta \in \Omega^1_\lambda$, hence $G^{(1)}_\theta = G_\theta$, and every $x \in G_\theta - \mathrm{Cent}\, G_\theta$ has at least μ conjugates (see 3.8). Hence every subgroup of G_θ of power $\leqslant \theta$ is included in some explicit θ-subgroup of G_θ (e.g. see AP1.3). Now we define $K^\theta_i \subseteq G_\theta$ $(i < \theta^+)$ by induction on i, $|K^\theta_i| = \theta$, K^θ_i increasing with i.

If K^θ_j $(j < i)$ have been defined, we can define by induction on $\beta < \gamma(G)$ a subgroup $H^\theta_{i,\beta}$ of $\mathrm{Cm}_G(\bigcup_{j<i} K_j \cup \bigcup_{\gamma<\beta} H^\theta_{j,\gamma})$, which is a γ_β-required group, for some γ_β, $\beta \leqslant \gamma_\beta < \gamma(G)$ which is not included in $\mathrm{Cent}[\mathrm{Cm}_{G_\theta}(\bigcup_{j<i} K^\theta_j \cup \bigcup_{\gamma<\beta} H^\theta_{j,\gamma})]$ (this is when $\beta = 0$).

Let K^θ_i be a θ-subgroup of G_θ (of power θ) which includes $\bigcup_{j<i} K^\theta_j \cup \bigcup_{\beta<\gamma(G)} H^\theta_{j,\gamma}$. The only serious problem is why K^θ_i is not conjugate (in G) to some K^θ_j $(j < i)$. This is guaranteed by $G \in \mathscr{P}^2_\lambda$ (see 5.9).

(2) The proof is similar replacing (iv) by

(iv)′ Moreover for $i \neq j$, $K^\theta_i/\mathrm{Cent}\, G$, $K^\theta_j/\mathrm{Cent}\, G$ are not conjugate in $G/\mathrm{Cent}\, G$.

Now we make:

6.11. Hypothesis. For some $\kappa < \mu$, $\prod_{\theta<\mu,\theta\geqslant\kappa} \theta^+ < \lambda$.

6.12. Fact. *If μ is limit, $\mathcal{P}_\lambda \neq \emptyset$, then*
(1) *$\mu < \aleph_\mu$, so μ is singular.*
(2) *For unboundedly many $\theta < \mu$, $2^\theta < 2^{\theta^+}$.*
(3) *$\mu < 2^{<\mu} < (2^{<\mu})^{\mathrm{cf}\,\mu} = 2^\mu$.*
(4) *If $G \in \mathcal{P}^0_\lambda$, for no normal subgroup N, $2^{<\mu} \leqslant (G:N) < \lambda$.*

Proof. By cardinal arithmetic we can prove (1), (2), (3). As for (4) by 1.4(1), we know that $\mathrm{nc}_{\leqslant\mu}(G/N) \leqslant \mathrm{nc}_{\leqslant\mu}(G)$, and we apply 1.2(3) to G/N for the cardinal $\mathrm{cf}(\mu)$ (as $(2^{<\mu})^{\mathrm{cf}\,\mu} = 2^\mu = \lambda$).

7. The number of direct summands is small

Later, at some crucial point, the number of direct summands of $G \in \Omega_\lambda$ (or the power of $\mathrm{BA}'_G(\mathrm{Min}\, G)$ for $G \in \mathcal{P}_\lambda$) will become important. If it is $< \mu$, we know that for 'quite many' subgroups H of G of power $< \mu$, their direct summands are exactly those induced by direct summands of G. This helps in proofs like 6.5 when we want in each θ to have θ^+ subgroups in H_S. Here we shall prove that this is always the case when μ is a successor cardinal.

7.1. Theorem. *Suppose $\mu = \kappa^+$, if $G \in \Omega^1_\lambda$, then* $\mathrm{BA}(G)$ *has power $< \mu$.*

For singular μ we need more elaborate information involving the existence of many nonconjugates of $[\theta, \kappa)$-groups.

7.2. Theorem. *Suppose $\theta < \kappa < \mu$, $G_1 \in \mathcal{P}^1_\lambda$, $G = \mathrm{Min}\, G_1$, $\mathrm{BA}'_{G_1}(G)$ has power $> \kappa$, and $2^\kappa \geqslant \mu$. Then G has 2^{κ^+} $[\theta, \kappa^{++})$-subgroups, which are pairwise nonconjugate in G_1.*

We want to prove the theorems together. For this in 7.1 let $G_1 = G$, $\mathcal{B} = \mathrm{BA}'(G)$ and so clearly $\mathrm{BA}'_{G_1}(G) = \mathrm{BA}'(G)$ includes $\mathrm{BA}(G)$. For 7.1 let $\theta = \aleph_0$ if $\aleph_0 < \kappa$ and otherwise $\theta = 1$. So always $\theta < \kappa$. For 7.2 let $\mathcal{B} = \mathrm{BA}'_{G_1}(G)$. We are assuming G is a counterexample and eventually get a contradiction. So we are assuming $|\mathcal{B}| > \kappa$, and note that $2^\kappa \geqslant \mu$ for both theorems.

We shall use 4.8 freely.

7.3. Fact. *There are χ pairwise disjoint nontrivial $L_\alpha \in \mathcal{B}$ s.t.,* (a) *$\chi \leqslant \kappa$ and for some uniform ultrafilter $\mathcal{D}$ over χ, $\prod_{\alpha<\chi} |\mathcal{B} \restriction L_\alpha| = \prod_{\alpha<\chi} |\mathcal{B} \restriction L_\alpha| / \mathcal{D}$ is at least κ^+ or* (b) $\chi = \kappa^+$.

Proof. If $\mathscr{B}$ has $>\kappa$ atoms, we finish (as case (b) holds). If not let W be the ideal of $\mathscr{B}$ generated by the atoms. We define by induction on α, L_α such that:

(i) $L_\alpha \in \mathscr{B} - W$.

(ii) L_α is disjoint to L_β for $\beta < \alpha$ (as members of $\mathscr{B}$, so $L_\alpha \cap L_\beta \subseteq \operatorname{Cent} G$).

(iii) Under conditions (i) and (ii), the power of $\{L \in \mathscr{B} : L \subseteq L_\alpha\}$ is minimal.

(iv) There are infinitely many pairwise disjoint L'_α satisfying (i), (ii) and disjoint to L_α or $|\{L \in \mathscr{B} : L \subseteq L_\alpha\}| \leq \kappa$.

Suppose α is the first cardinal such that we cannot define L_α. Let $W^* = \{L \in \mathscr{B} : L$ disjoint to every L_β $(\beta < \alpha)\}$, clearly W^* is an ideal of $\mathscr{B}$, and

$$\kappa^+ \leq |\mathscr{B}| \leq \prod_{\beta<\alpha} |\mathscr{B} \restriction L_\beta| + |W^*| + \aleph_0$$

because the function $F : \mathscr{B} \to \prod_{\beta<\alpha} \mathscr{B} \restriction L_\beta$ defined by $F(L) = \langle L \cap L_\beta : \beta < \alpha \rangle$, satisfies: $[F(L) = F(L') \Rightarrow (L - L') \cup (L' - L) \in W^*]$. As α is maximal $|W^*| \leq |W| + \aleph_0$ but we have assumed $|W| \leq \kappa$, so $\prod_{\beta<\alpha} |\mathscr{B} \restriction L_\beta|$ is at least κ^+. By (iii), $|\mathscr{B} \restriction L_\alpha|$ is nondecreasing, and by (iv), α is limit; lastly by (i), $|\mathscr{B} \restriction L_\alpha|$ is infinite.

If $\alpha \geq \kappa^+$, case (b) holds; so assume $\alpha < \kappa^+$. Now we can find an ultrafilter on α as required, (see [1]: some regular ultrafilter) and replace α by $\chi \stackrel{\text{def}}{=} |\alpha|$.

7.4. Fact. *Always $\chi \leq \kappa$ (if G is a counterexample).*

Proof. Easy.

7.5. Fact. *Suppose $M \in \mathscr{B}$. Then for every $A \subseteq G$, $|A| < \mu$ there is an explicit κ-group $P \subseteq M \cap \operatorname{Min} \operatorname{Cm}_G(A)$, such that M is the minimal member of $\mathscr{B}$ which includes P, and even $P/\operatorname{Cent} P$ is an explicit κ-group, and $|P| \leq \kappa$.*

Proof. We define by induction on $\alpha < \kappa^+$, K_α such that:

(1) K_α is a subgroup of $M \cap \operatorname{Min} \operatorname{Cm}_G(A)$.

(2) K_α is an explicit κ-group, and even $K_\alpha/\operatorname{Cent} K_\alpha$ is an explicit κ-subgroup.

(3) For some disjoint nonzero I_α, $J_\alpha \in \mathscr{B}$, $M = I_\alpha + J_\alpha$, $K_\alpha \subseteq I_\alpha$, $\bigcup_{\beta<\alpha} K_\beta \subseteq J_\alpha$.

At stage α, we choose, if possible, I_α, $J_\alpha \in \mathscr{B}$, $\bigcup_{\beta<\alpha} K_\beta \subseteq J_\alpha$, $I_\alpha \subseteq M$, $I_\alpha \cap J_\alpha$ is abelian and I_α is not abelian. If this is possible, then $I_\alpha \in \Omega^1_\lambda$ (by 4.8), hence $I_\alpha/\operatorname{Cent} I_\alpha \in \Omega_\lambda$, so there is no problem to choose K_α. [The presence of A does not change much; we can replace it by $A_1 = \{\operatorname{End}^{I_\alpha}_{L_\gamma}(a) : a \in A\}$ and then use $\operatorname{Min} \operatorname{Cm}_{I_\alpha}(A_1)$ instead of I_α as $I_\alpha \in \Omega^1_\lambda$.] If there are no such I_α and J_α, then $\sum_{\beta<\alpha} K_\beta$ satisfies the requirements of P in 7.5.

If K_α is defined for every $\alpha < \kappa^+$, then we let for $S \subseteq \kappa^+$, $H_S = \langle K_\alpha : \alpha \in S \rangle_G$. Clearly the H_S's are as required in 7.2 (or contradict 7.1). Note that no $x_l \in I_{\alpha_l} - \operatorname{Cent} I_{\alpha_l}$ (for $l = 1, 2$; $\alpha_1 \neq \alpha_2$) are conjugates even in G_1.

7.6. Fact. *Suppose $M_\gamma \in \mathscr{B}$, $M_\gamma \subseteq L_\gamma \in \mathscr{B}$ for $\gamma < \chi$, the L_γ's are pairwise disjoint, and $A \subseteq G$, $|A| < \mu$.*

Then we can find P_γ, $(\gamma<\chi)$ *and* $g\in G$ *such that*

(a) $P_\gamma\subseteq M_\gamma\cap \operatorname{Min}\operatorname{Cm}_G(A)$ *has cardinality* $\leqslant\kappa$.

(b) M_γ *is the minimal member of* $\mathcal{B}$ *which includes* P_γ.

(c) P_γ *is an explicit* κ*-group, and* $P_\gamma/\operatorname{Cent} P_\gamma$ *too.*

(d) $\square^g(P_\gamma)$ *commutes with* P_γ.

(e) $g\in \operatorname{Min}\operatorname{Cm}_G(A)$, *and if* $L_\gamma\subseteq L\in\mathcal{B}$ *for* $\gamma<\chi$, *then* $g\in L$ *(for a specific* L*).*

Remark. Note that $g\in G$, not $g\in G_1$.

Proof. We can define by induction on $\alpha<\mu$ a group $P_{\alpha,\gamma}$ for $\gamma<\chi$ such that

(i) $P_{\alpha,\gamma}=P^{(1)}_{\alpha,\gamma}\subseteq M\cap \operatorname{Min}\operatorname{Cm}_G(A\cup\bigcup_{\beta<\alpha}P_{\beta,\gamma})$.

(ii) $P_{\alpha,\gamma}$ is an explicit κ-group as well as $P_{\alpha,\gamma}/\operatorname{Cent} P_{\alpha,\gamma}$.

(iii) M_γ is the minimal semi-direct summand of G which include $P_{\alpha,\gamma}$.

This is possible by applying 7.5 to $A'=A\cup\bigcup_{\beta<\alpha}P_{\beta,\gamma}$. Clealy $P_{\alpha,\gamma}$ $(\gamma<\chi,\ \alpha<\mu)$ are pairwise commuting subgroups of $\operatorname{Min}\operatorname{Cm}_G(A)$, which belong to Ω^1_λ.

For every $S\subseteq\mu$ let $K_S=\langle P_{\gamma,\alpha}:\gamma<\chi,\ \alpha\in S\rangle_G$, if for $S\neq T$, K_S and K_T are not conjugate in $\operatorname{Min}\operatorname{Cm}_G(A)$, then we get a contradiction: as $\operatorname{Min}\operatorname{Cm}_G(A)\in\Omega^1_\lambda$, it has up to conjugation less than λ subgroups $H=H^{(1)}$ of power $\leqslant\mu$. Otherwise there is $g\in\operatorname{Min}\operatorname{Cm}_G(A)$ such that $\square^g$ maps K_S onto K_T, and there is $\alpha\in S-T$. Let $P_\gamma=P_{\alpha,\gamma}$ for $\gamma<\chi$. (We can replace g by $\operatorname{End}^L_G(g)$.)

Now P_γ commutes with K_T, hence with $\square^g\left(\bigcup_\beta P_\beta\right)$. So we finish 7.6.

7.7. Fact. *Let* L_γ $(\gamma<\chi)$ L *be as in* 7.6, $\mu_1=\operatorname{Min}\{\mu,\ |\prod_{\gamma<\chi}(\mathcal{B}\restriction L_\gamma)|\}$ $(\chi<\mu_1$ *of course). There are for* $\alpha<\kappa^+$, K_α, $P_{\alpha,\gamma}$ $(\gamma<\chi)$ B_α, g_α *and sequences* $\langle M_{\alpha,\gamma}:\gamma<\chi\rangle$ *such that*:

(i) $K_\alpha=\langle B_\alpha,\bigcup_{\gamma<\chi}P_{\alpha,\gamma}\rangle_G$.

(ii) *The* K_α*'s are pairwise commuting.*

(iii) $P_{\alpha,\gamma}\subseteq M_{\alpha,\gamma}$, $P_{\alpha,\gamma}$ *an explicit* κ*-group.*

(iv) $M_{\alpha,\gamma}$ *is the minimal member of* $\mathcal{B}$ *which includes* $P_{\alpha,\gamma}$.

(v) $\square^{g_\alpha}$ *maps* $\bigcup_{\gamma<\chi}P_{\alpha,\gamma}$ *to a subgroup of* G *which commutes with it.*

(vi) $M_{\alpha,\gamma}\subseteq L_\gamma$, $g_\alpha\in L$.

(vii) *For* $\alpha<\beta<\kappa^+$, $\{\gamma: M_{\alpha,\gamma}\neq M_{\beta,\gamma}\}\in\mathcal{D}$.

(viii) $g_\alpha\in B_\alpha\subseteq G$, $|B_\alpha|=\theta$, *and if* $\theta>1$ *then* $B^{(1)}_\alpha=B_\alpha$, B_α *a* θ*-group.*

Proof. First we can define $\langle M_{\alpha,\gamma}:\gamma<\chi\rangle$ for $\alpha<\kappa^+$ to satisfy (vii). Then we define by induction on α, K_α, g_α, $P_{\alpha,\gamma}$ $(\gamma<\chi)$ using 7.6 with $A=\bigcup_{\beta<\alpha}K_\beta$, and then we define B_α.

From now we shall use g_α, $P_{\alpha,\gamma}$ of 7.7.

7.8. Proof of Theorem 7.1: when $2^\chi>\kappa$. In fact here 7.5 is irrelevant and condition (vi) in 7.7 too. By Engelking and Karlowicz [3] there are subsets $T_\alpha\subseteq\chi$ (for $\alpha<\mu=\kappa^+$) such that no one is included in a finite union of the others and a finite set. Let $K^*_\alpha = \langle g_\alpha, P_{\alpha,\gamma} : \gamma < \chi,\ \gamma \in T_\alpha \rangle_G$. Let for $S \subseteq \mu$, $H_S = \langle K^*_\alpha : \alpha \in S \rangle_G$, clearly H_S is a subgroup of G of power $\leq \mu$. Suppose $S_0 \neq S_1$ but $a \in G$, $\Box^a$ maps H_{S_0} onto H_{S_1}, and suppose $\alpha \in S_0$, $\alpha \notin S_1$.

So $\Box^a g_\alpha \in H_{S_1}$, hence there are $\beta_1, \ldots, \beta_n \in S_1$ and $\gamma_1, \ldots, \gamma_m < \chi$, such that

$$\Box^a g_\alpha \in \left\langle \{g_{\beta_k} : k = 1, n\} \cup \bigcup_{k=1}^{m} (L_{\gamma_k} \cap H_{S_1}) \right\rangle_G .$$

By the choice of the T_γ's for some γ

$$\gamma \in T_\alpha - \bigcup_{k=1}^{n} T_{\beta_k} - \{\gamma_1, \ldots, \gamma_m\}.$$

Now g_α does not commute with some elements of $L_\gamma \cap H_{S_0}$, (remember $P_{\alpha,\gamma}$) but $\Box^a g_\alpha$ (by its representation) commutes with every member of $L_\alpha \cap H_{S_1}$. As $\Box^a$ maps L_α onto itself, we get a contradiction. So we finish 7.8, as $\kappa^+ = \mu$.

From now on let $M_{\alpha,\gamma}$, B_α $(\gamma < \chi,\ \alpha < \kappa^+)$ be as in 7.7, $K_\alpha \stackrel{\text{def}}{=} \langle B_\alpha, P_{\alpha,\gamma} : \gamma < \chi,\ \gamma \in T_\alpha \rangle_G$, and for $S \subseteq \kappa^+$, $H_S \stackrel{\text{def}}{=} \langle K_\alpha : \alpha \in S \rangle_G$.

