JOURNAL OF ALGEBRA 117, 198–226 (1988)

Representation of the Affine Superalgebras $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$ and Their Subalgebras $A^{(2)}_{2l}$, $A^{(2)}_{2l-1}$ by Vertex Operators

GEORGE GOLITZIN*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by Melvin Hochster

Received November 5, 1986

The structure theory of standard modules of affine Lie algebras, given by J. Lepowsky and R. L. Wilson in [LW], is stated for representations of affine superalgebras. As an application, the standard modules of level one for the superalgebras $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$ and their affine subalgebras $A^{(2)}_{2l}$, $A^{(2)}_{2l-1}$ are constructed explicitly. These modules are realized as the tensor product of symmetric and exterior algebras with an irreducible representation of a certain finite 2-group. The affine superalgebra acts on this space by tensor products of vertex operators, operators of Clifford type, and elements of the 2-group. As a corollary, the spin representations of the Lie algebras B_l and D_l are obtained from the 2-group representation.

INTRODUCTION

This paper gives a construction of certain irreducible representations of the affine superalgebras $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$ and their subalgebras $A^{(2)}_{2l}$, $A^{(2)}_{2l-1}$. The representations are of fundamental highest weight, and are precisely those on which the unique central element (suitably normalized) acts as unity.

Relative to a particular Heisenberg subalgebra $\tilde{\mathfrak{t}}'$, the representations decompose as

$$V = \mathscr{S} \otimes \Omega,$$

where the first factor is the symmetric algebra on $\tilde{\mathfrak{t}}'_-$ and the second is the

* This research supported in part by NSF Grant DMS 86-00037.

0021-8693/88 $3.00

vacuum space $\{v \in V \mid \tilde{\mathfrak{t}}'_{+} \cdot v = 0\}$. In the "super" case the vacuum space has the structure

$$\Omega = L \otimes M,$$

where L is an infinite-dimensional exterior algebra and M is a representation space for a finite 2-group $\mathscr{P}$. The superalgebra is represented on

$$V = \mathscr{S} \otimes L \otimes M$$

by the tensor product of vertex operators on the first factor, Clifford-type operators on the second, and elements of $\mathscr{P}$ on the third. In the non-super case the picture is much the same, except that the vacuum space is now just M.

The space M also affords a construction of the spin representations of the Lie algebras B_l and D_l. These algebras are represented by elements of the group algebra on $\mathscr{P}$.

The paper draws heavily on two sources, [K2] and [LW]. In [K2] Kac introduced and classified the affine superalgebras. In Section 1 we review basic facts about superalgebras and include that classification theorem; this has the dual purpose of setting notation and establishing a context for what follows. In [LW] Lepowsky and Wilson introduced the Z-algebras, of which we make free use here. The Z-algebra approach was extended to the affine superalgebras in [G] through some minor alterations. In Section 2 we state the central result of that theory (Theorem 2). While the paper's main result (Theorem 3) could probably be written down without reference to Theorem 2, we include this general theorem because it is convenient to use, and because we hope to refer to it in future work. Since the presentation of Section 2 so closely follows that of [LW], we refer the reader to that paper for proofs and more thorough motivation.

In Section 3 we realize the algebras $A^{(4)}(0, 2l)$ and $A^{(2)}(0, 2l-1)$ by writing down their Chevalley generators in terms of the underlying finite-dimensional algebra $A(0, n)$ and its Cartan automorphism. (The subalgebra fixed by this automorphism is B_l or D_l, $n = 2l$ or $2l-1$, respectively.) This information is necessary in Section 4, where we construct the representations and calculate their highest weights.

The representations given here were first constructed (in much different fashion) in [FF]. Essentially the same construction of $A^{(2)}_{2l}$, $A^{(2)}_{2l-1}$ has appeared in [FLM], where, more generally, the twisted affine algebras $\hat{\mathfrak{g}}$ are constructed for $\mathfrak{g}$ a Lie algebra of type A, D, or E.

1. Preliminaries

We review basic material concerning superalgebras, particularly those of affine type. For the original exposition, see [K1] for the finite-dimensional theory and [K2] for the infinite-dimensional theory.

We take the complex numbers $\mathbb{C}$ as ground field. An algebra A is called a *superalgebra* if it is graded by the additive group $\mathbb{Z}/2\mathbb{Z}$. Thus

$$A = A_0 \oplus A_1$$

and multiplication of homogeneous elements respects the grading. Given $a \in A$, $\alpha = 0$ or $1 \pmod 2$, write

$$\deg_2(a) = \alpha,$$

the ($\mathbb{Z}/2\mathbb{Z}$-) *degree of* a. Elements of A_0 are called *even*, elements of A_1 *odd*. A *homomorphism* $f\colon A \to B$ *of superalgebras* is a homomorphism of algebras such that

$$f(A_\alpha) \subset B_\alpha,$$

$\alpha = 0$ or $1 \pmod 2$. A *Lie superalgebra* (LSA) is a superalgebra $\mathfrak{g}$ with product, denoted by a bracket $[\ ,\ ]$, satisfying

$$[a, b] = -(-1)^{\alpha\beta}[b, a]$$

and

$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]],$$

for all $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$, $c \in \mathfrak{g}$, $\alpha, \beta = 0$ or $1 \pmod 2$. In particular $\mathfrak{g}_0$ is an ordinary Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. An associative superalgebra may be given the structure of LSA with bracket

$$[a, b] = ab - (-1)^{\alpha\beta}\, ba,$$

for $a \in A_\alpha$, $b \in A_\beta$.

Let V be a vector space over $\mathbb{C}$, $V = V_0 \oplus V_1$ some decomposition. Then

$$\text{End } V = (\text{End } V)_0 \oplus (\text{End } V)_1,$$

where

$$(\text{End } V)_\alpha = \{a \in \text{End } V \mid a \cdot V_\beta \subset V_{\alpha+\beta},\ \beta = 0, 1 \pmod 2\},$$

$\alpha = 0$ or 1 (mod 2). This makes End V into an associative superalgebra. We denote the associated LSA by $l(V)$ or $l(V_0, V_1)$. The notion of *representation* or *module* of an associative or Lie superalgebra should now be clear; in either category a homomorphism $\phi: V \to V'$ is assumed to be graded in the sense that $\phi \cdot V_\alpha \subset V'_{\sigma\alpha}$, where $\sigma: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a bijection. The analog for $l(V_0, V_1)$ of the trace form is called the *supertrace form* and is defined as follows: given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in l(V_0, V_1)$, set

$$\text{str}(A) = \text{trace } a - \text{trace } d;$$

given $A, B \in l(V_0, V_1)$, set

$$(A, B) = \text{str}(AB).$$

To an $n \times n$ matrix A and a subset $\tau \subset \{1, ..., n\}$ of indices one associates the *contragredient LSA* $\mathfrak{g}(A, \tau)$ as in [K1]. This algebra is characterized in

PROPOSITION 1. *Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{h} \subset \mathfrak{g}_0$ a commutative subalgebra, $e_1, ..., e_n$, $f_1, ..., f_n$ elements of $\mathfrak{g}$, and let $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\} \subset \mathfrak{h}$, $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ be linearly independent sets such that*

$$e_i, f_i \in \begin{cases} \mathfrak{g}_1, & \text{if } i \in \tau, \\ \mathfrak{g}_0, & \text{if } i \notin \tau. \end{cases}$$

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee,$$

$$[h, e_i] = \langle \alpha_i, h \rangle e_i, \qquad [h, f_i] = -\langle \alpha_i, h \rangle f_i,$$

$h \in \mathfrak{h}$, $i, j = 1, ..., n$. Suppose that e_i, f_i $(i = 1, ..., n)$ and $\mathfrak{h}$ generate $\mathfrak{g}$ as an LSA, and that $\mathfrak{g}$ has no nonzero ideals which intersect $\mathfrak{h}$ trivially. Finally, set $A = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j=1}^{i,j=n}$, and suppose that dim $\mathfrak{h} = 2n - \text{rank}(A)$. *Then $\mathfrak{g} = \mathfrak{g}(A, \tau)$, the LSA associated to the pair (A, τ).*

(Proposition 1 appears in the "non-super" context as Proposition 1.4 in [K3].)

The pair (A, τ) is called a *generalized Cartan matrix* (GCM) if it satisfies

$$a_{ij} \in \mathbb{Z}, \qquad a_{ij} \leqslant 0, \qquad \text{for all } i, j;$$

$$a_{ii} = \begin{cases} 1, & \text{if } i \in \tau \\ 2, & \text{if } i \notin \tau; \end{cases}$$

$$a_{ij} \neq 0 \qquad \text{if and only if} \quad a_{ji} \neq 0, \qquad \text{for all } i, j.$$

In view of the second condition we may dispense with τ in the notation when A is a GCM. The algebra $\mathfrak{g}(A)$ is then called a *Kac–Moody superalgebra*. (We remark that the matrix A is normalized so that its transpose is again a GCM. This varies slightly from the original exposition

[K2], and causes some statements (and notably the Dynkin diagrams) to differ from those appearing in the literature.)

When A is a *symmetrizable* GCM (i.e., DA is symmetric for some invertible diagonal matrix D), the associated Kac–Moody superalgebra enjoys a satisfactory representation theory culminating in the Weyl–Kac character formula [K2]. Much more can be said, however, about the representations of $\mathfrak{g}(A)$ when A is also positive semi-definite of corank one. The algebra $\mathfrak{g}(A)$ is then called *affine*, and its Dynkin diagram appears on one of the following lists. The diagrams consist of $l+1$ nodes; the ith node is clear if $i \notin \tau$ and dark if $i \in \tau$. The ith and jth nodes are connected by $\max(|a_{ij}|, |a_{ji}|)$ segments. If $|a_{ij}| > |a_{ji}|$, an arrow points to the ith node. The integers written next to the nodes give the coefficients of linear dependence of the corresponding columns of the matrix.