We have decided in the beginning that for 7.1, $\theta = \aleph_0$ except when $\kappa = \aleph_0$, but when $\kappa = \aleph_0$, necessarily $\chi = \aleph_0$, $2^\chi > \kappa$; so from now on we deal with $\theta \geq \aleph_0$. We prove that there are many nonconjugate subgroups getting a contradiction.

7.9. Fact. $K_\alpha = K_\alpha^{(1)}$.

Proof. As each $P_{\alpha,\gamma}$ is a κ-group, $K_\alpha^{(1)}$ includes $P_{\alpha,\gamma}^{(1)} = P_{\alpha,\gamma}$ and $B_\alpha^{(1)} = B_\alpha$, but K_α is generated by those elements, hence $K_\alpha = K_\alpha^{(1)}$.

7.10. Notation. (1) For every $I \in \mathrm{BA}'(K_\alpha)$, $\gamma < \chi$ let $\mathrm{Pro}_\gamma(I)$ be the ideal of $M \in \mathscr{B}$, $M \subseteq L_\gamma$, and in $G/\mathrm{Cent}\, G$, $\mathrm{End}^G_M(I)/\mathrm{Cent}\, G$ has power $\leq \theta$.

(2) $\mathrm{Set}_\alpha = \{\langle \mathrm{Pro}_\gamma(I) : \gamma < \chi \rangle / \mathscr{D} : I \in \mathrm{BA}'(K_\alpha)\}$ (the division by $\mathscr{D}$ just means that we shall count them up to equality mod $\mathscr{D}$).

7.11. Fact. (1) $\mathrm{Pro}_\gamma(K_\alpha)$ *is the ideal generated by* $L_\gamma - M_{\alpha,\gamma}$ *(subtraction, in* $\mathscr{B}$*).*

(2) *If* $I, J \subseteq G$, *then* $\mathrm{Pro}_\gamma(\langle I, J \rangle_G) = \mathrm{Pro}_\gamma(I) \cap \mathrm{Pro}_\gamma(J)$.

7.12. Fact. (1) *If* $K_\alpha = I +' J$, $g_\alpha \in I$, *then* $|J/\mathrm{Cent}\, J| \leq \theta$.

(2) *If* $K_\alpha = I +' J$, $g_\alpha \in \langle I \cup \bigcup_{m=1}^{n} (L_{\gamma_m} \cap K) \rangle_G$, *then for* $\gamma \in \chi - \{\gamma_1, \ldots, \gamma_n\}$, $\mathrm{End}_{L_\gamma/\mathrm{Cent}\, G}(J/\mathrm{Cent}\, J)$ *has power* $\leq \theta$.

Proof. (1) As $g_\alpha \in I$, for every $x \in K_\alpha$, $g_\alpha x g_\alpha^{-1} x^{-1} \in I$. Let for $x \in K_\alpha$, $x = x^I + x^J$, $x^I \in I$, $x^J \in J$. Now for $x, y \in P_{\alpha,\gamma}$, x and $g_\alpha y g_\alpha^{-1}$ commute (see 7.5), hence x^J, $(g_\alpha y g_\alpha^{-1})^J$ commute and $(g_\alpha y g_\alpha^{-1})^J\ \mathrm{Cent}\, G = g_\alpha^J y^J (g_\alpha^{-1})^J\ \mathrm{Cent}\, G = y^J\ \mathrm{Cent}\, G$ (as $g_\alpha \in I$), hence x^J, $g_\alpha^J y^J (g_\alpha^{-1})^J \in y^J\ \mathrm{Cent}\, G$ commute. However, as $P_\alpha = P_\alpha^{(1)}$, and the map $x \mapsto x^J\ \mathrm{Cent}\, K_\alpha$ which we call h, is a homomorphism from P_α into

$\langle \text{Cent}\, K_\alpha \cup \{x^J : x \in P_\alpha\}\rangle_G/\text{Cent}\, K_\alpha$, clearly $(\text{Range}\, h)^{(1)} = \text{Range}\, h$ and Range h is a commutative group (by the previous sentence), hence $x^J \in \text{Cent}\, K_\alpha$ for $x \in P_\alpha$. As K_α is generated by $P_\alpha \cup B_\alpha$, J is generated by $\{x^J : x \in B_\alpha\} \cup \text{Cent}\, J$, hence $J/\text{Cent}\, J$ has power $\leq \theta$.

(2) The proof is similar.

7.13. Fact. Set_α *has power* $\leq \theta$.

Suppose not and so let for $i < \theta^+$, $K_\alpha = I_i +' J_i$ be distinct semi-decompositions with $\langle \text{Pro}_\gamma(I_i) : \gamma < \chi\rangle/\mathcal{D}$ pairwise distinct; let $g_\alpha = a^i_\alpha + b^i_\alpha$, $a^i_\alpha \in I_i$, $b^i_\alpha \in J_i$. Let $a^i_\alpha = \prod_{m=1}^{n(i)} x_{\alpha,i,m}$, $x_{\alpha,i,m}$ is from B_α if m is even, and from $P_{\alpha,\gamma(i,m)}$ if m is odd. W.l.o.g., $n(i) = n$ and $x_{\alpha,i,2m} = x_{\alpha,i}$ for $1 \leq 2m \leq n$. So for $i, j < \theta^+$, $a^i_\alpha(a^j_\alpha)^{-1}$ and $b^i_\alpha(b^j_\alpha)^{-1}$ belongs to $K_\alpha \cap \sum \{L_\gamma : \gamma = \gamma(i,m)$ or $\gamma = \gamma(j,m)$ where $1 \leq m \leq n\}$.

Now by 4.7, $K_\alpha = I_0 \cap J_1 +' I_0 \cap J_0 +' I_1 \cap J_0 +' I_1 \cap J_1$. By 7.12(2) for all but finitely many γ's, $\text{Pro}_\gamma(I_0 \cap J_1) = \text{Pro}_\gamma(I_1 \cap J_0) = \{I \in \text{BA}'_{G_1}(G) : I \leq L_\gamma\}$, hence by 7.11(2), $\text{Pro}_\gamma(I_0) = \text{Pro}_\gamma(I_1)$ for all but finitely many γ. This contradicts their choice.

7.13A. Fact. *W.l.o.g., for* $\alpha < \beta < \kappa^+$, $\text{Set}_\alpha \cap \text{Set}_\beta \subseteq \text{Set}$ *where* Set *is a set of power* κ.

Proof. By 7.13 and a lemma of Fodor (see AP2.3) there is a stationary $S \subseteq \{\delta < \kappa^+ : \text{cf}\ \delta = \theta^+\}$ and $\beta < \kappa^+$ such that for every $\alpha \in S$, $\text{Set}_\alpha \cap (\bigcup_{i<\alpha} \text{Set}_i) \subseteq \bigcup_{i<\beta} \text{Set}_i$. By renaming we get the first phrase.

7.14. Proof of Theorem 7.1. Let for $S \subseteq \kappa^+$, $H_S = \langle K_\alpha : \alpha \in S\rangle_G = \sum'_{\alpha\in S} K_\alpha$. Suppose $S_0 \neq S_1 \subseteq \mu$, $a \in G_1$, $\Box^a$ maps H_{S_0} onto H_{S_1} and $|S_1 - S_0| = |S_0 - S_1| = \kappa^+$. We shall get a contradiction and this clearly suffices.

So $\sum'_{\alpha\in S_1} K_\alpha = H_{S_1} = \Box^a K_{S_0} = \sum'_{\beta\in S_0} \Box^a K_\beta$. As $K^{(1)}_\alpha = K_\alpha$, by 4.7,

$$H_{S_1} = \sum_{\alpha\in S_1, \beta\in S_0}{}' K_\alpha \cap \Box^a K_\beta$$

and for $\alpha \in S_1$, $K_\alpha = \sum'_{\beta\in S_0} K_\alpha \cap \Box^a K_\beta$.

So for $\alpha \in S_1$, for some finite $w(\alpha) \subseteq S_0$, $g_\alpha \in \sum'_{\beta\in w(\alpha)} K_\alpha \cap \Box^a K_\beta$. As

$$K_\alpha = \Big(\sum_{\beta\in w(\alpha)}{}' K_\alpha \cap \Box^a K_\beta\Big) +' \Big(\sum_{\beta\in S_0 - w(\alpha)}{}' K_\alpha \cap \Box^a K_\beta\Big),$$

so by 7.12(1) the cardinality of $(\sum'_{\beta\in S_0-w(\alpha)} K_\alpha \cap \Box^a K_\beta)/\text{Cent}\, K_\alpha$ is $\leq \theta$.

Hence $\text{Pro}_\gamma(\sum'_{\beta\in S_0-w(\alpha)} K_\alpha \cap \Box^a K_\beta) = \{L \in \mathcal{B} : L \subseteq L_\gamma\}$, hence by 7.11(2)

$$\Big\langle \text{Pro}_\gamma(K_\alpha) : \gamma < \chi \Big\rangle = \Big\langle \text{Pro}_\gamma\Big(\sum_{\beta\in w(\alpha)}{}' K_\alpha \cap \Box^a K_\beta\Big) : \gamma < \chi\Big\rangle$$

$$= \Big\langle \bigcap_{\beta\in w(\alpha)} \text{Pro}_\gamma(K_\alpha \cap \Box^a K_\beta) : \gamma < \chi\Big\rangle.$$

Let $u(\alpha)=\{\beta\in w(\alpha)$: for some $Y\in\mathscr{D}$ for every $\gamma\in Y$: $\mathrm{Pro}_\gamma(K_\alpha\cap\Box^a K_\beta)\neq\{L\in\mathscr{B}: L\subseteq L_\gamma\}\}$. So $u(\alpha)$ is finite and for $\beta\in u(\alpha)$, $\langle\mathrm{Pro}_\gamma(K_\alpha\cap\Box^a K_\beta)$: $\gamma<\chi\rangle/\mathscr{D}$ belongs to Set_α and it is also clear that it belongs to Set_β. As $u(\alpha)\subseteq w(\alpha)\subseteq S_0$, for $\alpha\in S_1-S_0$ this implies that $\langle\mathrm{Pro}_\gamma(K_\alpha\cap\Box^a K_\beta)$: $\gamma<\chi\rangle/\mathscr{D}$ belongs to Set. By 7.11(2), $\{\langle\mathrm{Pro}_\gamma(K_\alpha\cap\Box^a K_\beta)$: $\gamma<\chi\rangle/\mathscr{D}$: $\beta\in u(\alpha)\}$ determines $\langle\mathrm{Pro}_\gamma(K_\alpha)$: $\gamma<\chi\rangle/\mathscr{D}$. As $|S_1-S_0|=\kappa^+>|\mathrm{Set}|$ for some $\alpha(1)\neq\alpha(2)\in S_1-S_0$, $\langle\mathrm{Pro}_\gamma(K_{\alpha(1)})$: $\gamma<\chi\rangle/\mathscr{D}=\langle\mathrm{Pro}_\gamma(K_{\alpha(2)})$: $\gamma<\chi\rangle$. But this contradicts 7.11(2).

So we have finished proving 7.1 and let us now prove 7.2.

7.15. Lemma. *Let L_γ ($\gamma<\chi$), L be as in 7.6 ($\chi\leqslant\kappa$ of course). Then there are for $\alpha<\kappa^+$, K_α, B_α, $P_{\alpha,\gamma}$ ($\gamma<\chi$) B_α, g_α and sequences $\langle M_{\alpha,\gamma}:\gamma<\chi\rangle$ s.t.:*

(i) $K_\alpha=\langle B_\alpha\cup\bigcup_{\gamma<\chi}(K_\alpha\cap L_\gamma)\rangle_G$.
(ii) *The K_α's are pairwise commuting.*
(iii) *$P_{\alpha,\gamma}\subseteq K_\alpha$, and $P_{\alpha,\gamma}$, $K_\alpha\cap M_{\alpha,\gamma}$ are explicit κ-groups.*
(iv) *$M_{\alpha,\gamma}$ is the minimal member of $\mathscr{B}$ which includes $P_{\alpha,\gamma}$.*
(v) *$\Box^{g_\alpha}$ maps $\bigcup_{\gamma<\chi}P_{\alpha,\gamma}$ to a subgroup of G commuting with $\bigcup_{\gamma<\chi}P_{\alpha,\gamma}$*
(vi) *$M_{\alpha,\gamma}\subseteq L_\gamma$, $g_\alpha\in L$ and $\mathrm{End}_G^{L_\gamma-M_{\alpha,\gamma}}(K_0)$ has cardinality $\leqslant\theta$.*
(vii) *For $\alpha<\beta<\kappa^+$, $\{\gamma<\chi$: $M_{\alpha,\gamma}\neq M_{\beta,\gamma}\}\in\mathscr{D}$.*
(viii) *$g_\alpha\in B_\alpha$, $|B_\alpha|=\theta$, B_α is an explicit θ-group.*
(ix) *K_α is a $[\theta,\kappa)$-group.*
(x) *K_α is nice (hence $L_\gamma\cap K_\alpha\in BA'(K_\alpha)$ for $\gamma<\chi$) where*

7.15A. Notation. K is called nice when: if $a\in K_\alpha$, $\gamma<\chi$, then some $a'\in\mathrm{End}_G^{L_\gamma}(a)$ is in K_α, and also $K=K^{(1)}$.

Proof. We define first $M_{\alpha,\gamma}$ ($\gamma<\chi$) for $\alpha<\kappa^+$ as in 7.7. Then we define K_α, B_α, $P_{\alpha,\gamma}$ ($\gamma<\chi$) B_α, g_α by induction on α. For each α, choose $P_{\alpha,\gamma}$ ($\gamma<\chi$) g_α as in 7.7. then we define by induction on $i<\theta$, $B_{\alpha,i}$, $K_{\alpha,i}$ s.t.:

(1) $B_{\alpha,i}$, $K_{\alpha,i}$ are increasing with i, $g_\alpha\in B_{\alpha,i}$, $|B_{\alpha,i}|\leqslant\theta$, $|K_{\alpha,i}|\leqslant\kappa$ and $\mathrm{End}_G^{L_\gamma-M_{\alpha,\gamma}}(K_{\alpha,i})$ has power $\leqslant\theta$.

(2) $K_{\alpha,i}=\langle B_{\alpha,i}\cup\bigcup_{\gamma<\chi}(K_{\alpha,i}\cap L_\gamma)\rangle_G\subseteq\mathrm{Min}\,\mathrm{Cm}_G\bigcup_{\beta<\alpha}K_\beta$.

(3) For $i=5j+5$, for every $y\in K_{\alpha,i}\cap L_\gamma-\mathrm{Cent}\,L_\gamma$ the set $\{\Box^a y$: $a\in K_{\alpha,i+1}\cap L_\gamma\}$ has power $\geqslant\theta$.

(4) For $i=5j+1$, $K_{\alpha,i}\cap L_\gamma\subseteq(K_{\alpha,i+1}\cap L_\gamma)^{(1)}$.

(5) For $i=5j+2$, $x\in B_{\alpha,i}$, $\gamma<\chi$, there is $y\in\mathrm{End}_G^{L_\gamma}(x)\cap(K_{\alpha,i+1}\cap L_\gamma)$ and $x\in(B_{\alpha,i+1})^{(1)}$.

(6) For $i=5j+3$, $x\in B_{\alpha,i}$ if $\{\gamma<\chi$: $\mathrm{End}_G^{L_\gamma}(x)\nsubseteq\mathrm{Cent}\,L_\gamma\}$ is infinite, then for infinitely many such γ's $|\{\Box^g x$: $g\in K_{\alpha,i+1}\cap L_\gamma\}|\geqslant\theta$.

(7) For $i=5j+4$, $x\in B_{\alpha,i}$, $x\notin\mathrm{Cent}\,G$, if $w=\{\gamma<\chi$: $\mathrm{End}_G^{L_\gamma}x\nsubseteq\mathrm{Cent}\,L_\gamma\}$ is finite, let M_x be the complement of $\sum'\{L_\gamma:\gamma\in w\}$ in $\mathrm{BA}'(G,G_1)$. Then (a) for some $x^\cdot\in M_x$, $x_\gamma\in L_\gamma$ (for $\gamma\in w$), $x=x^\cdot\prod_{\gamma\in w}x^\gamma$ and $x^\cdot$, $x^\gamma\in B_{\alpha,i+1}$ for $\gamma\in w$; (b) x has θ conjugates in $B_{\alpha,i+1}$.

(8) $K_{\alpha,0} = \langle g_\alpha, \bigcup_{\gamma<\chi} P_{\alpha,\gamma}\rangle_G$. There are no special problems in the definition. For (3) and (4) operate separately on $K_{\alpha,i} \cap M_{\alpha,\gamma}$ and on $K_{\alpha,i} \cap (L_\gamma - M_{\alpha,\gamma})$ (subtraction $-$ in $\mathscr{B}$).

Now $B_\alpha = \bigcup_{i<\theta} B_{\alpha,i}$, $K_\alpha = \bigcup_{i<\theta} K_{\alpha,i}$ are as required. (In the construction for each γ, work for $M_{\alpha,\gamma}$, $L_\gamma - M_{\alpha,\gamma}$ separately).

Note that $K_\alpha = \langle B_\alpha, \bigcup_{\gamma<\chi}(K_\alpha \cap L_\gamma)\rangle_G$ by (2). Also $K_\alpha^{(1)}$ includes $K_\alpha \cap L_\gamma$ (by (4)), and B_α (by (5)) and by the previous sentence $K_\alpha = K_\alpha^{(1)}$.