The representation theory of the affine (super)algebras is particularly rich due to the realization of these algebras as (essentially) central extensions of loop algebras. We review this realization below.

TABLE Aff1

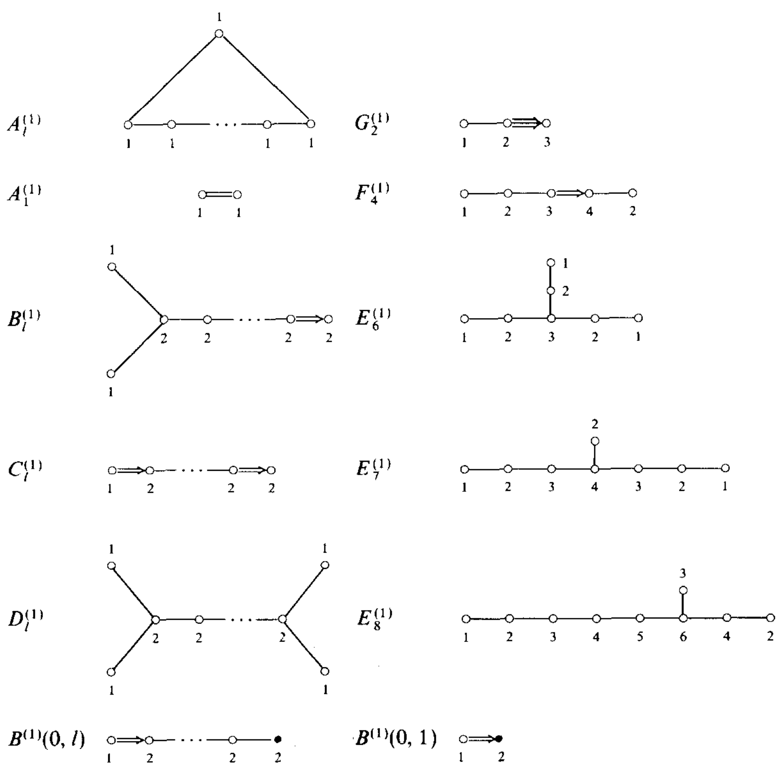


TABLE Aff2

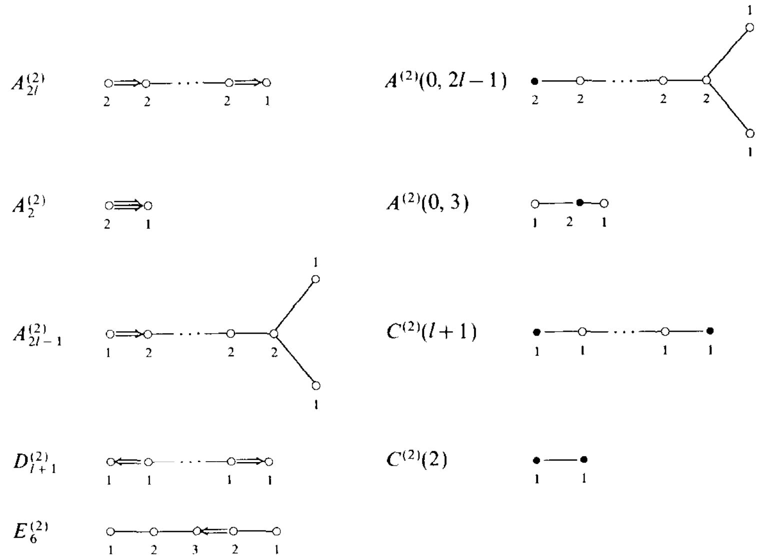
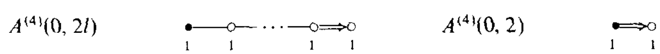


TABLE Aff3

TABLE Aff4



Let $\mathfrak{g}$ now denote one of the finte-dimensional Lie algebras of type $A-G$, or one of the LSA's $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, or $G(3)$. (With the exception of $A(n, n)$, which has a one-dimensional center, these are all the simple, finite-dimensional, contragredient LSA's: see [K1, Theorem 3].) In this paper we are especially interested in $A(0, n)$: this is the subalgebra of $l(V_0, V_1)$ of elements of supertrace zero, where V_0 is one-dimensional and V_1 is $(n+1)$-dimensional over $\mathbb{C}$.