Let us check that every $x \in K_\alpha - \mathrm{Cent}\, K_\alpha$ has $\geqslant \theta$ conjugates in K_α. If for some $\gamma < \chi$, $\mathrm{End}_G^{L_\gamma}(x) \not\subseteq \mathrm{Cent}\, L_\gamma$, then (3) (and (5)) take care of this. Otherwise let $x = \prod_{l=1}^n x_l a_l$, $x_l \in B_\alpha$, $a_l \in K_\alpha \cap L_{\gamma_l}$. Let $M_a = \sum' L_{\gamma_l}$, M_b its complement in $\mathrm{BA}'(G, G_1)$. Clearly $\mathrm{End}_G^{M_b}(x) = \mathrm{End}_G^{M_b}(\prod_{l=1}^n x_l)$; and as we are assuming $(\forall \gamma < \chi)\, \mathrm{End}_G^{L_\gamma}(x) \subseteq \mathrm{Cent}\, L_\gamma$ for $\gamma \notin \{\gamma_1, \ldots, \gamma_n\}$, $\mathrm{End}_G^{L_\gamma}(\prod_{l=1}^n x_l) \subseteq \mathrm{Cent}\, L_\gamma$. So (6) applies except when $w = \{\gamma < \chi\colon \mathrm{End}_G^{L_\gamma}(\prod_{l=1}^n x_l) \not\subseteq \mathrm{Cent}\, L_\gamma\}$ is finite. If $w \not\subseteq \{\gamma_1, \ldots, \gamma_n\}$ we finish by (7)(a) and (5) (first phrase applies to any $\gamma \in w - \{\gamma_1, \ldots, \gamma_n\}$).

By (7)(a), $\prod_{l=1}^n x_l = x^\cdot \prod_{\gamma\in w} x^\gamma$, $x^\gamma \in \mathrm{End}_G^{L_\gamma}(x) \cap K_\alpha$, $x^\cdot \in B_\alpha$. Together (using the properties of direct decomposition) $(x^\cdot)^{-1}x \in K_\alpha \cap \sum_{l=1}^n L_{\gamma_l}$, hence $x = \prod_{l=1}^n y_l$, $y_l \in K_\alpha \cap L_{\gamma_l}$, $y_0 \in G - \sum_{l=1}^n L_{\gamma_l}$ (the subtraction in $\mathscr{B}$) and $y_0 \in B_\alpha$, $y_l \in K_\alpha \cap L_{\gamma_l}$ for $l = 1, n$. By (7)(b), $\{\Box^g y_0 : g \in B_\alpha\}$ has power $\geqslant \theta$, and then easily $\{\Box^g x : g \in B_\alpha\}$ has power θ, except when $y_0 \in \mathrm{Cent}\, G$. Also if $y_l \notin \mathrm{Cent}\, G$ $(l = 1, n)$, then $\{\Box^g y_l : g \in K_\alpha \cap L_\gamma\}$ has power $\geqslant \theta$ giving the conclusion. So we fail only if $x = \prod_{l=1}^n y_l \in \mathrm{Cent}\, G$ but we assumed $x \notin \mathrm{Cent}\, G$.

Also the other properties are easy.

7.15B. Definition. For $x \in K \subseteq G(K = K^{(1)})$ let $\mathrm{sv}_\gamma(x, K) = \{M \in \mathscr{B}\colon M \subseteq L_\gamma$, and for some $K_1 \in \mathrm{BA}'(K)$, $K_1 \cap x\, \mathrm{Cent}\, K \neq \emptyset$ and $K_1 \subseteq L_\gamma - M\}$.

$\mathrm{sv}(x, K) = \langle \mathrm{sv}_\gamma(x, K) : \gamma < \chi\rangle / \mathscr{D}$.

7.16. Fact. (1) *If* $K = K^{(1)}$, $x \in K - \mathrm{Cent}\, K$, *then* $\{I \in \mathrm{BA}'(K) : I \cap x\, \mathrm{Cent}\, K \neq \emptyset\}$ *is a filter of the Boolean algebra* $\mathrm{BA}'(K)$.

(2) $\mathrm{sv}_\gamma(x, K)$ *is an ideal of* $\mathscr{B} \restriction L$.

Proof. (1) Note that $(x\, \mathrm{Cent}\, K) \cap I = \emptyset$ is equivalent to: x commutes with the complement of I in $\mathrm{BA}'(K)$. Clearly $\mathrm{sv}_\gamma(x, K)$ is upward closed. Suppose M_a, M_b belong to $\mathrm{sv}_\gamma(x, K)$. We can find $M_l \in \mathrm{BA}'(K)$ for $l = 1, 2, 3, 4$, $K = \sum_{l=1}^{4}{}' M_l$, $M_a = M_1 + M_2$, $M_b = M_1 + M_3$. We can find $x_l \in M_l$ (for $l = 1, 2, 3, 4$) such that $x = \sum_{l=1}^4 x_l$. The checking is easy.

(2) Left to the reader.

7.17. Fact. *If* x, $y \in K$, $x\, \mathrm{Cent}\, K = y\, \mathrm{Cent}\, K$ *or even* $\mathrm{End}_G^{L_\gamma}(x) = \mathrm{End}_G^{L_\gamma}(y)$, *then* $\mathrm{sv}_\gamma(x, K) = \mathrm{sv}_\gamma(y, K)$.

7.18. Fact. *If* $K = K^{(1)} = K_a +' K_b$, $x \in K_a$, *then* $\mathrm{sv}_\gamma(x, K) = \mathrm{sv}_\gamma(x, K_a)$, $\mathrm{sv}(x, K) = \mathrm{sv}(x, K_a)$.

7.19. Fact. *For every $x \in K_\alpha$ for some $z \in B_\alpha$,* $\mathrm{sv}(x, K_\alpha) = \mathrm{sv}(z, K_\alpha)$.

Proof. Let $x = \prod_{l=1}^{n} x_l a_l$, $x_l \in B_\alpha$, $a_l \in K_\alpha \cap L_{\gamma_l}$ $(\gamma_l < \chi)$ (possibly by (i) of 7.15.) Let K_a be $(\sum_{l=1}^{n} L_{\gamma_l})$, and K_b its complement in $\mathrm{BA}'(G)$. As K is nice, $K = K_a \cap K +' K_b \cap K$. So let $x = x_a + x_b$, $x_a \in K_a$, $x_b \in K_b$. By the choice of x_l, a_l, γ_l $(l = 1, n)$ $(x_a \operatorname{Cent} K) \cap B_\alpha \neq \emptyset$.

Let $y = \prod_{l=1}^{n} x_l$; clearly $y \in B_\alpha$. Next choose $x_l' \in \mathrm{End}_G^{K_b}(x)$, then clearly

$$\prod_{l=1}^{m} x_l \operatorname{Cent} K_b = \prod_{l=1}^{n} x_l' \operatorname{Cent} K_b$$

and so $\prod_{l=1}^{n} x_l'$ belongs to $\mathrm{End}_G^{K_b}(y)$ and also to $\mathrm{Eng}_G^{K_b}(x)$ and *if* $\gamma < \chi$, $\gamma \notin \{\gamma_1, \ldots, \gamma_n\}$, *then* $\mathrm{End}_G^{L_\gamma}(y)$, $\mathrm{End}_G^{L_\gamma}(\prod_{l=1}^{n} x_l')$ and $\mathrm{End}_G^{L}(x)$ are equal (they are all of the form $z \operatorname{Cent} L_\gamma$), hence $\mathrm{sv}_\gamma(x, K) = \mathrm{sv}_\gamma(y, K)$. As the filter $\mathscr{D}$ is non-principal (no finite set belongs to it), clearly $\mathrm{sv}(y, K) = \mathrm{sv}(x, K)$ and the proof is complete.

7.20. Fact. *If* $K = K^{(1)} = \sum_{l<m} K^l$, $x_l \in K^l$, $x = \sum_{l<m} x_l$, *then*:

(1) $\mathrm{sv}_\gamma(x, K) = \bigcap_{l<m} \mathrm{sv}_\gamma(x_l, K)$.

(2) *From* $\langle \mathrm{sv}(x_l, K_l) : l < m \rangle$ *we can compute* $\mathrm{sv}(x, K)$.

Proof. Clearly (2) follows from (1), and (1) is straightforward.

7.21. Fact. $\mathrm{sv}(g_\alpha, K_\alpha)$ *for* $\alpha < \kappa^+$ *are distinct.*

Proof of 7.2. Let $\mathrm{SV}_\alpha = \{\mathrm{sv}(x, K_\alpha) : x \in K_\alpha\}$. So by Fact 7.19, $\mathrm{SV}_\alpha = \{\mathrm{sv}(x, K_\alpha) : x \in B_\alpha\}$, hence has power $\leq \theta < \kappa$. Let $\mathrm{SV}_\alpha^{\mathrm{a}}$ be $\bigcup_{\beta<\alpha} \mathrm{SV}_\beta$. Let $\mathrm{SV}_\alpha^{\mathrm{b}}$ be $\{\mathrm{sv}(x, K_\gamma)$: for some m and γ_l $(l < m)$, $\mathrm{sv}(x_l, K_{\gamma_l}) \in \mathrm{SV}_\alpha^{\mathrm{a}}$ and for every γ, $\mathrm{sv}(x, K_\gamma)$ is computed from them as in 7.20(2)$\}$.

Clearly $|\mathrm{SV}_\alpha^{\mathrm{a}}| \leq \kappa$ for $\alpha < \kappa^+$, and even $|\mathrm{SV}_\alpha^{\mathrm{b}}| \leq \kappa$. Also $\mathrm{SV}_\alpha^{\mathrm{a}} \subseteq \mathrm{SV}_\alpha^{\mathrm{b}}$, and $\mathrm{SV}_\alpha^{\mathrm{b}}$ is increasing and continuous.

By AP2.1 (Fodor's Lemma) for some unbounded $S \subseteq \kappa^+$ and $\alpha(*) < \kappa^+$, for every $\alpha \in S$, $\mathrm{SV}_\alpha \cap \mathrm{SV}_\alpha^{\mathrm{b}} \subseteq \mathrm{SV}_{\alpha(*)}^{\mathrm{b}}$. By 7.21 w.l.o.g. $\alpha \in S \Rightarrow \mathrm{sv}(g_\alpha, K_\alpha) \notin \mathrm{SV}_\alpha^{\mathrm{b}}$.

Now suppose $T_1, T_2 \subseteq S$, $\alpha \in T_1 - T_2$, $g \in G_1$ and $\square^g$ maps $\langle \bigcup (K_\beta : \beta \in T_1\} \rangle_G$ onto $\langle \bigcup \{K_\beta : \beta \in T_2\} \rangle_G$; we shall get a contradiction. Thus finishing the proof of 7.2. By 4.3(2), $\square^g K_\alpha = \sum'_{\beta \in T_2} (\square^g K_\alpha \cap K_\beta)$, so there are n, $\beta_1 < \cdots < \beta_n \in T_2$ and $g^l \in \square^g K_\alpha \cap K_{\beta_l}$ s.t.:

$$\square^g g_\alpha = \sum_{l=1}^{n} g^l.$$

If $\beta_1, \ldots, \beta_n < \alpha$, then $\mathrm{sv}(g_\alpha, K_\alpha) \in \mathrm{SV}_\alpha^{\mathrm{b}}$ is a contradiction to the choice of S.

If $\beta_l \geq \alpha$, then $\beta_l > \alpha$ (as $\alpha \notin T_2$), and $(\square^g K_\alpha) \cap K_{\beta_l}$ is conjugate (in G_1) to a direct summand of K_α, hence by 7.18, $\mathrm{sv}(g^l, (\square^g K_\alpha) \cap K_{\beta_l}) \in \mathrm{SV}_{\alpha+1}^{\mathrm{a}} \subseteq \mathrm{SV}_{\beta_l}^{\mathrm{b}}$, but also by 7.18, $\mathrm{sv}(g^l, K_{\beta_l}) = \mathrm{sv}(g^l, (\square^g K_\alpha) \cap K_{\beta_l})$. As $\beta_l \in T_2 \subseteq S$, $\mathrm{sv}(g^l, K_{\beta_l}) \in \mathrm{SV}_{\alpha(*)}^{\mathrm{b}}$. So for each l, $\mathrm{sv}(g^l, K_{\beta_l}) \in \mathrm{SV}_\alpha^{\mathrm{b}}$, hence again $\mathrm{sv}(g_\alpha, K_\alpha) \in \mathrm{SV}_\alpha^{\mathrm{b}}$, contradiction to the choice of S.

8. The end for μ successor

8.1. Lemma. *Suppose* $\mu = \kappa^+$. *If* $G \in \Omega^1_\lambda$, *then* $\mathrm{cg}(G) \leq \kappa$.

Proof. It is enough to prove this for $G \in \Omega_\lambda$. We choose by induction on $i < \mu$, a_i such that:

(1) $a_i \notin \langle a_j : j < i \rangle^{\mathrm{cg}}_G$, $a_i \in \mathrm{Cm}_G\{a_j : j < i\}$.

(2) Let $n_i = \mathrm{Min}\{n : n > 0, (a_i)^n \in \{a_i : j < i\rangle^{\mathrm{cg}}_G\}$ (and $n_i = 0$ if there is no such n). Then n_i is minimal but > 0 if possible.

Suppose first a_i is defined for $i < \mu$. Clearly, n_i is zero or a prime. Let for each i, $w_i \subseteq i$ be finite, such that $(a_i)^{n_i} \in \langle a_j : j \in w_i \rangle^{\mathrm{cg}}_G$. By Fodor's Lemma (see AP 2) there are a stationary $S \subseteq \kappa^+ = \mu$ and w such that $w_i = w$, $n_i = n(*)$ for $i \in S$. Now let $N = \langle a_j : j \in w \rangle^{\mathrm{cg}}_G$, so clearly N is a normal subgroup of G, but $|G/N| = \lambda$ (if $G = N$, we get our conclusion, otherwise remember $G \in \Omega_\lambda$). Hence, by 1.6, $G/N \in \mathcal{P}_\lambda$ (in fact $G/N \in \Omega_\lambda$). For any $T \subseteq S$ let $H_T = \langle a_i : i \in T \rangle_G$, so it is enough to prove that: if $\alpha \in T_1 - T_2$, then for no $g \in G$, $\Box^g$ maps H_{T_1} onto H_{T_2} or even a_α into H_{T_2}. If this occurs let (remember H_{T_2} is commutative):

$$(*) \qquad \Box^g a_\alpha = \prod_{l=1}^{k} (a_{\beta_l})^{m(l)} \prod_{l=1}^{m} ((a_{\gamma_l})^{n(*)})^{k(l)},$$

where $m(l) \neq 0$ and $[n(*) > 0 \Rightarrow 0 < m(l) < n(*)]$. We know $a_\alpha \notin \langle a_j : j < \alpha \rangle^{\mathrm{cg}}_G$, hence $\Box^g a_\alpha \notin \langle a_j : j < \alpha \rangle^{\mathrm{cg}}_G$, and as $(a_{\gamma_l})^{n(*)} \in \langle a_j : j \in w \rangle^{\mathrm{cg}}_G$, $w \subseteq \alpha$, clearly for some l, $\beta_l \geq \alpha$, $1 \leq l \leq k$. As $\alpha \notin T_2$, $\beta_l > \alpha$, and choose a maximal such β_l, and w.l.o.g., it is β_k. Now

$$(a_{\beta_k})^{-m(k)} = (\Box^g a_\alpha)^{-1} \prod_{l=1}^{k-1} (a_{\beta_l})^{k(l)} \prod_{l=1}^{m} ((a_{\gamma_l})^{n(*)})^{k(l)},$$

hence $(a_{\beta_k})^{m(k)} \in \langle a_j : j < \beta_k \rangle^{\mathrm{cg}}_G$, contradicting $0 < m(k) < n(*)$ or $m(k) \neq 0$ (by (2)).

We conclude that for some $i < \mu$ we cannot find an a_i. Clearly there is no a_i satisfying (1), so $\mathrm{Cm}_G(\{a_j : j < i\})$ is included in $N = \langle a_j : j < i \rangle^{\mathrm{cg}}_G$. By 1.12 this implies $(G : N) < \lambda$. But $G \in \Omega_\lambda$, hence $G = N$, so we finish.

8.2. Theorem. *The main theorem holds when* $\mu = \kappa^+$.

Proof. Choose $G \in \mathcal{P}^1_\lambda$ with minimal $\gamma(G)$. We know that $\gamma(G) < \mu$ (by 5.7).

We define by induction on $i < \mu$, a subgroup K_i such that:

(1) $K_i = K_i^{(1)}$, K_i has power κ.

(2) K_i commutes with K_j for $j < i$.

(3) $\mathrm{BA}(K_i)$ has power $\leq \kappa$.

(4) No $I \in \mathrm{BA}(K_i)$ is conjugate in G to any $J \in \bigcup_{j<i} \mathrm{BA}(K_j)$.

This clearly suffices.

Suppose we have defined K_j for $j<i$; let $G_i = \operatorname{Min} \operatorname{Cm}_G(\bigcup_{j<i} K_j)$. We know that $G_i \in \Omega^1_\lambda$, $\operatorname{cg}(G_i) \leq \kappa$ (by 8.1), $|\operatorname{BA}(G_i)| \leq \kappa$ (by 7.1), and let $G_i = \langle A \rangle^{\mathrm{cg}}_{G_i}$, $|A| \leq \kappa$. Now we define by induction on $\alpha < \mu$, $M_\alpha \subseteq G_i$, such that:

(a) $A \subseteq M_0$, M_α has power $\leq \kappa$, M_α is increasing, $M_\alpha = M_\alpha^{(1)}$.

(b) $\operatorname{BA}(M_\alpha) = \{I \cap M_\alpha : I \in \operatorname{BA}(G_i)\}$.

(c) For every $\alpha < \mu$, $I \in \operatorname{BA}(G_i)$ and $\gamma < \gamma(G)$ there is a $H^i_\alpha = \sum'_{\beta<\gamma(G)} H^i_{\alpha,\beta}$, $H^i_{\alpha,\beta} \not\subseteq \operatorname{Cent} \operatorname{Cm}_{M_\alpha}(G_i)$ and for some γ_β, $\beta \leq \gamma_\beta < \gamma(G)$, $H^i_{\alpha,\beta}$ is a γ_β-required subgroup of $M_{\alpha+1} \cap \operatorname{Cm}_{G_i}(M_\alpha) \cap I$.

We first take care of (c)—remembering $\operatorname{Cm}_{G_i}(M_\alpha) \cap I \in \mathscr{P}^1_\lambda$ and $\gamma(G)$'s minimality—then of (b), (a) by 4.12, 4.13 (and see AP 1.3).

Now by (c) for no $I \in \operatorname{BA}(G_i)$ and $\alpha < \beta$ are $I \cap M_\alpha$, $I \cap M_\beta$ conjugate in G (by the minimality of $\gamma(G)$ and 5.9). As $\bigcup_{j<i} \operatorname{BA}(K_j)$ has power $\leq \kappa$, necessarily for each $I \in \operatorname{BA}(G_i)$ for some $\alpha_I < \mu$, for no α, $\alpha_I \leq \alpha < \kappa^+$, is $I \cap M_\alpha$ conjugate in G to some $J \in \bigcup_{j<i} \operatorname{BA}(K_j)$. As $\operatorname{BA}(G_i)$ has power $\leq \kappa$, $\alpha = \bigcup \{\alpha_I + 1 : I \in \operatorname{BA}(G_i)\}$ is smaller than μ. But now by (b), M_α is a satisfactory candidate for K_i, so we finish the construction of the K_i's hence of the theorem.

8.3. Hypothesis. *μ is a limit cardinal.*

8.4. Lemma. *Let $\theta^+ < \kappa$, $2^\kappa < 2^{\kappa^+}$. For every group G at least one of the following occurs*:

(1) *For some $A \subseteq G$, $|A| \leq \kappa$, $\operatorname{Min}_\theta \operatorname{Cm}_G A = \{e\}$.*

(2) *There are $K_\alpha \subseteq G$ for $\alpha < (2^\kappa)^+$, $|K_\alpha| \leq \kappa^+$, the subgroups $\langle K_\alpha, \operatorname{Cent} G\rangle_G$ are pairwise nonconjugate in G and $\bigcup \{K_\alpha : \alpha < (2^\kappa)^+\}$ has power $\leq \kappa^+$.*

(3) *There are $K_\alpha \subseteq G$ for $\alpha < (2^\kappa)^+$, $|K_\alpha| \leq \kappa^+$, the subgroups $\langle K_\alpha, \operatorname{Cent} G\rangle_G$ are pairwise nonconjugate in G, and K_α is a semi-direct sum of $[\theta, \kappa^+)$-groups.*

Remark. We can replace κ^+ by an inaccessible cardinal.

Proof. This is really a repetition of the proof of 8.2. Let $\mathscr{P}^3 = \mathscr{P}^3_{\kappa,\theta}$ be the class of counterexamples (i.e., $G \in \mathscr{P}^3$ iff G does not satisfy (1), (2) (3)) and let

$$\mathscr{P}^4 = \{H : H^{(\infty)} \text{ is a semi-direct sum of } [\theta, \kappa^+)\text{-groups}\}.$$

(a) Each $G \in \mathscr{P}^3$ (or just G satisfies not (1)) has an abelian subgroup of cardinality κ^+. [Why? We can choose by induction on $i < \kappa$ $a_i \in G$ such that $a_i \notin A_i \overset{\text{def}}{=} \{a_j : j < i\}$, $a_i \in \operatorname{Cm}_G A_i$; if we succeed to carry the definition we clearly prove the assertion. Suppose a_j is defined for $j < i$, then $H_i \overset{\text{def}}{=} \operatorname{Min}_\theta \operatorname{Cm}_G A_i$ is not trivial (as G, being a member of $\mathscr{P}^3$, does not satisfy (1)), also $H_i = \operatorname{Min}_\theta H_i$ by its definition, so H_i cannot be commutative (see 3.10(1)). Choose $a_i \in H_i - \operatorname{Cent} H_i$, easily a_i is as required.]

(b) Each $G \in \mathscr{P}^3$ (or just G satisfies not (d) nor (2)) has cardinality $\geq 2^{(\kappa^+)}$. [Why? By (a) G has a commutative subgroup H of cardinality κ^+. We know that

H has $2^{(\kappa^+)}$ distinct subgroups, so if $|G|<2^{(\kappa^+)}$, $2^{(\kappa^+)}$ of them are nonconjugate in pairs; so (2) holds for G, contradiction.]

(c) If $H \subseteq G$, $G \in \mathscr{P}^3$, $H \in \mathscr{P}^4$, $|H| \leq \kappa$, then $\mathrm{Cm}_G(H) \in \mathscr{P}^3$ [by (d) below for $i=1, 2, 3$, we know that if $\mathrm{Cm}_G(H)$ satisfies (i), then so does G].

(d) Suppose $H \subseteq G$, $|H| \leq \kappa$.

(i) If $\mathrm{Cm}_G(H)$ satisfies (1), then so does G.

(ii) If $\mathrm{Cm}_G(H)$ satisfies (2), then so does G.

(iii) If $\mathrm{Cm}_G(H)$ satisfies (3) and $H \in \mathscr{P}^4$, then G satisfies (3).

[Proof: check.]

(e) If $G \in \mathscr{P}^3$ (or just G does not satisfy (2)), then $|\mathrm{Cent}(G)| \leq \kappa$, $(G:G^{(1)}) \leq \kappa$. [Proof: easy.]

(f) If $G \in \mathscr{P}^3$ (or just does not satisfy (2)), then G has no strictly decreasing sequence of normal subgroups of length κ^+. [Proof: see the proof of 3.1.]

(g) If G fails (2), N a subgroup of G, $(G:N) \leq 2^\kappa$, then N fails (2). [Proof: check.]

(h) If G fails (2), $\sigma \leq \kappa$, then $(G: \mathrm{Min}_\sigma\, G) \leq 2^\kappa$. [Proof: let h, $\langle N_i : i \leq \alpha^* \rangle$ be as in 3.10(4). By (f) above $C = \{\zeta \leq \beta : h(\zeta) = \zeta\}$ has cardinality $<\kappa^+$. Also if $\zeta \in C$, $(G:N_\zeta) \leq 2^\kappa$, then by (g) above N_ζ fails (2) hence by (f) applied to N_ζ the set $\{i : h(i) = \zeta\}$ has cardinality $\leq \kappa^+$. We conclude that $\alpha^* < \kappa^+$. By 3.10(4) $(N_i : N_{i+1}) \leq 2^\kappa$, so we can easily show that $\alpha < \kappa^+$, $(G:N_\alpha) \leq 2^\kappa$, as required.]

(i) If G fails (3), N a subgroup of G, $(G:N) \leq 2^\kappa$, then N fails (3). [Proof: check.]

(j) If G fails (1), (2), N a subgroup of G, $(G:N) \leq 2^\kappa$, then N fails (1). [Proof: suppose N satisfies (1), then for some $A \subseteq N$, $\mathrm{Min}_\theta\, \mathrm{Cm}_N(A) = \{e_G\}$. Let $K = \mathrm{Cm}_G(A)$, so $\mathrm{Cm}_N(A) = K \cap N$. We know that K fails (1) and (2) (by (d) above) and that $(K : K \cap N) \leq (G:N) \leq 2^\kappa$, hence $K \cap N$ fails (2) (by (g) above). So by (h) above $(K \cap N : \mathrm{Min}_\theta\, K \cap N) \leq 2^\kappa$. But $\mathrm{Min}_\theta(K \cap N) = \mathrm{Min}_\theta(\mathrm{Cm}_G(A) \cap N) = \mathrm{Min}_\theta\, \mathrm{Cm}_N(A) = \{e\}$, hence $(K \cap N) \leq 2^\kappa$. As $(K : K \cap N) \leq (G:N) \leq 2^\kappa$, clearly $|K| \leq 2^\kappa$, so by (b) K satisfies (1) or (2), contradicting a statement above.]

(k) If $G \in \mathscr{P}^3$, N a subgroup of G, $(G:N) \leq 2^\kappa$, then $N \in \mathscr{P}^3$. [Proof: by (g), (i), (j) above.]

(l) If $G \in \mathscr{P}^3$, $\sigma \leq \kappa$, then $\mathrm{Min}_\sigma\, G \in \mathscr{P}^3$, $(G : \mathrm{Min}_\sigma(G)) \leq 2^\kappa$. [Proof: by (h) and (f) above.]

Let $\sigma \overset{\mathrm{def}}{=} \theta^+$, so $\theta < \sigma < \kappa$.

Choose $G \in \mathscr{P}^3$ with minimal $\gamma(G)$. W.l.o.g. $G = \mathrm{Min}_\kappa\, G$. Choose by induction on $\alpha < \kappa^+$, a group $K_\alpha \subseteq \mathrm{Cm}_G(\bigcup_{\beta<\alpha} K_\beta)$, $|K_\alpha| \leq \kappa$, $\mathrm{BA}'(\mathrm{Min}_\sigma\, K_\alpha, K_\alpha)$ has power $\leq \kappa$ and no $I \in \bigcup_{\beta<\alpha} \mathrm{BA}'(\mathrm{Min}_\sigma\, K_\beta, K_\beta)$, $J \in \mathrm{BA}'(\mathrm{Min}_\sigma\, K_\alpha, K_\alpha)$ are conjugate in G. If K_α is defined for each $\alpha < \kappa^+$, we easily get a contradiction by having (2).

So we assume K_{α_0} cannot be defined. Next we define by induction on non-limit $\gamma < \kappa^+$, for each $\eta \in {}^{\gamma}2$ a subgroup H_η of G s.t.:

(i) H_η is a $[\theta, \kappa^+)$-group.

(ii) $\bigcup_{\alpha<\alpha_0} K_\alpha \subseteq H_{\langle\,\rangle}$.

(iii) H_η is included in $\mathrm{Min}_\sigma \mathrm{Cm}_G \bigcup \{H_{\eta \restriction \gamma}: \gamma < l(\eta) \text{ non-limit}\}$.

(iv) If $\eta \in {}^\gamma 2$, $H_{\eta^\wedge\langle 0\rangle}$, $H_{\eta^\wedge\langle 1\rangle}$ satisfies: if we define H_ν $(\gamma < l(\nu) < \kappa^+)$ in any way satisfying (i), (ii), (iii), and $\nu_0, \nu_1 \in {}^{\kappa^+}2$, $\nu_l \restriction \gamma = \eta^\wedge\langle l\rangle$ and $g \in \mathrm{Cm}_G \bigcup \{H_{\eta\restriction\beta}: \beta < \gamma \text{ non-limit}\}$, then $\Box^g$ does not map $\{H_{\nu_0\restriction\beta}: \beta < \kappa^+ \text{ non-limit}\}$ onto $\{H_{\nu_1\restriction\beta}: \beta < \kappa^+ \text{ non-limit}\}$.

If we succeed we can get (3) by the weak diamond by AP 3.2. So suppose $H_{\eta\restriction\gamma}$ $(\gamma \leqslant l(\eta)$ non-limit) are defined, but not $H_{\eta^\wedge\langle 0\rangle}$, $H_{\eta^\wedge\langle 1\rangle}$. Let $G_\eta = \mathrm{Cm}_G \bigcup \{H_{\eta\restriction\gamma}: \gamma \leqslant l(\eta) \text{ non-limit}\}$. So $G_\eta \in \mathcal{P}^3$, $\mathrm{Min}_\sigma G_\eta \in \mathcal{P}^3$, $\mathrm{Min}_\theta G_\eta \in \mathcal{P}^3$.

We first note

8.4A. Fact. *If* $\mathrm{Min}_{\sigma(1)} G = G$, *N is a normal subgroup of G*, $(\mathrm{Cm}_G A) \subseteq N$, $\aleph_0 + |A| \leqslant \kappa(1)$, $(G:N) \geqslant \lambda(1)$, $\lambda(1) > 2^{\kappa(1)}$, *then there are* $H_i \subseteq G$ *for* $i < \lambda(1)$, *nonconjugate in pairs*, H_i *an explicit* (Min$\{\sigma(1), \kappa(1)\}$)*-group*. [Proof: like 1.12.]

8.4B. Fact. *If* $\mathrm{Min}_{\sigma(1)} G = G$, $\mathrm{cg}(G) > \kappa(1)$, *for no* $A \subseteq G$, $[|A| \leqslant \kappa(1) \wedge \mathrm{Cm}_G A \subseteq \langle A\rangle_G^{\mathrm{cg}}]$, *then for some* $a_i \in G$ $(i < \kappa(1)^+)$ *the groups* $\{\langle a_i : i \in S\rangle_G : S \subseteq \kappa(1)^+\}$ *are pairwise nonconjugate.* [Proof: like 8.1.]

8.4C. Fact. *If* $G = \mathrm{Min}_\sigma G \in \mathcal{P}^3$ *has minimal* $\gamma(G)$, *then: for some* $A \subseteq G$, $|A| \leqslant \kappa$, *for no H*, $A \subseteq H \subseteq G$, $|\mathrm{BA}'(\mathrm{Min}_\sigma \mathrm{Cm}_G H)| \leqslant \kappa$, *and* $\mathrm{cg}(\mathrm{Min}_\sigma \mathrm{Cm}_G H) \leqslant \kappa$. [Proof: like 8.2.]

8.4D. Fact. *If* $G = \mathrm{Min}_\sigma G \in \mathcal{P}^3$ $[I \in \mathrm{BA}'(G) \Rightarrow I \in \mathcal{P}^3]$, *then* $|\mathrm{BA}'(G)| \leqslant \kappa$. [Proof: like 7.1.]

8.4E. Fact. *If* $G = \mathrm{Min}_\sigma G \in \mathcal{P}^3$, *then for some explicit* $[\sigma, \kappa^+)$*-group* $H \subseteq G$, $[I \in \mathrm{BA}'(G),\ I \notin \mathcal{P}^3 \Rightarrow I \cap \mathrm{Min}_\sigma \mathrm{Cm}_G H \subseteq \mathrm{Cent}\ I]$.

Proof. We choose by induction on $\alpha < \kappa^+$, a_α, I_α such that:

(i) $a_\alpha \in I_\alpha \in \mathcal{B} = \{I \in \mathrm{BA}'(G): I \notin \mathcal{P}^3\}$,

(ii) $a_\alpha \notin \langle \bigcup_{\beta<\alpha} I_\beta\rangle_G$ *and* $a_\alpha \in \mathrm{Cm}_G \bigcup_{\beta<\alpha} I_\beta$,

(iii) under (i), (ii), $n_i = \mathrm{Min}\{n > 0: a_\alpha^n \in \langle \alpha_\beta : \beta < \alpha\rangle_G\}$ is minimal.

If we succeed we continue as in 8.1 and get that G satisfies (1), contracting $G \in \mathcal{P}^3$ (note that $\langle \bigcup_{\beta<\alpha} I_\beta\rangle_G$ is a normal subgroup of G as each I_β $(\beta < \alpha)$ is). If not, say $a_{\alpha(*)}$ not defined, we can choose (by 3.4F below) A_α, $A_\alpha \subseteq I_\alpha$, $\mathrm{Cm}_{A_\alpha} I_\alpha = \{e\}$ for $\alpha < \alpha(*)$ and an explicit $[\sigma, \kappa^+)$-group H, $\{a_\alpha : \alpha < \alpha(*)\} \cup \bigcup\{A_\alpha : \alpha < \alpha(*)\} \subseteq H \subseteq G$. H is as required.

8.4F. Fact. *If* $I \in \mathrm{BA}'(G)$, $G \in \mathcal{P}^3$, $I \notin \mathcal{P}^3$, *then* (1) *of* 8.4 *fails for I.* [Proof: check.]

Now we return to deriving a contradiction from the impossibility to define $H_{\eta^\wedge\langle 0\rangle}$, $H_{\eta^\wedge\langle 1\rangle}$.

By 8.4C applied to $G_a \overset{\mathrm{def}}{=} \mathrm{Min}_\sigma G_\eta$ we get an A as there. So there is an explicit $[\sigma, \kappa^+)$-group H_a, $A \subseteq H_a \subseteq G_a$. Let $G_b = \mathrm{Min}_\sigma \mathrm{Cm}_{G_a} H_a$. By the choice of A, $|\mathrm{BA}'(G_b)| > \kappa$ or $\mathrm{cg}(G_b) > \kappa$. By 8.4A, 8.4B, $\mathrm{cg}(G_b) \leqslant \kappa$, hence $|\mathrm{BA}'(G_b)| > \kappa$.

By 8.4D [applied to G_b] there is $I \in \mathrm{BA}'(G_b)$, $I \notin \mathscr{P}^3$. Let H_b^0 be as in 3.4E (for G_b). Let $H_b^1 \subseteq I$ be an explicit θ-group.

We choose $H_{\eta^\wedge\langle\beta\rangle} = H_a + H_b^l$. We leave the checking to the reader.

9. The end for μ limit

9.1. Definition. (1) Suppose $G \in \mathscr{P}_\lambda^1$, $\aleph_1 < \theta < \mu$. Then H is called a $[\theta, \kappa)$-special subgroup of G if $H = \sum_{i=1}^{\prime 2} H_i$ where

(a) $H_1^{(1)} = H_1$, $H_2^{(1)} = H_2$, H_1 commutes with H_2.

(b) $|H_1| + |H_2| < \kappa$.

(c) H_1 is a semi-direct sum of groups each of power $\leqslant \aleph_1$

(d) H_2 is a $[\theta, \kappa)$-group, $H \subseteq G$.