Let v be an automorphism of $\mathfrak{g}$ of order m; write

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_{(i)} \tag{1.1}$$

for the decomposition of $\mathfrak{g}$ into eigenspaces for v. Let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in an indeterminate t, and let $\bar{\mathfrak{g}}(\nu)$ be the subalgebra of $\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$ given by

$$\bar{\mathfrak{g}}(\nu)=\sum_{i\in\mathbb{Z}}\mathfrak{g}_{(i \bmod m)}\otimes t^i.$$

This is an LSA with induced $\mathbb{Z}/2\mathbb{Z}$ grading ($\deg_2 t=0$) and bracket

$$[x\otimes t^i, y\otimes t^j]=[x, y]\otimes t^{i+j},$$

for $x\in\mathfrak{g}_{(i \bmod m)}$, $y\in\mathfrak{g}_{(j \bmod m)}$, $i, j\in\mathbb{Z}$. We define an extension $\tilde{\mathfrak{g}}(\nu)$ of $\bar{\mathfrak{g}}(\nu)$ by a one-dimensional center and "degree" operator d:

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}(\nu)=\bar{\mathfrak{g}}(\nu)\oplus\mathbb{C}c\oplus\mathbb{C}d, \tag{1.2}$$

where $\deg_2(c)=\deg_2(d)=0$, and the bracket is given by

$$\begin{aligned}[c, x\otimes t^i]&=[c, d]=0,\\ [d, x\otimes t^i]&=ix\otimes t^i,\\ [x\otimes t^i, y\otimes t^j]&=[x, y]\otimes t^{i+j}+i\delta_{i,-j}(x, y)\,c,\end{aligned} \tag{1.3}$$

for $x\in\mathfrak{g}_{(i \bmod m)}$, $y\in\mathfrak{g}_{(j \bmod m)}$, $i, j\in\mathbb{Z}$, where (,) is the (super)trace form on $\mathfrak{g}$.

The correspondence between the algebras $\tilde{\mathfrak{g}}(\nu)$ and the affine superalgebras $\mathfrak{g}(A)$ of Tables Aff1–Aff4 is given in the following theorem of Kac ([K2, Proposition 1.2]).

THEOREM 1 (Kac). *Let $\mathfrak{g}(A)$ be a Lie superalgebra from tables* Aff1–Aff4 *with Dynkin diagram $L^{(k)}$, and let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra of type L. Then for every node p_s of $L^{(k)}$ with numerical mark a_s there exists an automorphism ν of $\mathfrak{g}$ of order $m=ka_s$ such that*

(i) *$\mathfrak{g}_{(0)}$ is a contragredient Lie superalgebra of type $L^{(k)}-\{p_s\}$;*

(ii) $\mathfrak{g}(A)\approx\tilde{\mathfrak{g}}(\nu)$.

2. LEPOWSKY–WILSON STRUCTURE THEORY OF REPRESENTATIONS

In this section we restate a structure theorem of Lepowsky and Wilson in the context of affine superalgebras. By giving the equivalence of two categories (defined below) the theorem reduces the construction of representations of affine superalgebras $\tilde{\mathfrak{g}}$ to the construction of spaces Ω with operators $Z_i(\beta)$ ($i \in \mathbb{Z}$, β a root of $\mathfrak{g}$) satisfying relations of the form

$$a(\zeta_1/\zeta_2)\left(\sum Z_i(\alpha)\,\zeta_1^i\right)\left(\sum Z_j(\beta)\,\zeta_2^j\right)$$
$$+\,b(\zeta_2/\zeta_1)\left(\sum Z_j(\beta)\,\zeta_2^j\right)\left(\sum Z_i(\alpha)\,\zeta_1^i\right)$$
$$= c(\zeta_1/\zeta_2)\sum Z_i(\alpha+\beta)\,\zeta_2^i + d(\zeta_1/\zeta_2),$$

where ζ_1, ζ_2 are indeterminates, $c(\zeta)$ and $d(\zeta)$ are doubly infinite formal series, and (typically) $a(\zeta)$ and $b(\zeta)$ are binomial series. While the general form of these relations may appear at first to offer no simplification of the original problem, in particular cases the relations admit pleasing solutions which serve both to eludicate the structure of the algebra at hand and to connect this theory with many areas of mathematics and physics.

The equivalence theorem as we state it appeared in the "non-super" context in [LW]. It was extended to the "super" case in [G], where a construction of the basic modules of the superalgebra $C^{(2)}(2)$ was given. We refer the reader to [LW] for the proof; the extension to the super case is rather straightforward.