(e) No semi-direct summand of H_1 is included in $\mathrm{Min}\,\mathrm{Cm}_G(H_2)$.

(2) If $\kappa = \theta^+$, we write "a θ-specal subgroup of G".

9.2. Claim. *Suppose $G \in \mathscr{P}_\lambda^1$, H^i a $[\theta_i, \mu)$-special subgroup of $G_i = \mathrm{Min}\,\mathrm{Cm}_G(\bigcup_{j<i} H^j)$ for $i < \mathrm{cf}\,\mu$; and $i < j \Rightarrow |H^i| < \theta_j$, and $H = \sum'_{i<\mathrm{cf}\,\mu} H^i$. Then from H we can reconstruct the H^i's (modulo the θ_i's).*

Proof. We reconstruct them by induction on i. Let $G_i = \mathrm{Cm}_G(\bigcup_{j<i} H_j)$. In stage i let $H^i = H_1^i + H_2^i$ (as in the definition). So H_2^i is the maximal semi-direct summand $[\theta_i, \theta_{i+1})$-subgroup of $H \cap \mathrm{Cm}_G(\bigcup_{j<i} H_j)$ and $H_1^i = \langle I \cap H : I$ a semi-direct summand of $\mathrm{Cm}_{G_i}(H_2^i)$, $|I \cap H| \leqslant \aleph_1$, no direct summand of which is included in $\mathrm{Min}\,\mathrm{Cm}_{G_i}(H_2^i)\rangle_{G_i}$.

9.3. Lemma. (1) *Suppose $G \in \mathscr{P}_\lambda^1$, $\aleph_1 < \theta < \mu$. Then G has a $[\theta, \mu)$-special subgroup H such that for some $\kappa < \mu$, $\mathrm{Min}\,\mathrm{Cm}_G\, H$ does not have 2^κ $[\kappa, \mu)$-special subgroups nonconjugate (in $\mathrm{Min}\,\mathrm{Cm}_G\, H$) in pairs.*

(2) *If H_i is a $[\theta, \kappa)$-special subgroup of $\mathrm{Cm}_G(\bigcup_{j<i} H_j)$ for $i < \alpha$ where $G \in \mathscr{P}_\lambda^1$, then $\langle H_i : i < \alpha\rangle_G$ is a $[\theta, \kappa)$-special subgroup of G (provided that its power is $<\kappa$).*

(3) *If $G \in \mathscr{P}_\lambda^1$, H a $[\theta, \kappa)$-special subgroup of $\mathrm{Min}\, G$, then H is a $[\theta, \kappa)$-special subgroup of G.*

(4) *Any $[\theta, \kappa)$-subgroup is a $[\theta, \kappa)$-special subgroup (for $\theta > \aleph_1$).*

Proof. (1) By 6.12 there are strictly increasing $\kappa(i)$ $(i < \mathrm{cf}\,\mu)$, $\mu = \sum_{i<\mathrm{cf}\,\mu} \kappa(i)$, $\mathrm{cf}\,\mu + \theta < \kappa(0)$, $\kappa(i) < \mu$, $\mu < 2^{\kappa(i)} < 2^{\kappa(i)^+}$. We assume that the conclusion fails, and we define by induction on $i < \mathrm{cf}\,\mu$, for every $\eta \in \prod_{j<i} 2^{\kappa(j)}$ an ordinal $i_\eta < \mathrm{cf}\,\mu$ and subgroups H_η, H'_η of G such that:

(i) H_η commutes with $H_{\eta \restriction j}$ for $j < l(\eta)$, and $H'_\eta = \langle H_{\eta \restriction i} : i \leqslant l(\eta)\rangle_G$.

(ii) H_η is an $[\kappa(i_\eta), \mu)$-special subgroup, of $\mathrm{Min}\,\mathrm{Cm}_G(\bigcup H_{\eta \restriction j} : j < l(\eta)\})$.

(iii) $\kappa(i_\eta) > \sum_{j<l(\eta)} |H_{\eta \restriction j}|$.

(iv) For $i = l(\eta)$, $\alpha < \beta < 2^{\kappa(i)}$, the subgroups $H'_{\eta^\wedge\langle\alpha\rangle}$, $H'_{\eta^\wedge\langle\beta\rangle}$ are nonconjugate in G.

By 9.2 this is enough (as H_η is a $[\kappa(i_\eta), |H_\eta|^+)$-special subgroup of $\operatorname{Min}\operatorname{Cm}_G(\bigcup\{H_{\eta\restriction j}: j<l(\eta)\})$; and so for $\eta\in\prod_{j<\operatorname{cf}\mu}2^{k(j)}$, $H'_\eta=\langle H_{\eta\restriction j}: j<\operatorname{cf}\mu\rangle$ are pairwise nonconjugate subgroups of G of power μ, contradiction to $G\in\mathcal{P}^1_\lambda$ by 1.13(2)). As the number of possible $\langle\kappa(i_{\eta\restriction j}): j<\operatorname{cf}\mu\rangle$ is $\leq 2^{\operatorname{cf}\mu}<2^\mu=\lambda$.

So suppose $H_{\eta\restriction j}$ are defined for $j\leq i=l(\eta)<\operatorname{cf}\mu$, and we shall define $H_{\eta^\wedge\langle\alpha\rangle}$, $i_{\eta^\wedge\langle\alpha\rangle}$. We let, for all α's, $i(*)=i_{\eta^\wedge\langle\alpha\rangle}$ be the first $i<\operatorname{cf}\mu$ such that $\kappa(i(*))>\sum_{j\leq i}|H_{\eta\restriction j}|$, and $2^{\kappa(i(*))}>(\operatorname{Cm}_G(\bigcup_{j<i}H_{\eta\restriction j}):\operatorname{Min}\operatorname{Cm}_G(\bigcup_{j<i}H_{\eta\restriction j}))$ (this is possible by 6.12(4)). Now as we have assumed that the lemma fails, and as by 9.3(2) $\sum'_{j\leq i}H_{\eta\restriction j}$ is a $[\theta,\mu)$-special subgroup of G, clearly there are $[\kappa(i(*)),\mu)$-special subgroups of $\operatorname{Min}\operatorname{Cm}_G(\bigcup_{j\leq i}H_{\eta\restriction j})\,H^\alpha_\eta$ $(\alpha<2^{\kappa(i(*))^+})$ which are pairwise nonconjugate in $\operatorname{Min}\operatorname{Cm}_G(\bigcup_{j\leq i}H_{n\restriction j})$. As

$$2^{\kappa(i(*))^+}>2^{\kappa(i(*))}>\left(\operatorname{Cm}_G\left(\bigcup_{j\leq i}H_{\eta\restriction j}\right):\operatorname{Min}\operatorname{Cm}_G\left(\bigcup_{j\leq i}H_{\eta\restriction j}\right)\right),$$

by the proof of 1.5 w.l.o.g., they are pairwise nonconjugate in $\operatorname{Cm}_G(\bigcup_{j\leq i}H_{\eta\restriction j})$ and by the proof of 1.9 w.l.o.g., $\langle\bigcup_{j\leq i}H_{\eta\restriction j}, H^\alpha_\eta\rangle_G$ are pairwise nonconjugate in G. So we can have our $H_{\eta^\wedge\langle\alpha\rangle}$ $(\alpha<2^{\kappa(l(\eta))})$ as required.

(2), (3), (4) Easy.

9.4. Lemma. *If $G\in\mathcal{P}^1_\lambda$, $\aleph_1<\sigma<\theta<\mu$ and G has no 2^{θ^+} explicit $[\sigma,\theta^{++})$-subgroups nonconjugate in pairs inside G, then there is an explicit $[\sigma,\theta^+)$-subgroup K of* $\operatorname{Min}_\sigma G$ *such that* $\operatorname{Min}_\sigma\operatorname{Cm}_G(K)$ *is included in $\langle K\rangle^{\mathrm{cg}}_G$ (hence by 3.12,* $\operatorname{Min} G\subseteq\langle K\rangle^{\mathrm{cg}}_{\operatorname{Min}_\sigma G}$).

Proof. Suppose that the conclusion fails. Then we can define by induction on $i<\theta^+$, an element $x_i\in\operatorname{Min}_\sigma\operatorname{Cm}_G(\bigcup_{j<i}K_j)$ not in $\langle\bigcup_{j<i}K_j\rangle^{\mathrm{cg}}_G$, and then choose an explicit σ-subgroup K_i of $\operatorname{Min}_\sigma\operatorname{Cm}_G(\bigcup_{j<i}K_j)$ to which x_i belongs. (Remember that by 3.10, $\operatorname{Min}_\sigma\operatorname{Cm}_G(\bigcup_{j<i}K_j)$ is an explicit σ-group, and of course is in $\mathcal{P}^1_\lambda$.)

Let $\mathrm{Set}_i=\{\{gyg^{-1}: g\in G\}: y\in K_i\}$, it has power $\leq|K_i|=\sigma$. Let $\mathrm{Set}^i=\{\{gyg^{-1}: g\in G\}: y\in\langle\bigcup_{j<i}K_j\rangle^{\mathrm{cg}}_G\}$, so Set^i is increasing (in i) continuously, and $\bigcup_{j<i}\mathrm{Set}_j\subseteq\mathrm{Set}_i\not\subseteq\mathrm{Set}^i$. By a lemma of Fodor (see AP 2.3) for some $\alpha(*)<\theta^+$ and unbounded $S(*)\subseteq\theta^+$, for every $\beta\in S(*)$, $\mathrm{Set}_\beta\cap\mathrm{Set}^\beta\subseteq\mathrm{Set}^{\alpha(*)}$.

Now for any set $S\subseteq S(*)$ let:

$$H_S=\langle K_i: i\in S\rangle_G.$$

Now for $S\neq T$, H_S, H_T cannot be conjugates, for suppose $\alpha\in S-T$, $a\in G$ and $\Box^a$ maps H_T onto H_S, then $H_T=\sum'_{i\in T}K_i=\sum'_{j\in S}\Box^a K_j$, hence by 4.7 for some $T_1\subseteq T$, $\Box^a K_\alpha=\sum'_{j\in T_1}(\Box^a K_\alpha)\cap H_j$, hence for some finite $T_2\subseteq T_1$, $\Box^a x_\alpha=\prod_{i\in T_2}y_i$, $y_i\in(\Box^a K_\alpha)\cap K_i$. Let $T_3=\{i\in T_2: y_i\notin\langle\bigcup_{j<\alpha(*)}K_j\rangle^{\mathrm{cg}}_G\}$. If $T_3\subseteq\alpha$, then $\{gx_\alpha g^{-1}: g\in G\}\in\mathrm{Set}^\alpha$, contradicting the choice of x_α. So $T_3\not\subseteq\alpha$, and let i be the maximal member of T_3, but then as y_j $(j\in T_1)$ are pairwise commuting, $\{gy_ig^{-1}: g\in G\}\in\mathrm{Set}^i$ again. So $\{H_S: S\subseteq S(*)\}$ contradicts a hypothesis, hence we have proven the lemma.

9.5. Lemma. *If $\aleph_1 < \sigma < \theta < \mu < 2^\theta < 2^{\theta^+}$, $G \in \mathscr{P}^1_\lambda$ and G has no $(2^\theta)^+$ explicit $[\sigma, \theta^{++})$-subgroups nonconjugate in pairs inside G, then there is an explicit $[\sigma, \theta^+)$-subgroup K of* $\mathrm{Min}_\sigma G$, *such that*

$$\mathrm{cg}(\mathrm{Min}_\sigma \mathrm{Cm}_G(K), \mathrm{Cm}_G(K)) \leqslant \theta.$$

Proof. Suppose not. Then we shall define by induction on $\alpha < \theta^+$, for every $\eta \in {}^{(\alpha+1)}2$ an explicit $[\sigma, \theta^+)$-subgroup K_η such that:

(i) Let $G_\eta = \mathrm{Cm}_G(\bigcup \{K_{\eta \restriction (\beta+1)} : \beta + 1 \leqslant l(\eta)\})$.

(ii) $K_{\eta\hat{}\langle m \rangle} \subseteq \mathrm{Min}_\sigma G_\eta$ for $m = 0, 1$.

(iii) If $\nu \in {}^\alpha 2$, then some $x_\eta \in K_{\eta\hat{}\langle 1 \rangle}$ does not belong to $\langle K_{\eta\hat{}\langle 0 \rangle}, \mathrm{Min}_\sigma G_{\eta\hat{}\langle 0 \rangle}\rangle^{\mathrm{cg}}_{G_\eta}$.

Suppose $K_{\eta \restriction (\beta+1)}$, $\beta + 1 \leqslant l(\eta)$ are defined already. As $\langle K_{\eta \restriction (\beta+1)} : \beta + 1 \leqslant l(\eta)\rangle_G$ cannot satisfy the conclusion of Lemma 9.5, for no $A \subseteq \mathrm{Min}_\sigma G_\eta$ of power $\leqslant \theta$ does $\mathrm{Min}_\sigma G_\eta \subseteq \langle A \rangle^{\mathrm{cg}}_{G_\eta}$. Also G_η has no $(2^\theta)^+$ explicit $[\sigma, \theta^{++})$-subgroups nonconjugate in pairs inside G_η: for if H_i $(i < (2^\theta)^+)$ are such subgroups, then by the proof of 1.9 w.l.o.g., the subgroups $\langle \bigcup \{K_{\eta \restriction (\beta+1)} : \beta + 1 \leqslant l(\eta)\} \cup H_i \rangle_G$ for $i < (2^\theta)^+$ are nonconjugate in pairs inside G, contradiction to a hypothesis. So by Lemma 9.4, (applied to G_η) there is $K \subseteq \mathrm{Min}_\sigma G_\eta$, an explicit $[\sigma, \theta^+)$-group, such that $\mathrm{Min}_\sigma \mathrm{Cm}_{G_\eta}(K) \subseteq \langle K \rangle^{\mathrm{cg}}_{G_\eta}$ and we let $K_{\eta\hat{}\langle 0 \rangle} = K$. But as mentioned above, $\mathrm{Min}_\sigma G_\eta$ is not included in $\langle K_{\eta\hat{}\langle 0 \rangle}\rangle^{\mathrm{cg}}_{G_\eta}$, and choose $x_\eta \in \mathrm{Min}_\sigma G_\eta - \langle K_{\eta\hat{}\langle 0 \rangle}\rangle^{\mathrm{cg}}_{G_\eta}$ and $K_{\eta\hat{}\langle 1 \rangle}$ a (explicit) σ-subgroup of $\mathrm{Min}_\theta G_\eta$ to which x_η belongs.

Now (ii) holds trivially, and (iii) holds by the choices of $K_{\eta\hat{}\langle 0 \rangle} = K$, x_η and $K_{\eta\hat{}\langle 1 \rangle}$. Let for $\eta \in {}^{(\theta^+)}2$, $H_\eta = \langle K_{\eta \restriction (\alpha+1)} : \alpha < \theta^+ \rangle$.

We can now apply AP3.2 alternatively to the following.

By a hypothesis of the lemma, there is $\{N_i : i < i^* \leqslant 2^\theta\}$, a list of subgroups of G, so that each H_η $(\eta \in {}^{(\theta^+)}2)$ is conjugate inside G to one of them. So let $\square^{g_\eta}$ map H_η onto $N_{i(\eta)}$, $g_\eta \in G$. By a set-theoretic statement called 'the weak diamond' which holds for θ^+ as we have assumed that $2^\theta < 2^{\theta^+}$ (see AP 3.1) we can conclude:

$(*)$ There are $\eta, \nu \in {}^{\theta^+}2$, and ρ and a limit ordinal δ, s.t. $\rho = \eta \restriction \delta = \nu \restriction \delta$, $\eta(\delta) \neq \nu(\delta)$ but $i(\eta) = i(\nu)$, $\square^{g_\eta} \restriction \langle K_{\rho \restriction (\alpha+1)} : \alpha < \delta \rangle_G = \square^{g_\nu} \restriction \langle K_{\rho \restriction (\alpha+1)} : \alpha < \delta \rangle_G$.

So $\square^{g_\eta^{-1} g_\nu}$ is an inner automorphism of G, which is the identity on $\langle K_{\rho \restriction (\alpha+1)} : \alpha < \delta \rangle_G$, hence $g_\eta^{-1} g_\nu$ belongs to Cm_G $(\langle K_{\rho \restriction (\alpha+1)} : \alpha < \delta \rangle_G)$, i.e., to G_ρ, and maps H_ν onto H_ρ. This is an easy contradiction.

9.6. Lemma. *Suppose $G \in \mathscr{P}^1_\lambda$, $\gamma(G)$ minimal, $\theta < \mu$. Then there are $\kappa < \mu$, $\kappa > \theta$ and $A \subseteq G$ of power $< \mu$, such that:*

$(*)$ *In* $\mathrm{Cm}_G(A)$ *we cannot find pairwise commuting subgroups K_α $(\alpha < \kappa^+)$ such that:*

(i) $|K_\alpha| < \mu$.

(ii) $\mathrm{BA}'(\mathrm{Min}_\theta(K_\alpha), K_\alpha)$ *has power* $< \mu$.

(iii) *For $I \in \mathrm{BA}'(K_\alpha^{(\infty)}, K_\alpha)$: if $|I| < \kappa$, then every $x \in I - \mathrm{Cent}\, I$ has $\leqslant \aleph_0$ I-conjugates (such I is called essentially countable).*

(iv) *For* $\alpha \neq \beta$, *no nonessentially-countable* $I \in \mathrm{BA}'(\mathrm{Min}_\theta K_\alpha, K_\alpha)$, $J \in \mathrm{BA}'(\mathrm{Min}_\theta K_\beta, K_\beta)$ *are conjugates in G.*

(v) K_α *is not essentially countable.*

Proof. Let $\mu = \sum_{i<\mathrm{cf}\,\mu} \kappa_i^0$, $\theta < \kappa_i^0 < \mu$. We define by induction on $i < \mathrm{cf}\,\mu$, cardinals θ_i, κ_i and subgroups K_α^i $(\alpha < \theta_i^+)$ such that:

(a) $\sum_{j<i} \kappa_j < \theta_i < \mu$, $\mathrm{cf}\,\mu + \kappa_i^0 < \theta_i$, $\mu < 2^{\theta_i} < 2^{\theta_i^+}$.