As in Section 1 let $\mathfrak{g}$ be a simple, finite-dimensional, contragredient LSA. Let ν be an automorphism of order m, isometric with respect to the (super)trace form $(\ ,\ )$. Let $\mathfrak{t}$ be a ν-invariant Cartan subalgebra; we identify $\mathfrak{t}$ with $\mathfrak{t}^*$ via the form $(\ ,\ )$ and denote by $\Phi \subset \mathfrak{t}$ the set of roots of $\mathfrak{g}$ relative to $\mathfrak{t}$. For each $\beta \in \Phi$ choose a nonzero root vector x_β. Define structure constants $\varepsilon(\alpha, \beta)$ for $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$ by

$$[x_\alpha, x_\beta] = \varepsilon(\alpha, \beta)\, x_{\alpha+\beta}. \tag{2.1}$$

Define constants $\eta(p, \beta)$ for $\beta \in \Phi$, $p \in \mathbb{Z}/m\mathbb{Z}$ by

$$\nu^p x_\beta = \eta(p, \beta)\, x_{\nu^p\beta}. \tag{2.2}$$

Recall from Section 1 the decomposition (1.1) of $\mathfrak{g}$ into eigenspaces for ν, and the construction (1.2) of the affine superalgebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(\nu)$. We will consider also the subalgebra

$$\tilde{\mathfrak{t}}' = [\tilde{\mathfrak{t}}, \tilde{\mathfrak{t}}] = \tilde{\mathfrak{t}}_- \oplus \mathbb{C}c \oplus \tilde{\mathfrak{t}}_+,$$

where

$$\tilde{\mathfrak{t}}_\pm = \sum_{i>0} \tilde{\mathfrak{t}}_{\pm i}$$

with respect to the $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$. From the bracket definition (1.3) it is clear that $\tilde{\mathfrak{k}}'$ is a graded Heisenberg subalgebra of $\tilde{\mathfrak{g}}$.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space; let $\zeta, \zeta_1, \zeta_2, \ldots$ be commuting indeterminates. Denote by $V\{\zeta_1, \zeta_2, \ldots\}$ the space of formal (doubly infinite) Laurent series in $\zeta_1, \zeta_2, \ldots$ with coefficients in V, i.e., the space of series

$$\sum_{i_1, i_2, \ldots \in \mathbb{Z}} v_{i_1 i_2 \cdots} \zeta_1^{i_1} \zeta_2^{i_2} \cdots$$

with $v_{i_1 i_2 \cdots} \in V$. Note we allow infinitely many coefficients to be nonzero regardless of whether the i_j are greater or less than zero. We give $V\{\zeta_1, \zeta_2, \ldots\}$ a $\mathbb{Z}/2\mathbb{Z}$-grading via the coefficients. Given a map $\pi: V \to V'$ denote again by π the induced map $V\{\zeta_1, \zeta_2, \ldots\} \to V'\{\zeta_1, \zeta_2, \ldots\}$.

Given $x \in \mathfrak{g}$, write $x = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} x_{(i)}$ for the decomposition of x into eigenvectors for v, and set

$$x(\zeta) = \sum_{i \in \mathbb{Z}} (x_{(i \bmod m)} \otimes t^i)\, \zeta^i. \tag{2.3}$$

Also of importance are the series $\delta(\zeta)$, $(D\delta)(\zeta) \in \mathbb{C}\{\zeta\}$:

$$\delta(\zeta) = \sum_{i \in \mathbb{Z}} \zeta^i, \qquad (D\delta)(\zeta) = \sum_{i \in \mathbb{Z}} i\zeta^i. \tag{2.4}$$

The equivalence theorem concerns modules in the category $\mathscr{C}_k$, by definition the category of $\tilde{\mathfrak{g}}$-modules V such that

(i) c acts by the scalar k on V;

(ii) d acts diagonally on V, so that $V = \coprod_{z \in \mathbb{C}} V_z$, where $V_z = \{v \in V \mid d \cdot v = zv\}$;

(iii) for every $z \in \mathbb{C}$, there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$ one has $V_{z+i} = (0)$.

Fix $k \in \mathbb{C}^*$ and let $(V, \pi) \in \mathscr{C}_k$, where $\pi: \tilde{\mathfrak{g}} \to \operatorname{End} V$ is the map giving the representation. For each $\beta \in \Phi$ define series in $(\operatorname{End} V)\{\zeta\}$ by

$$E^{\pm}(\beta, \zeta, \pi) = \exp\left(\pm \sum_{j \geq 1} \pi(\beta_{(\pm j)} \otimes t^{\pm j})\, \zeta^{\pm j} \Big/ jk \right),$$
$$Z(\beta, \zeta, \pi) = E^{-}(\beta, \zeta, \pi)\, \pi x_\beta(\zeta)\, E^{+}(\beta, \zeta, \pi),$$

where $x_\beta(\zeta)$ is given by (2.3). We will sometimes write

$$Z(\beta, \zeta, \pi) = \sum Z_i(\beta, \pi)\, \zeta^i.$$

Because of the truncation property (iii) of V, these are well defined elements of $(\text{End } V)\{\zeta\}$. Writing any of these four series as $Y = \sum Y_i \zeta^i$, we have Y_i homogeneous of degree i with respect to the gradation (ii) of V defined by d.