(b) Conditions (i)–(v) of the lemma hold with $\bigcup \{K_\beta^j : \beta < \theta_j^+, j < i\}$, θ_i, K_α^i $(\alpha < \theta_i^+)$ standing for A, κ, K_α $(\alpha < \theta^+)$.

(c) $\theta_i^+ < \kappa_i < \mu$ and K_α^i has power $< \kappa_i$.

If the lemma fails there is no problem to proof it by induction on i: First define θ_i by (a), then K_α^i $(\alpha < \theta_i^+)$ by the failure of the lemma, then we replace $\langle K_\alpha^i : \alpha < \theta_i^+ \rangle$ by a subsequence of the same power so that we can define κ_i by (c).

Now for any $\bar{S} = \langle S_i : i < \mathrm{cf}\,\mu \rangle$, $S_i \subseteq \theta_i^+$, we let $H_S = \langle K_\alpha^i : \alpha \in S_i,\ i < \mathrm{cf}\,\mu \rangle_G$. Clearly the number of possible $\bar{S}$,s is 2^μ, and by (iv) and (iii) (as used in (b)) we can prove that they are pairwise nonconjugate.

Remark. Remember that we should be careful to be able to know from which $\langle K_\alpha^i : \alpha \in S_i \rangle$ a semi-summand comes.

9.7. Definition. $\mathrm{Min}[H, \chi] = \bigcap \{N : N$ a normal subgroup of H, $(H:N) < \chi\}$; let $\mathrm{Min}^\kappa H = \mathrm{Min}[H, (2^\kappa)^+]$.

9.7A. Fact. (1) $\mathrm{Min}[H, \chi]$ *is a characteristic subgroup of H.*

(2) $(H:\mathrm{Min}[H, \chi]) < \chi$ *and* $(\mathrm{Min}(H, \chi):\mathrm{Min}[\mathrm{Min}(H, \chi), \chi]) < \chi$ *implies* $\mathrm{Min}[\mathrm{Min}(H, \chi), \chi] = \mathrm{Min}[H, \chi]$.

(3) *Also, if* $2^\chi \geq \mu > \chi$, $G \in \mathscr{P}_\lambda^1$, *then* $\mathrm{Min}[G, \chi] \in \mathscr{P}_\lambda^1$ *and* $\mathrm{Min}^\chi G \subseteq \mathrm{Min}_{\chi^+} G$.

(4) *Also, if* $2^\chi \geq \mu > \chi$, $G \in \mathscr{P}_\lambda^1$, $A \subseteq G$, $|A| \leq \chi$, *N a normal subgroup of G including* $\mathrm{Min}^\chi \mathrm{Cm}_G A$, *then* $\mathrm{Min}^\chi G \subseteq N$ (see 1.12, 3.12).

9.8. Lemma. (1) *For* $G_1 \in \Omega_\lambda^1$ *and* $\sigma < \mu$ *there is a subgroup* $H_1 \subseteq G_1$ *and* θ, $\sigma \leq \theta < \mu$, $2^\theta < 2^{\theta^+}$, $|H_1| \leq \theta$ *s.t.* $G \overset{\mathrm{def}}{=} \mathrm{Min}\,\mathrm{Cm}_{G_1}(H_1)$ *satisfies*:

$(*)_\theta$ *If H is a subgroup of G,* $|H| + \theta \leq \kappa < \mu$, *then* $(\mathrm{Cm}_G H : \mathrm{Min}[\mathrm{Cm}_G H, (2^\kappa)^+]) \leq 2^\kappa$ *and* $\mathrm{Min}[\mathrm{Min}[\mathrm{Cm}_G H, (2^\kappa)^+], (2^\kappa)^+] = \mathrm{Min}[\mathrm{Cm}_G H, (2^\kappa)^+]$.

(2) *If* $G_1 \in \Omega_\lambda^1$, $\sigma < \mu$, *and* G_1 *satisfies* $(*)_\sigma$ *of* 9.8(1), *then for some* $[\sigma, \mu)$*-special subgroup* $H_1 \subseteq G_1$ *and* θ, $\sigma \leq \theta < \mu$, $2^\theta < 2^{\theta^+}$, $|H_1| \leq \theta$, *and* $G \overset{\mathrm{def}}{=} \mathrm{Min}\,\mathrm{Cm}_{G_1}(H_1)$ *satisfies:*

$(*)_\theta$ *If H is a* $[\theta, \mu)$*-special subgroup of G,* κ *a cardinal* $|H| + \theta \leq \kappa < \mu$, $2^\kappa < 2^{\kappa^+}$ *and* $I \in \mathrm{BA}'(\mathrm{Min}_\theta \mathrm{Cm}_G H)$, $|I| < \lambda$, *then there is* $A \subseteq I$, $|A| \leq \kappa$ *s.t.* $\mathrm{Min}_\theta \mathrm{Cm}_I A = \{e\}$.

Proof. Suppose G_1, σ form a counterexample (to 9.8(1) or 9.8(2)). Let $\mu=\sum_{i<\mathrm{cf}\,\mu}\kappa_i^0$, $\kappa_i^0<\mu$. We define by induction on $i<\mathrm{cf}\,\mu$, cardinals θ_i, κ_i and subgroups H^i, I^i such that:

(a) $\sum_{j<i}\kappa_j<\theta_i$, $\theta_i^+<\kappa_i$, $\sigma+\mathrm{cf}\,\mu+\kappa_i^0<\theta_i$, $\mu<2^{\theta_i}<2^{\theta_i^+}$ and $2^{\kappa_i}<2^{\kappa_i^+}$.

(b) $H^i\subseteq \mathrm{Cm}_G(\bigcup_{j<i}H^j)$ is a subgroup of $G_i=\mathrm{Min}(\mathrm{Cm}_G\bigcup_{j<i}H^j)$, $|H^i|\leqslant\kappa_i$ and for 9.8(2) H^i is $[\theta_i,\kappa_i^+)$-special group

(c) $(\mathrm{Cm}_G(\bigcup_{j<i}H^j):\mathrm{Min}\,\mathrm{Cm}_G(\bigcup_{j<i}H^j))$ is $\leqslant 2^{\theta_i}$.

(d) For each i, (α) or (β), for 9.8(1), 9.8(2), respectively, holds

$(\alpha)_i$ $(\mathrm{Cm}_{G_i}H^i:\mathrm{Min}^{\kappa_i}\mathrm{Cm}_{G_i}H^i)>2^{\kappa_i}$ or

$(\mathrm{Min}^{\kappa_i}\mathrm{Cm}_{G_i}H^i:\mathrm{Min}^{\kappa_i}\mathrm{Min}^{\kappa_i}\mathrm{Cm}_{G_i}H^i)>2^{\kappa_i}$,

$(\beta)_i$ $I^i\in\mathrm{BA}'(\mathrm{Min}_{\theta_i}\mathrm{Cm}_{G_i}H^i,\ \mathrm{Cm}_{G_i}H^i)$, $|I^i|<\lambda$ and for no $A\subseteq I^i$, $|A\leqslant\kappa_i$ and $\mathrm{Min}_{\theta_i}\mathrm{Cm}_{I^i}(A)=\{e\}$.

If for some $i<\mathrm{cf}\,\mu$ we have defined for every $j<i$ but cannot find H^i, I^i, θ_i, κ_i, then clearly we have gotten the desired conclusion of 9.8 (note we can choose θ_i satisfying (a) and (c) by 6.12(4), now we look for H^i, (I^i) and κ^i and for (d)(α) remember 9.7A(3)).

So we suppose we have carried out the definition.

Case I: *We will prove* 9.8(1). Let $S_0=\{i:i<\mathrm{cf}\,\mu\}$ and let K_i be $\mathrm{Cm}_{G_i}H^i$ if $(\mathrm{Cm}_{G_i}H^i:\mathrm{Min}^{\kappa_i}\mathrm{Cm}_{G_i}H^i)>2^{\kappa_i}$ and $K_i=\mathrm{Min}^{\kappa_i}\mathrm{Cm}_{G_i}H^i$ otherwise. First note

9.8A. Observation. *For each $i\in\mathrm{cf}\,\mu$ there are normal subgroups $N_{i,\alpha}$ $(\alpha<\kappa_i^+)$ of K_i, $\mathrm{Min}\,K_i\subseteq N_{i,\alpha}$ and $N_{i,\alpha}$ is strictly decreasing with α.*

We choose by induction on $i<\mathrm{cf}\,\mu$ elements $a_{i,\alpha}$ $(\alpha<\kappa_i^+)$ of K_i and ordinals $\gamma_{i,\alpha}<\kappa_i^+$, s.t.:

(i) $a_{i,\alpha}\in N_{i,\gamma_{i,\alpha}}-N_{i,\gamma_{i,\alpha}+1}$

(ii) $\langle\bigcup_{j\leqslant i}H^j\cup\{a_{j,\beta}:j\in S_0\cap i,\ \beta<\kappa_j^+$ or $j=i,\ \beta<\alpha\}\rangle_G$ is disjoint to $N_{i,\gamma_{i,\alpha}}-N_{i,\gamma_{i,\alpha}+1}$.

This is done by induction on $\alpha<\kappa_i^+$.

Now for each $i<\mathrm{cf}\,\mu$ let $\{u^i_\xi:\xi<(2^{\kappa_i})^+\}$ be a list of distinct subsets of κ_i^+. For each ξ, $\zeta<(2^{\kappa_i})^+$ choose, if possible, $V^j_{\xi,\zeta}$ $(j\in S_0-\{i\})$ subsets of κ_j^+ and $g_{\xi,\zeta}\in G$ s.t. $\square^{g_{\xi,\zeta}}$ maps $\langle\bigcup_{j<\mathrm{cf}\,\mu}H^j\cup\{a_{j,\beta}:\beta\in V^j_{\xi,\zeta}$ and $j\neq i\}\cup\{a_{i,\alpha}:\alpha\in u^i_\xi\}\rangle_G$ onto $\langle\bigcup_{j<\mathrm{cf}\,\mu}H^j\cup\{a_{j,\beta}:\beta\in V^j_{\xi,\zeta}$ and $j\neq i\}\cup\{a_{i,\alpha}:\alpha\in u^i_\zeta\}\rangle_G$. Let $T_\xi=\{\zeta<(2^{\kappa_i})^+:$ $g_{\xi,\zeta}$, $V^j_{\xi,\zeta}$ are defined$\}$.

Now $|T_\xi|\leqslant 2^{\kappa_i}$: otherwise for some $w\subseteq T_\xi$ of power $(2^{\kappa_i})^+$, for all $\zeta\in w$, $\langle V^j_{\xi,\zeta}:j\in S_0\cap i\rangle_G$ and $\square^{g_{\xi,\zeta}^{-1}}\langle\bigcup_{j\leqslant i}H^j\cup\{a_{j,\beta}:j\in S_0\cap i,\ \beta\in V^j_{\xi,\zeta}\}\rangle$ are the same (the former has $\leqslant 2^{\kappa_i}$ possibilities and the latter has $\leqslant|\bigcup_{j<\mathrm{cf}\,\mu}H^j\cup\{a_{j,\beta}:\beta<\kappa_j^+,\ j<\mathrm{cf}\,\mu\}|^{\kappa_i}\leqslant\mu^{\kappa_i}=2^{\kappa_i}$ possibilities).

So let $V^j_{\xi,\zeta}=V^j$ for $j\in S_0\cap i$, $\zeta\in w$. So, for ζ_1, $\zeta_2\in w$, we have $g_{\xi,\zeta_1}g^{-1}_{\xi,\zeta_2}\in\mathrm{Cm}_G\langle\bigcup_{j\leqslant i}H^j\cup\{a_{j,\beta}:j\in S_0\cap i,\ \beta\in V^j\}\rangle_G$. Choose $\xi_0\in w$, then (by (c)) for some $\zeta_1\neq\zeta_2\in w$, $g_{\xi,\xi_0}g^{-1}_{\xi,\zeta_1}K_i=g_{\xi,\xi_0}g^{-1}_{\xi,\zeta_2}K_i$, hence $(g_{\xi,\xi_0}g^{-1}_{\xi,\zeta_2})^{-1}g_{\xi,\xi_0}g^{-1}_{\xi,\zeta_1}\in K_i$ but this is $g_{\xi,\zeta_2}g^{-1}_{\xi,\zeta_1}$.

Now $\square^{g_{\xi,\zeta_2}g^{-1}_{\xi,\zeta_1}}$ is the identity on $\bigcup_{j\le 1} H^j \cup \{a_{j,\beta} : j \in S_0 \cap i,\ \beta \in V^j\}$ and necessarily maps

$$\left\langle \bigcup_{j\le i} H^j \cup \{a_{j,\beta} : j \in S_0 \cap i,\ \beta \in V^j\} \cup \{a_{i,\alpha} : \alpha \in u^i_{\zeta_1}\} \cup G_{i+1} \right\rangle_G$$

onto

$$\left\langle \bigcup_{j\le i} H^j \cup \{a_{j,\beta} : j \in S_0 \cap i,\ \beta \in V^j\} \cup \{a_{i,\alpha} : \alpha \in u^i_{\zeta_2}\} \cup G_{i+1} \right\rangle_G$$

(remember $a_{j,\alpha} \in G_{i+1}$ for $j > i$).

But $g_{\xi,\zeta_2} g^{-1}_{\xi,\zeta_1} \in K_i$, so we get a contradiction as in 3.1.

We have finished Case I.

We will prove now 9.8(2), hence $(\beta)_i$ always happens. Here we shall define for each i subgroups K^i_α $(\alpha < (2^{\kappa_i})^+)$ of I^i s.t.

(A) K^i_α has power $\le \kappa_i^+$.

(B) Either (α) no member of $\mathrm{BA}'((K^i_\alpha)^{(\infty)})$ is a $[\aleph_0, \aleph_2)$-group or $[\theta_j, \kappa_j)$-group for $j < i$, or

(β) $\bigcup \{K^i_\alpha : \alpha < (2^{\kappa_i})^+\}$ has power $\le \kappa_i^+$.

(C) For $\alpha < \beta < (2^{\kappa_i})^+$, $\langle K^i_\alpha \cup \mathrm{Cent}\, I^i\rangle_{I_i}$, $\langle K^i_\beta \cup \mathrm{Cent}\, I^i\rangle_{I_i}$ are not conjugate in I^i.

This (and even more, in (B)(α)) is possible by 8.4. As $I \in \mathrm{BA}'(\mathrm{Min}_{\theta_i} \mathrm{Cm}_{G_i} H^i)$, clearly $\langle K^i_\alpha \cup \mathrm{Cent}\, I^i\rangle_{I^i}$ $(\alpha < (2^{\kappa_i})^+)$ are nonconjugate in pairs in Min_{θ_i} $(\mathrm{Cm}_{G_i}(H^i))$. By 9.7A(3), as $(\mathrm{Cm}_{G_i}(H^i) : \mathrm{Min}_{\theta_i}$ $(\mathrm{Cm}_{G_i}(H^i))$ is $\le 2^{\kappa_i}$ (because G_1 satisfies $(*)_\sigma$) w.l.o.g. $\langle K^i \cup \mathrm{Cent}\, I^i\rangle_{I^i}$, for $\alpha < (2^{\kappa_i})^+$ are nonconjugate in pairs in $\mathrm{Cm}_{G_i}(H^i)$ and by (c) even in $\mathrm{Cm}_G(\bigcup_{j\le i} H^j)$.

Note that the groups $\langle H^i, \bigcup_\alpha K^i_\alpha\rangle_G$ for $i < \mathrm{cf}\,\mu$ are pairwise commuting. Now for g a function, $\mathrm{Dom}\, g \subseteq (\mathrm{cf}\,\mu)$, $g(i) < (2^{\kappa_i})^+$, let $K_g = \langle \{H^j : j < \mathrm{cf}\,\mu\} \cup \{K^i_{g(i)} : i \in \mathrm{Dom}\, g\}\rangle_G$.

It is easy to check that $K^i_{g(i)} \subseteq K_g \cap I^i \subseteq \langle K^i_{g(i)},\ \mathrm{Cent}\, I^i\rangle$ for $i \in \mathrm{Dom}\, g$. Remember that $\langle \alpha_j : j < i\rangle$ is the function h, $h(j) = \alpha_j$. Let $S = \{i < \mathrm{cf}\,\mu$: in (B), (α) occurs$\}$.

Case II: *S has power* $\mathrm{cf}\,\mu$. For notational simplicity assume $S = \mathrm{cf}\,\mu$. Let $i < \mathrm{cf}\,\mu$. Let for $\alpha < (2^{\kappa_i})^+, S_\alpha$ be the set of $\beta < (2^{\kappa_i})^+$ such that for some $\gamma(j)$, $\alpha(j) < (2^{\kappa_j})^+$ for $j < \mathrm{cf}\,\mu$, $j \ne i$ the groups $K_{\eta_\beta} = K_{\langle \alpha(j) : j < i\rangle ^\wedge \langle \alpha\rangle ^\wedge \langle \alpha(j) : i < j < \mathrm{cf}\,\mu\rangle}$ and $K_{\nu_\alpha} = K_{\langle \gamma(j) : j < i\rangle ^\wedge \langle \beta\rangle ^\wedge \langle \gamma(j) : i < j < \mathrm{cf}\,\mu\rangle}$ are conjugates in G by $\square^{g^{-1}_\beta}$. These groups have cardinality $\le \mu \le 2^{\kappa_i}$. Now we shall check (by (c) and cardinality considerations) that $|S_\alpha| \le 2^{\kappa_i}$.