As $\tilde{\mathfrak{t}}'$-module,

$$V = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega_V,$$

where

$$\Omega_V = \{v \in V \mid \tilde{\mathfrak{t}}_+ \cdot v = 0\},$$

the $\tilde{\mathfrak{t}}'$-*vacuum space* of V. One shows that the operators $Z_i(\beta)$ commute with the action of $\tilde{\mathfrak{t}}'$ on V and hence preserve Ω. They also satisfy certain (rather complicated) relations (2.5(iv–vii) below) which in some sense characterize V as $\tilde{\mathfrak{g}}$-module. This motivates the definition of the category $\mathscr{D}_k$ which follows.

Let

$$\mathfrak{b} = \mathfrak{t}_{(0)} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

an abelian Lie subalgebra of $\tilde{\mathfrak{g}}$. (Recall $\mathfrak{t}_{(0)}$ is the subset of $\mathfrak{t}$ fixed by ν.) Let S be the set

$$S = \mathfrak{b} \cup (\mathbb{Z} \times \Phi).$$

An *S-module* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$W = W_0 \oplus W_1,$$

together with a $\mathfrak{b}$-module structure

$$\sigma \colon \mathfrak{b} \to \text{End } W$$

and a map

$$Z \colon \mathbb{Z} \times \Phi \to \text{End } W$$
$$(i, \beta) \mapsto Z_i(\beta)$$

such that $Z_i(\beta)$ is an even (resp., odd) operator on W if and only if β is an even (resp., odd) root of $\mathfrak{g}$. (One says a root β is *even* or *odd* if the root space $\mathfrak{g}_\beta$ lies in $\mathfrak{g}_0$ or $\mathfrak{g}_1$, respectively.) Given an S-module W, set

$$Z(\beta, \zeta) = \sum_{i \in \mathbb{Z}} \mathbb{Z}_i(\beta)\, \zeta^i \in (\text{End } W)\{\zeta\}.$$

For $k \in \mathbb{C}^*$, denote by $\mathcal{D}_k$ the category of S-modules (W, σ, Z) such that

(2.5) (i) $\sigma(c) = k$,

(ii) $W = \coprod_{z \in \mathbb{C}} W_z$ with respect to $\sigma(d)$,

(iii) for all $z \in \mathbb{C}$ there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$ one has $W_{z+i} = (0)$, and such that for all $\beta \in \Phi$,

(iv) $Z_i(\beta)$ has operator degree i on W with respect to the gradation (ii),

(v) for $\alpha \in \mathfrak{t}_{(0)}$, $[\sigma\alpha, Z(\beta, \zeta)] = (\alpha, \beta)\, Z(\beta, \zeta)$,

(vi) for $p \in \mathbb{Z}/m\mathbb{Z}$, $Z(\beta, \omega^p \zeta) = \eta(p, \beta)\, Z(\nu^p \beta, \zeta)$,

where ω is a primitive mth root of unity, fixed for the discussion. Finally we require that the $Z_i(\beta)$ satisfy the *generalized (anti-)commutation relations*

(vii) for all $\alpha, \beta \in \Phi$,

$$\prod_{p \in \mathbb{Z}/m\mathbb{Z}} (1 - \omega^{-p}\zeta_1/\zeta_2)^{(\nu^p\alpha, \beta)/km}\, Z(\alpha, \zeta_1)\, Z(\beta, \zeta_2)$$

$$-(-1)^{d_\alpha d_\beta} \prod_{p \in \mathbb{Z}/m\mathbb{Z}} (1 - \omega^{-p}\zeta_2/\zeta_1)^{(\nu^p\beta, \alpha)/km}\, Z(\beta, \zeta_2)\, Z(\alpha, \zeta_1)$$

$$= m^{-1} \sum{}' \eta(p, \alpha)\, \varepsilon(\nu^p\alpha, \beta)\, Z(\nu^p\alpha + \beta, \zeta_2)\, \delta(\omega^{-p}\zeta_1/\zeta_2)$$

$$- m^{-1}(x_{-\beta}, x_\beta) \sum{}'' \eta(p, \alpha)(\pi(\beta_{(0)})(\zeta_2)\, \delta(\omega^{-p}\zeta_1/\zeta_2)$$

$$- k(D\delta)(\omega^{-p}\zeta_1/\zeta_2)),$$

where $\sum'$ is over $p \in \mathbb{Z}/m\mathbb{Z}$ such that $\nu^p\alpha + \beta \in \Phi$, $\sum''$ is over $p \in \mathbb{Z}/m\mathbb{Z}$ such that $\nu^p\alpha + \beta = 0$, and $d_\gamma = \deg_2(x_\gamma)$ for $\gamma = \alpha, \beta$. Recall that the constants $\varepsilon(\alpha, \beta)$ and $\eta(p, \beta)$ are given by (2.1) and (2.2), respectively, and the series $\delta(\zeta)$ and $(D\delta)(\zeta)$ by (2.4).

We define a functor $\Omega: \mathcal{C}_k \to \mathcal{D}_k$. Given (V, π) in $\mathcal{C}_k$ with $k \neq 0$, set

$$\Omega(V) = \Omega_V,$$

$$\sigma(b) = \pi(b)|_{\Omega_V}, \qquad \text{for } b \in \mathfrak{b},$$

$$Z_i(\beta) = Z_i(\beta, \pi)|_{\Omega_V}, \qquad \text{for } i \in \mathbb{Z}, \beta \in \Phi.$$

Given a morphism $f: V \to V'$, one has $f(\Omega_V) \subset \Omega_{V'}$; set

$$\Omega(f) = f|_{\Omega_V}.$$

For the proof that (Ω_V, σ, Z) is in $\mathcal{D}_k$ we refer the reader to [LW, Theorem 3.10, Proposition 4.7].

We now define the "reverse" functor $\Lambda: \mathscr{D}_k \to \mathscr{C}_k$. Set

$$\tilde{\mathfrak{t}}^+ = \mathfrak{b} \oplus \tilde{\mathfrak{t}}_+ = \sum_{i \geqslant 0} \tilde{\mathfrak{t}}_i.$$

Denote by $\mathbb{C}(k)$ the one-dimensional $\tilde{\mathfrak{t}}^+$-module on which c acts by the scalar k and $\tilde{\mathfrak{t}}_+ \oplus \mathfrak{t}_{(0)} \oplus \mathbb{C}d$ acts trivially. Consider the induced $\tilde{\mathfrak{t}}$-module

$$K(k) = \mathfrak{u}(\tilde{\mathfrak{t}}) \otimes_{\mathfrak{u}(\tilde{\mathfrak{t}}^+)} \mathbb{C}(k).$$

By the Poincaré–Birkhoff–Witt theorem we may identify $K(k)$ with the symmetric algebra $\mathscr{S}(\tilde{\mathfrak{t}}_-)$ as $\tilde{\mathfrak{t}}$-module. Let $(W, \sigma, Z) \in \mathscr{D}_k$. Then

$$\operatorname{Ind}(W) = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes_{\mathbb{C}} W$$

is a $\tilde{\mathfrak{t}}$-module with action π given by

$$\pi(c) = k, \qquad \pi(d) = d \otimes 1 + 1 \otimes d,$$
$$\pi(a) = 1 \otimes \sigma(a), \qquad \text{for} \quad a \in \mathfrak{t}_{(0)},$$
$$\pi(h) = h \otimes 1,$$

Note that $\operatorname{Ind}(W)$ inherits a $\mathbb{Z}/2\mathbb{Z}$-gradation from W, and that in $\operatorname{End}(\mathscr{S}(\tilde{\mathfrak{t}}_-))\{\zeta\}$ the series $E^{\pm}(\beta, \zeta)$ are defined for each $\beta \in \Phi$. The representation π of $\tilde{\mathfrak{t}}$ extends to a representation of $\tilde{\mathfrak{g}}$ on $V = \operatorname{Ind}(W)$ by setting

$$\pi x_\beta(\zeta) = E^-(-\beta, \zeta)\, E^+(-\beta, \zeta) \otimes Z(\beta, \zeta);$$

in fact (V, π) is in $\mathscr{C}_k$ (see [LW, Proposition 5.3]). Hence we may define a functor $\Lambda: \mathscr{D}_k \to \mathscr{C}_k$ by setting

$$\Lambda(W, \sigma, Z) = (V, \pi) = (\operatorname{Ind}(W), \pi).$$

For a morphism $g: W \to W'$ in $\mathscr{D}_k$, let

$$\Lambda(g): \operatorname{Ind}(W) \to \operatorname{Ind}(W')$$

be the induced map. We can now state

THEOREM 2 (equivalence theorem of Lepowsky and Wilson). *For $k \in \mathbb{C}^*$, the functors*

$$\Omega: \mathscr{C}_k \to \mathscr{D}_k$$
$$\Lambda: \mathscr{D}_k \to \mathscr{C}_k$$

define exact equivalences of categories.

3. Realization of $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$

In this section we give an explicit realization (à la Theorem 1) of $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$ as the affine algebras associated with the finite-dimensional superalgebras $A(0, 2l)$, $A(0, 2l-1)$ and their Cartan automorphisms. We will need this information to identify the representations constructed in Section 4.