Suppose $\eta \ne \nu$, $g \in G$ and $\square^g$ maps K_η onto K_ν. What can be $\square^g H^j$ (for $j < \mathrm{cf}\,\mu$)? As H^j is in $\mathrm{BA}'(K^{(\infty)}_\eta)$, clearly $\square^g H^j \in \mathrm{BA}'(K^{(\infty)}_\nu)$. So

$$\square^g H^j = \sum_{\xi<\mathrm{cf}\,\mu}{}' H^\xi \cap \square^g H^j +' \sum_{\xi<\mathrm{cf}\,\mu}{}' (K^\xi_{\nu(\xi)})^{(\infty)} \cap \square^g H^j.$$

Now for $\xi > j$, $K^{\xi}_{\nu(\xi)}$ has no semi-direct direct summand which is also a (nonzero) semi-direct summand of $\Box^{g_\beta} H^j$ (by (b) and (B)). As $(\Box^g H^j)^{(1)} = \Box^g H^j$, clearly

$$\Box^{g_\beta} H^j \subseteq \sum_{\xi<\mathrm{cf}\,\mu}{}' H^{\xi} + \sum_{i\leqslant j}{}' K^{\xi}_{\nu(\xi)}.$$

So $\bigcup_{\beta\in S_\alpha}\bigcup_{j\leqslant i}\Box^{g_\beta}H^j$ has cardinality $\leqslant \mu + 2^{\kappa_i}$. So $|S_\alpha| > 2^{\kappa_i}$ implies for some $\beta(1)\neq\beta(2)$, $\Box^{g_{\beta(1)}}$ and $\Box^{g_{\beta(2)}}$ agree on $\bigcup_{j\leqslant i} H^j$ and we get an easy contradiction to the paragraph after (C).

It is also easy to check that $\beta\in S_\alpha \Leftrightarrow \alpha\in S_\beta$, hence by replacing (successively for each i) $\langle K^i_\alpha : \alpha < (2^{\kappa_i})^+\rangle$ by a subsequence, w.l.o.g. $S_\alpha \subseteq \{\alpha\}$ for every $\alpha < (2^{\kappa_i})^+$.

Case III: *Not* I *nor* II. So assume for notational simplicity that S is empty.

We shall define by induction on $i < \mathrm{cf}\,\mu$, a subset T_i of $(2^{\kappa_i})^+$ of cardinality $(2^{\kappa_i})^+$ s.t.:

$(*)$ If η, $\nu\in\prod_{i<\mathrm{cf}\,\mu}(2^{\kappa_i})^+$, $\eta(i)$ and $\nu(j)$ are distinct members of T_i, then K_η and K_ν are not conjugate in G.

Clearly this suffices.

Let us for each $\alpha\neq\beta<(2^{\kappa_i})^+$ choose, if possible, $\eta_{\alpha,\beta}$, $\nu_{\alpha,\beta}\in\prod_{i<\mathrm{cf}\,\mu}(2^{\kappa_i})^+$, $\eta_{\alpha,\beta}(i)=\alpha$, $\nu_{\alpha,\beta}(i)=\beta$, and $g_{\alpha,\beta}\in G$ such that $\Box^{g_{\alpha,\beta}}$ maps $K_{\eta_{\alpha,\beta}}$ onto $K_{\nu_{\alpha,\beta}}$. Let for $\alpha<(2^{\kappa_i})^+$, $W_\alpha=\{\beta : \eta_{\alpha,\beta}, \nu_{\alpha,\beta}, g_{\alpha,\beta}$ are defined.$\}$ Clearly it suffices to prove $|W_\alpha|\leqslant 2^{\kappa_i}$.

So suppose $|W_\alpha|\geqslant(2^{\kappa_i})^+$. Now $\Box^{g^{-1}_{\alpha,\beta}}$ maps $\bigcup_{j\leqslant i}H^j$ into $\langle\bigcup_{j<\mathrm{cf}\,\mu}H^j\cup\bigcup_j\bigcup_{\gamma<\kappa_j^+}K^j_\gamma\rangle$, which has cardinality $\leqslant\mu$ (by (β) of (B), as S is empty). So the number of such maps is $\leqslant\mu^{\kappa_i}\leqslant 2^{\kappa_i}<|W_\alpha|$, hence for some $W\subseteq W_\alpha$, $|W|\geqslant(2^{\kappa_i})^+$, and the image of K^i_β under $\Box^{g^{-1}_{\alpha,\beta}}$ and also $\Box^{g^{-1}_{\alpha,\beta}}\restriction(\bigcup_{j\leqslant i}H^j)$ are the same for all $\beta\in W$. Choose distinct $\beta(1)$, $\beta(2)$ in W. So $\Box^{g_{\alpha,\beta(2)}g^{-1}_{\alpha,\beta(1)}}$ is an inner automorphism of $\mathrm{Cm}_G(\bigcup_{j\leqslant i}H^j)$, mapping $K^i_{\beta(1)}$ onto $K^i_{\beta(2)}$, contradiction (as this inner automorphism necessarily maps $\mathrm{Min}_{\theta_i}\mathrm{Cm}_G(\bigcup_{j\leqslant i}H^j)$ on to itself, as well as $\mathrm{Min}\,\mathrm{Cm}_G(\bigcup_{j\leqslant i}H^j)$ and $\bigcup\{I : I\in \mathrm{BA}'(\mathrm{Min}_{\theta_i}\mathrm{Cm}_G(\bigcup_{j\leqslant i}H^j)),\ |I|<\lambda\}$).

So $|W_\alpha|\leqslant 2^{\kappa_i}$, and we can define T_i for each $i<\mathrm{cf}\,\mu$, hence $\{K_\eta : \eta\in\prod_{i<\mathrm{cf}\,\mu}T_i\}$ is a family of $\prod_{i<\mathrm{cf}\,\mu}(2^{\kappa_i})^+ = 2^\mu=\lambda$ nonconjugate subgroups of G of power $\leqslant\mu$. Clearly adding the center to each changes nothing, so we get a contradiction.

9.9. Remark. (1) In 9.8 (1) and (2) if $\theta\leqslant\theta_1<\mu$, then $(*)_\theta$ implies $(*)_{\theta_1}$.

(2) Also if $H\subseteq G$, $|H|<\mu$, $G\in\Omega^1_\lambda$, G satisfies $(*)_\theta$ of 9.8(1), *then* $\mathrm{Min}\,\mathrm{Cm}_G H$ satisfies $(*)_{\theta_1}$ of 9.8(1) for $\theta_1=\theta+|H|$, $(\mathrm{Cm}_G H : \mathrm{Min}\,\mathrm{Cm}_G H)\leqslant 2^{\theta_1}$.

9.10. Proof of the Main Theorem (in the Remaining Case). Choose $G_0\in\mathcal{P}^1_\lambda$ with minimal $\gamma(G_0)$. By 9.8(1) (and 9.9(1)) for some $\theta_0<\mu$, $H_0\subseteq G_0$, $|H_0|\leqslant\theta_0$ and $G_1\overset{\mathrm{def}}{=}\mathrm{Min}\,\mathrm{Cm}_{G_0}H_0$ satisfies $(*)$ of 9.8(1) with θ_0, and $2^{\theta_0}\geqslant\mu$, $\theta_0>\aleph_3$. By Lemma 9.3(1) for some $\theta_1<\mu$, $\theta_1>\theta_0^+$, and $H_1\subseteq G_1$, $|H_1|<\mu$ and $G_2=\mathrm{Min}\,\mathrm{Cm}_{G_1}H_1$ does not have 2^{θ_1} $[\theta_1,\mu)$-special subgroups pairwise nonconjugate in G_2 and H_1 is a $[\theta_0, \theta_1)$-special subgroup of G_1. By 9.9(2), G_2 satisfies $(*)_{\theta_2}$ of 9.8(2) where $\theta_1 < \theta_2 < \mu$ (remember 6.12(4)).

Now apply 9.8(2) to $G_2 \stackrel{\text{def}}{=} \text{Min Cm}_{G_1}(H_1)$ and get $\theta > \theta_2$ and $H_2 \subseteq G_2$, $|H_2| \leq \theta < \mu$ s.t. $(*)$ of 9.8(2) holds for $G \stackrel{\text{def}}{=} \text{Min Cm}_{G_2}(H_2)$, $2^\theta < 2^{\theta^+}$, and $\theta_2 < \theta < \mu$ and H_2 is a $[\theta_2, \theta)$-subgroup of G_2. It is easy to see that:

(α) $G \in \Omega^1_\lambda$, $2^\theta > \mu$, $\theta > \aleph_3$.

(β) $\gamma(G)$ is $\gamma(G_0)$ hence is minimal.

(γ) G, θ satisfies $(*)$ of 9.8(2), and: for $A \subseteq G$, $|A| < \mu$, and κ, $2^\kappa < 2^{\kappa^+}$, $(\text{Cm}_{G_2} H_2 \colon G) + \theta + |A| \leq \kappa < \mu$ implies $(\text{Cm}_G A : \text{Min}^\kappa \text{Cm}_G A) \leq 2^\kappa$ and $\text{Min}^\kappa(\text{Min}^\kappa \text{Cm}_G A) = \text{Min}^\kappa \text{Cm}_G A$.

(δ) G does not have 2^θ $[\theta, \mu)$-special subgroups pairwise nonconjugate in G (use the choice of G_2, note that by its choice and 9.3(3), H_2 is a $[\theta_1, \theta)$-special subgroup, now use 9.3(3), (4) and 1.9's proof to get a contradiction).

By Lemma 9.6 for some κ_0, $A \subseteq G$, $|A| < \mu$, $\kappa_0 > \theta$ $(*)$ of 9.6 holds, and choose κ, $\kappa_0 + |A| + \theta < \kappa < \mu < 2^\kappa < 2^{\kappa^+}$. In $\text{Cm}_G(A)$ choose a maximal sequence $\langle K_\alpha : \alpha < \alpha_0 \rangle$ of subgroups satisfying (i)–(v) of 9.6$(*)$ (except their number) and $|K_\alpha| \leq \kappa$, $|\text{BA}'(\text{Min}_\theta K_\theta, K_\alpha)| \leq \kappa$. So clearly $\alpha_0 < \kappa^+$.

Let $H_{\langle\,\rangle}$ be an explicit $[\theta, \kappa^+)$-subgroup of G such that $A \cup \bigcup_{\alpha<\alpha_0} K_\alpha \subseteq H_{\langle\,\rangle}$ (see AP1.3). Now we define by induction on $\beta < \kappa^+$, for every $\eta \in {}^{(\beta+1)}2$, subgroups H_η such that

(1) H_η has power $\leq \kappa$.

(2) H_η is a $[\theta, \kappa^+)$-special subgroup of $\text{Cm}_G(\bigcup \{H_{\eta \restriction (i+1)} : i + 1 < l(\eta)\} \bigcup H_{\langle\,\rangle})$.

(3) For no $\eta \in {}^\beta 2$, $g \in \text{Cm}_G(\bigcup_{i<\beta} H_{\eta \restriction (i+1)} \cup H_{\langle\,\rangle})$ does $\Box^g$ map $H_{\eta^\wedge\langle 0\rangle}$ into $\langle H_{\eta^\wedge\langle 1\rangle}, \text{Cm}_G(\bigcup_{i\leq\beta} H_{(\eta^\wedge\langle 1\rangle) \restriction (i+1)} \cup H_{\langle\,\rangle}) \rangle_G$.

Or at least

(3′) For no $\eta \in {}^\beta 2$, $g \in \text{Cm}_G(\bigcup_{i<\beta} H_{\eta \restriction (i+1)} \cup H_{\langle\,\rangle})$ and for $l = 0, 1$, $\gamma \in (\beta, \kappa^+)$, H^l_γ a $[\theta, \kappa^+)$-special subgroup of $\text{Cm}_G(H_{\langle\,\rangle} \bigcup_{i<\beta} H_{\eta \restriction (i+1)} \cup \bigcup_{\alpha<\gamma} H^l_\alpha)$ does $\Box^g$ map $\langle H_{\eta^\wedge\langle 0\rangle} \cup \bigcup \{H^0_\gamma : \beta + 1 < \gamma < \kappa^+\} \rangle_G$ onto $\langle H_{\eta^\wedge\langle 0\rangle} \cup \bigcup \{H^1_\gamma : \beta + 1 < \gamma < \kappa^+\} \cup \{H_{\eta \restriction \gamma} : \gamma \leq \beta \text{ non-limit}\} \rangle_G$.

If we succeed we get an easy contradiction (by the weak diamond (AP3.2) as in the proof of 9.5) to δ (i.e. to the choice of G by Lemma 9.3). So for some $\eta \in {}^\beta 2$ we cannot choose $H_{\eta^\wedge\langle 0\rangle}$, $H_{\eta^\wedge\langle 1\rangle}$.

By (δ) above, by the proof of 1.9 (and 9.3(2)) $G_\eta \stackrel{\text{def}}{=} \text{Cm}_G(\bigcup_{i<\beta} H_{\eta \restriction (i+1)} \cup H_{\langle\,\rangle})$ does not have $(2^\kappa)^+$ $[\theta, \kappa^{++})$-special subgroups nonconjugate (in G_η) in pairs. So the hypothesis of Lemma 9.5 (with G_η, θ, κ here standing for G, σ, θ there) holds, hence there is an explicit $[\theta, \kappa^+)$-subgroup of K of $\text{Min}_\theta(G_\eta)$, such that $\text{cg}(\text{Min}_\theta \text{Cm}_{G_\eta}(K), \text{Cm}_{G_\eta}(K)) \leq \kappa$. So for some $B \subseteq \text{Min}_\theta \text{Cm}_{G_\eta}(K)$, $|B| \leq \kappa$ and $\text{Min}_\theta \text{Cm}_{G_\eta}(K) \subseteq \langle B \rangle^{\text{cg}}_{\text{Cm}_{G_\eta}(K)}$. So $N = \langle K, B \rangle^{\text{cg}}_{G_\eta}$ is a normal subgroup of G_η. If $\text{Min}_\theta G_\eta$ is not a subgroup of N, we can find an explicit $[\theta, \kappa^+)$-subgroup K' of G_η, $K' \subseteq \text{Min}_\theta G_\eta$, $K' \nsubseteq N$. So we could have chosen K, K' as $H_{\eta^\wedge\langle 1\rangle}$, $H_{\eta^\wedge\langle 0\rangle}$ respectively [as we just said: "$H_{\eta^\wedge\langle 0\rangle}$ is not included in this normal subgroup G_η that $H_{\eta^\wedge\langle 1\rangle} \cup \text{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})$ generates"] getting a contradiction to the choice of

η, so $\mathrm{Min}_\theta G_\eta \subseteq N$, hence

(i) $\mathrm{cg}(\mathrm{Min}_\theta G_\eta, G_\eta) \leqslant \kappa$.

By 7.2 and (δ),

(ii) $\mathrm{BA}'(\mathrm{Min}\, G_\eta, G_\eta)$ has power $\leqslant \kappa$.

We want to prove that $\mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta)$ contains 'nothing more' than $\mathrm{BA}'(\mathrm{Min}\, G_\eta, G_\eta)$. More specifically, suppose we can find noncommutative $I \in \mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta)$, $I \cap \mathrm{Min}\, G_\eta \subseteq \mathrm{Cent\,Min}\, G_\eta$. By ($\gamma$) above and 9.8(2) for some $H \subseteq I$, $|H| \leqslant \kappa$ and $\mathrm{Min}_\theta \mathrm{Cm}_I(\mathrm{H}) = \{e\}$. Let $H_{\eta\wedge\langle 0\rangle}$ be an explicit $[\theta, \kappa^+)$-subgroup of I contaning H, and $H_{\eta\wedge\langle 1\rangle}$ be a countable subgroup of $\mathrm{Min}_\theta I$, $(H_{\eta\wedge\langle 1\rangle})^{(1)} = H_{\eta\wedge\langle 1\rangle}$. This contradicts the choice of η (by (3)). (Below we do in detail such an argument.) So

(iii) For nonzero $I \in \mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta)$, $I \cap \mathrm{Min}\, G_\eta$ is a nonzero member of $\mathrm{BA}'(\mathrm{Min}\, G_\eta, G_\eta)$.

We can conclude

(iv) $\mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta)$ has power $\leqslant \kappa$.

Let M_i ($i < \kappa^+$) be an increasing continuous sequence of elementary submodels of G_η closed enough by AP 1.3, each of power $\leqslant \kappa$ as in the proof of 6.10. So 4.12, 4.13 apply (so e.g., $(\forall I)\ [I \in \mathrm{BA}'(\mathrm{Min}_\theta M_i, M_i) \rightarrow (\exists J)(J \cap \mathrm{Min}_\theta M_i = I \wedge J \in \mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta))]$.

As in the proof of 6.5, w.l.o.g. for $i \neq j$ no (nonzero) $I \in \mathrm{BA}'(\mathrm{Min}_\theta M_i, M_i)$, $J \in \mathrm{BA}'(\mathrm{Min}_\theta M_j, M_j)$ are conjugate in G.

Can there be i and $I \in \mathrm{BA}'(M_i^{(\infty)}, M_i)$, $|I| < \kappa_0$, I not essentially countable? If so, $I \cap \mathrm{Min}_\theta M_i \subseteq \mathrm{Cent\,Min}_\theta M_i$ [otherwise, note first that $I_1 \overset{\mathrm{def}}{=} I \cap \mathrm{Min}_\theta M_i \in \mathrm{BA}'(\mathrm{Min}_\theta M_i, M_i)$ (as $M_i^{(\infty)} = I +' J$, where I, J are normal in M_i implies $\mathrm{Min}\, M_i^{(\infty)}) = \mathrm{Min}_\theta I +' \mathrm{Min}_\theta J$, and $\mathrm{Min}_\theta I$, $\mathrm{Min}_\theta J$ are normal in M_i). Second note that for some $I_2 \in \mathrm{BA}'(\mathrm{Min}_\theta G_\eta, G_\eta)$, $I_2 \cap M_i = I_1$. Third by (iii), as I_1 is not abelian, so is $I_2 \cap \mathrm{Min}\, G_\eta$, so necessarily $I_2 \in \Omega_\lambda^1$, but then I_2 has an element with $> \kappa$ conjugates. Hence M_i contains such element x, so $x \in I_2 \cap M_i \subseteq I_1 \cap M_i = I$, contradicting the essential countability of I.]

Hence $I \cap \mathrm{Min}_\theta G_\eta \subseteq \mathrm{Cent\,Min}_\theta G_\eta$. Let $\langle L_\gamma : \gamma < \gamma_0 \rangle$ be a maximal sequence of countable pairwise commuting subgroups of I satisfying $L_i = L_i^{(1)}$, and let $L_{\gamma_0} = \mathrm{Cm}_I(\bigcup L_\gamma : \gamma < \gamma_0\})$. We can find (remember I is not essentially countable)

$$H_{\eta\wedge\langle 1\rangle} = \langle L_\gamma : \gamma < \gamma_0 \rangle_G, \qquad H_{\eta\wedge\langle 0\rangle} = H^{(1)}_{\eta\wedge\langle 0\rangle} \subseteq I,$$

$|H_{\eta\wedge\langle 0\rangle}| = \aleph_1$, some $y \in H_{\eta\wedge\langle 0\rangle}$ has $\aleph_1$ conjugated in it. A contradiction to the choice of η will now be derived. Clearly $H_{\eta\wedge\langle l\rangle}$ are $[\theta, \kappa)$-special subgroups so we have to prove (3) or (3'). First we can assume that the M_i's were chosen such that: for every $x \in M_i$,

$$|\{\Box^g x : g \in G\}| \geqslant \kappa \quad \Rightarrow \quad |\{\Box^g x : g \in M_i\}| = \kappa,$$

$$|\{\Box^g x : g \in G\}| \leqslant \kappa \quad \Rightarrow \quad \{\Box^g x : g \in M_i\} \subseteq M_i.$$

As any member x of I has $< \kappa_0 < \kappa$ conjugated in M_i, necessarily $(\forall g \in G_\eta)\ \Box^g x \in M_i$, so I is a normal subgroup of G_η, so any inner automorphism of G_η

maps I onto itself. Note $\mathrm{Cm}_I(H_{\eta^\wedge\langle 1\rangle})$ contains no nontrivial subgroup $L = L^{(1)}$ (by γ_0's maximality), hence $(\mathrm{Cm}_I(H_{\eta^\wedge\langle 1\rangle}))^{(\infty)} = \{e\}$. Now $\langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})\rangle_G \cap I = \langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_I(H_{\eta^\wedge\langle 1\rangle})\rangle_I$ and $\langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_I(H_{\eta^\wedge\langle 1\rangle})\rangle_G^{(\infty)} = H_{\eta^\wedge\langle 1\rangle}$ is essentially countable. Hence $(I \cap \langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})\rangle_G)^{(\infty)}$ is essentially countable.

Now suppose $g \in G_\eta$ and $\square^g$ maps $H_{\eta^\wedge\langle 0\rangle}$ into $\langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})\rangle_G$; we know $\square^g$ maps I into I hence $H_{\eta^\wedge\langle 0\rangle}$ into I, hence it maps $H_{\eta^\wedge\langle 0\rangle}$ into $I \cap \langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})\rangle_G$; hence it maps $(H_{\eta^\wedge\langle 0\rangle})^{(\infty)}$ into $(I \cap \langle H_{\eta^\wedge\langle 1\rangle}, \mathrm{Cm}_{G_\eta}(H_{\eta^\wedge\langle 1\rangle})\rangle_G)^{(\infty)}$. But the former is $H_{\eta^\wedge\langle 0\rangle}$ whereas the latter is (see above) essentially countable. But $H_{\eta^\wedge\langle 1\rangle}$ is not essentially countable, (see paragraph after (δ) in the beginning of the proof), contradiction to the existence of g. By η's choice there are no i, I as above.

Now we shall show that for at least one $i < \kappa^+$, M_i can serve as K_{α_0}. Now (i) is trivial; (ii) we have; for (iii), we have proven that in the previous paragraph, now (v) is trivial. As for (iv) if it fails for every $j < \kappa^+$, there are $I_j \in \mathrm{BA}'(\mathrm{Min}_\theta\, M_j, M_j)$, $\beta_j < \alpha_0$, and $J_j \in \mathrm{BA}(\mathrm{Min}_\theta\, K_{\beta_j}, K_{\beta_j})$ such that I_j, J_j are conjugates in G. By a statement after defining M_i, there are $I_j^* \in \mathrm{BA}'(\mathrm{Min}_\theta\, G_\eta, G_\eta)$, $I_j^* \cap \mathrm{Min}_\theta\, M_i = I_j$. By (iv) in this proof w.l.o.g. $I_j^* = I^*$. But $\bigcup_{\alpha<\alpha_0} \mathrm{BA}'(\mathrm{Min}_\theta\, K_\alpha, K_\alpha)$ has power $\leqslant \kappa$, so for some $j_1 \neq j_2$, $J_{j_1} = J_{j_2}$, hence I_{j_1} and I_{j_2} are conjugates in G, contradiction as in the proof of 6.10. The two other demands on K_{α_0} [K_α and $\mathrm{BA}'(\mathrm{Min}_\theta(K_\alpha), K_\alpha)$ have power $\leqslant \kappa$] hold too.

So we have gotten a contradiction to the choice of α_0, thus finishing.

10. A Generalization

10.1. Theorem. *If G is a group and* $(\forall \kappa < \mu)\ 2^\kappa < |G|$, *then* $\mathrm{nc}_{\leqslant\mu}(G) \geqslant 2^\mu$.

Proof. The proof is a repetition of the proof of Theorem 0.1. By 1.2(3), we can assume $|G| = |G|^\mu$ and so by Theorem 0.1 we can assume $|G| > 2^\mu$. Let $\lambda_1 = 2^\mu$, $\lambda_2 = |G|$, $\lambda = \langle \lambda_1, \lambda_2 \rangle$. So $\lambda_2 = \lambda_2^\mu > \lambda_1 = 2^\mu$ and it is enough to prove $\mathcal{P}_\lambda = \{G' : |G'| = \lambda_2,\ \mathrm{nc}_{\leqslant\mu}(G') < 2^\mu\}$ is empty.

Remarks. The proof was gotten by successive corrections resulting in lengthening of the proof; maybe even by the same ideas we can get a shorter proof.

Appendix for non-logicians

AP 1. *Elementary submodels*

AP 1.1. Definition. M is an elementary submodel of N if M is a submodel of N and for every element $a_1, \ldots, a_n$ of M and first-order formula $\phi(x_1, \ldots, x_n)$.

$$M \models \phi[a_1, \ldots, a_n] \quad \text{iff} \quad N \models \phi[a_1, \ldots, a_n]$$

AP 1.2. The Downward Lowenheim–Skolem Theorem. *If A is a set of $\leqslant \lambda$ elements of model M and M has $\leqslant \lambda$ relations and functions, then M has an elementary submodel of power $\leqslant \lambda$ which includes A.*

Really, it is well known that AP 1.2 holds for logic stronger than first-order; and we use only very specific formulas. So what we need is

AP 1.3. Fact. *Let G be a group, κ a cardinal $< |G|$, H_i $(i < \kappa)$ subgroups of G. Then we can find functions F_i^n $(i < \kappa)$, F_i^n an n-place function from G to G, such that, if G^* is a non-empty subset of G closed under the F_i^n,s, then*

(a) *G^* is a subgroup of G.*

(b) *Suppose $x_1, \ldots, x_n$ are variables, $a_1, \ldots, a_m \in G^*$, Γ is a finite set whose elements have the form: equations (in $x_i, \ldots, x_n$, $a_1, \ldots, a_m$) inequalities (in $x_1, \ldots, x_m$, $a_1, \ldots, a_m$), and $x_k \in H_\alpha$, or $x_k \notin H_\alpha$. If Γ is solvable in G, then Γ is solvable in G^*.*

(c) *For $a \in G^*$, $\alpha < \kappa$, $|\{gag^{-1} : g \in H_\alpha \cap G^*\}| = \mathrm{Min}\{|G^*|, |\{gag^{-1} : g \in H_\alpha\}|\}$.*
For $a \in G^*$: (i) $a \in \mathrm{Cent}(G^*) \Rightarrow a \in \mathrm{Cent}\, G$,
(ii) $a \in \mathrm{Cent}^\infty(G^*) \Rightarrow a \in \mathrm{Cent}^\infty(G)$,
(iii) $a \in (G^*)^{(1)} \Rightarrow a \in G^{(1)}$,
(iv) $a \in (G^*)^{(\infty)} \Rightarrow a \in G^{(\infty)}$.

(d) *For $\theta \leqslant \kappa$, $a \in G^*$, $a \in \mathrm{Min}_\theta\, G \Rightarrow a \in \mathrm{Min}_\theta(G^*)$.*

(e) *Like* (c) *with H_i, $H_i \cap G^*$ instead of G, G^*.*

AP 1.4. *For G^n, F_i^n as in* AP 1.3, *the closure under the F_i^n's of a set of power κ has power κ.*

AP 2. *On Fodor's Lemma*

AP 2.1. Definition. For a regular uncountable cardinal λ, let $\mathcal{D}_\lambda$ be the filter generated by the closed unbounded subsets of λ (as an ordered set). Note: Every successor cardinal is regular.

A set $S \subseteq \lambda$ is called stationary if $\lambda - S \notin \mathcal{D}_\lambda$. Note that every stationary subset of λ has power λ, and λ is a stationary subset of λ.

By Fodor, we have the following

AP 2.2. Theorem. *If λ is regular and uncountable $S \subseteq \lambda$ is stationary, f a function from S into λ, $f(\alpha) < 1 + \alpha$, then on some stationary $T \subseteq S$, f is constant.*

Another way to phrase it is:

AP 2.2′. Theorem. *Let λ be regular and uncountable* (*e.g., a successor cardinal*), *$S \subseteq \lambda$ stationary. Suppose A_α is a set of power $< \lambda$* (*for $\alpha < \lambda$*). *If f is a function* *with domain S and for every $\alpha \in S$, $f(\alpha) \in \bigcup_{\beta<\alpha} A_\beta$, then f is constant on some stationary subset of S.*

Fodor uses his lemma to prove the existence of large free sets. We need the following variant.

AP 2.3. Conclusion. Suppose T_α is a set of power $< \theta$, for each $\alpha < \theta^+$. Then for some stationary $S \subseteq \theta^+$ (hence $|S| = \lambda^+$) and $\alpha(*) < \theta^+$, for every distinct β, γ from S, $T_\beta \cap T_\gamma \subseteq \bigcup_{\alpha<\alpha(*)} T_\alpha$ and for $\beta \in S$, $T_\beta \cap (\bigcup_{\gamma<\beta} T_v) \subseteq \bigcup_{\gamma<\alpha(*)} T_\gamma$.

AP 3. *On the weak diamond $\alpha(*)$*

The following is not as well known as AP 1 and AP 2. It is from Devlin and Shelah [2], and for $\chi > 2^\kappa$, [9, Ch. VIX, §1]. Note that A_i, B_η are used below only to omit some easy set theory in the applications.

AP 3.1. Theorem. *Suppose $2^\kappa < 2^{(\kappa^+)}$, χ a cardinal $\leq 2^\kappa$ or even $\chi^{\aleph_0} < 2^{\kappa^+}$ (or even less). Suppose further that for every sequence η of zeros and ones a set B_η is given, $|B_\eta| \leq \kappa$, $B_{\eta \restriction \alpha} \subseteq B_\eta$ for $\alpha < l(\eta)$, and for every $i < \chi$ a set A_i is given, $|A_i| \leq \kappa^+$. Lastly suppose that for each $\eta \in {}^{(\kappa^+)}2$, $i(\eta)$ is an ordinal $< \chi$ and f_η is a function from $\bigcup_{\alpha<\kappa^+} B_{\eta \restriction \alpha}$ into $A_{i(\eta)}$.*

Then we can find a limit $\delta < \kappa^+$, and sequences η, $v \in {}^{(\kappa^+)}2$ s.t.: $\eta \restriction \delta = v \restriction \delta$, $\eta(\delta) \neq v(\delta)$, $i(\eta) = i(v)$ and $f_\eta \restriction B_{\eta \restriction \delta} = f_v \restriction B_{v \restriction \delta}$.

AP 3.2. Corollary. *Suppose G is a group, $2^\kappa < 2^{\kappa^+}$, $\mu^{\aleph_0} < 2^{\kappa^+}$ and for $\eta \in {}^{(\kappa^+)>}2$, H_η is a subgroup of G of power $\leq \kappa$, $H_{\eta \restriction \alpha} \subseteq H_\eta$. If among $\{\bigcup_{\alpha<\kappa^+} H_{\eta \restriction \alpha}$: $\eta \in {}^{(\kappa^+)}2\}$ there are $\leq \mu$ nonconjugate subgroups of G, then for some η, $v \in {}^{(\kappa^+)}2$ and limit $\delta < \kappa^+$, for some $g \in \mathrm{Cm}_G(\bigcup_{\alpha<\delta} H_{\eta \restriction \alpha})$, $\Box^g$ maps $\bigcup_{\alpha<\kappa^+} H_{\eta \restriction \alpha}$ onto $\bigcup_{\alpha<\kappa^+} H_{v \restriction \alpha}$, $\eta \restriction \delta = v \restriction \delta$, $\eta(\delta) \neq v(\delta)$.*

Remark. We can assign a closed unbounded subset C_η of κ^+ for each $\eta \in {}^{(\kappa^+)}2$ and demand $\delta \in C_\eta \cap C_v$.

Remark. See the proof of 9.5, at the end, for the deduction of AP3.2 from AP3.1.

Final remarks. (1) It seems that the ideas of the end of the proof of 8.4 can be used to simplify the proofs toward the end of Section 9 (hence in Section 7). See below a shorter proof.

(2) It seems worthwhile to reorganize (and/or redo) the proof of Theorems 0.1, 10.1, as in the proof of 8.4 (particularly the beginning), i.e., to replace $\mathcal{P}$ by some more restrictive class (like those failing (1), (2), (3) respectively of 8.4).

A shorter proof. In 9.10 after having assumed H_η is defined but not $H_{\eta^\wedge\langle 0\rangle}$, $H_{\eta^\wedge\langle 1\rangle}$ and showing G_η does not have $(2^\kappa)^+$ $[\theta, \kappa^{++})$-special subgroups, nonconjugate in pairs in G_η, now $(G_\eta : \mathrm{Min}_\theta\, G_\eta) \leq 2^\kappa$ by (γ) (and 9.7A(3)). Hence $G'_\eta = \mathrm{Min}_\theta\, G_\eta$ does not have $(2^\kappa)^+$ $[\theta, \kappa^{++})$-special subgroups, nonconjugate in pairs in G^1_η. Now notice $\mathrm{cg}(G^1_\eta) \leq \kappa$ [by 9.4]. Next we shall prove (as in 8.4) $|\mathrm{BA}'(\mathrm{Min}\, G^1_\eta)| \leq \kappa$. Suppose there is $I \in \mathrm{BA}'(G^1_\eta)$, $I \cap \mathrm{Min}\, G^1_\eta \subseteq \mathrm{Cent\ Min}\ G^1_\eta$. We choose by induction on $\alpha < \kappa^+$, $I_\alpha \in \mathrm{BA}'(G^1_\eta)$, $L_\alpha \subseteq I_\alpha$, L_α an explicit $[\theta, \kappa^+)$-group,

$$L_\alpha \subseteq \mathrm{Min}_\theta\, \mathrm{Cm}_{G^1_\eta}\Big(\bigcup_{\beta<\alpha} L_\beta\Big), \qquad I_\alpha \cap \mathrm{Cm}_{G^1_\eta}\Big(\bigcup_{\beta\leq\alpha} L_\beta\Big) \subseteq \mathrm{Cent}\, I_\alpha.$$

We cannot succeed (as $\{\langle \bigcup_{\beta\in S} L_\beta\rangle_{G^1_\eta} : S \subseteq \kappa^+\}$ has power $> 2^\kappa$). If we have defined for every $\beta < \alpha$, $\alpha < \kappa^+$ and there is $I_\alpha \in \mathrm{BA}'(G^1_\eta)$, $|I_\alpha| < \lambda$, $I_\alpha \cap \mathrm{Min}_\theta\, \mathrm{Cm}_{G^1_\eta}(\bigcup_{\beta<\alpha} L_\beta) \not\subseteq \mathrm{Cent}\, I_\alpha$, we know

$$I_\alpha \cap \mathrm{Min}_\theta\, \mathrm{Cm}_{G^1_\eta}\Big(\bigcup_{\beta<\alpha} L_\beta\Big) \in \mathrm{BA}'\Big(\mathrm{Min}_\theta\Big(\mathrm{Cm}_{G^1_\eta}\Big(\bigcup_{\beta<\alpha} L_\beta\Big)\Big)\Big)$$

hence we know there is L_α as required (by (γ), 4.8(2)). So for some $\alpha < \kappa^+$, there is no such I_α. Let $K_{\eta^\wedge\langle 0\rangle} = \bigcup_\beta L_\beta$, $K_{\eta^\wedge\langle 1\rangle}$ a θ-subgroup of I_0. So $|\mathrm{BA}'(G^1_\eta)| \leq \kappa$. Now we do the last paragraph of 9.10.

Remark. So speciality is apparently not needed.

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