Let $n = 2l$ or $2l-1$, $\mathfrak{g} = A(0, n)$. Let $\beta_0, \beta_1, ..., \beta_n$ be the positive simple roots of $\mathfrak{g}$, with β_0 the odd root; denote by Φ the set of roots of $\mathfrak{g}$. It is convenient to choose the root vectors x_β as follows. Regard $\mathfrak{g}$ in its natural representation by $(n+2) \times (n+2)$ matrices of supertrace 0. Let x_{β_i} be the matrix with $(i, i+1)$-entry 1 and other entries 0; choose $x_{-\beta_i}$ similarly. For an arbitrary root $\beta = \pm\beta_i \pm \beta_{i+1} \pm \cdots \pm \beta_{i+k}$, set

$$x_\beta = [x_{\pm\beta_i}[x_{\pm\beta_{i+1}} \cdots [x_{\pm\beta_{i+k-1}}, x_{\pm\beta_{i+k}}] \cdots]].$$

If β is positive, x_β has a 1 in the appropriate position, and if β is negative, the nonzero entry is 1 if the height of β is odd and -1 if the height of β is even. Complete the basis of $\mathfrak{g}$ by setting

$$h_0 = \mathrm{diag}(1, 1, 0, ..., 0)$$
$$h_i = \mathrm{diag}(0, ..., 0, \underset{(i+1)}{1}, -1, 0, ..., 0),$$

$i = 1, ..., n$. Denote by $\mathfrak{t}$ the Cartan subalgebra spanned by $h_0, ..., h_n$. For the invariant form on $\mathfrak{g}$ we take

$$(a, b) = -(1/4)\,\mathrm{str}(ab),$$

$a, b \in \mathfrak{g}$.

Define a Cartan automorphism v of $\mathfrak{g}$ by

$$\begin{aligned} v x_\beta &= x_{-\beta}, && \beta \text{ even or } \beta \text{ odd negative,} \\ v x_\beta &= -x_{-\beta}, && \beta \text{ odd positive,} \\ v|_{\mathfrak{t}} &= -1. \end{aligned}$$

The automorphism v has order 4. We set

$$w_\beta = \begin{cases} x_\beta + x_{-\beta}, & \beta \text{ even} \\ x_\beta + \omega x_{-\beta}, & \beta \text{ odd positive} \end{cases}$$

$$\bar{w}_\beta = \begin{cases} x_\beta - x_{-\beta}, & \beta \text{ even positive} \\ x_\beta - \omega x_{-\beta}, & \beta \text{ odd positive,} \end{cases}$$

where ω is a primitive fourth root of unity. The eigenspaces for ν are

$$\begin{aligned}
\mathfrak{g}_{(0)} &= \operatorname{span}\{w_\beta \mid \beta \text{ even}\},\\
\mathfrak{g}_{(1)} &= \operatorname{span}\{w_\beta \mid \beta \text{ odd positive}\},\\
\mathfrak{g}_{(2)} &= \operatorname{span}(\{\bar{w}_\beta \mid \beta \text{ even positive}\} \cup \{h_i \mid i=0, ..., n\}),\\
\mathfrak{g}_{(3)} &= \operatorname{span}\{\bar{w}_\beta \mid \beta \text{ odd positive}\}.
\end{aligned}$$

We identify $\mathfrak{g}_{(0)}$ as B_l $(n=2l)$ or D_l $(n=2l-1)$ by writing down a Chevalley basis. Recall the definition of the structure constants $\varepsilon(\alpha, \beta)$, defined when $\alpha, \beta, \alpha+\beta \in \Phi$:

$$[x_\alpha, x_\beta] = \varepsilon(\alpha, \beta)\, x_{\alpha+\beta}.$$

It is easily seen that for even roots α, β one has

$$[w_\alpha, w_\beta] = \delta_{\alpha+\beta}\varepsilon(\alpha, \beta)\, w_{\alpha+\beta} + \delta_{\alpha-\beta}\varepsilon(\alpha, -\beta)\, w_{\alpha-\beta}, \tag{3.1}$$

where δ_γ equals one if $\gamma \in \Phi$ and zero otherwise.

We will use the following shorthand: for the simple positive roots β_i write $w_\beta = w_i$; for $\beta = \beta_i + \cdots + \beta_{i+k}$ write $w_\beta = w_{i,i+k}$. From (3.1) it is clear that $\operatorname{span}\{w_i \mid i \text{ odd}, 1 \leqslant i \leqslant n\}$ is a Cartan subalgebra of $\mathfrak{g}_{(0)}$. We may distinguish a root of $\mathfrak{g}_{(0)}$ by the l-tuple $(\delta(w_1), \delta(w_3), ...)$.

At this point we consider the cases $n=2l$ and $n=2l-1$ separately. First suppose $n=2l$. Set

$$\begin{aligned}
\delta_i &= (0, ..., 0, \underset{(i)}{1}, 1, 0, ..., 0), \qquad i=1, ..., l-1,\\
\delta_l &= (0, ..., 0, 2).
\end{aligned}$$

The δ_i are roots of $\mathfrak{g}_{(0)}$ with root vectors

$$\begin{aligned}
y_{\delta_i} &= (-w_{2i} - w_{2i-1,2i} + w_{2i,2i+1} + w_{2i-1,2i+1})/2,\\
y_{-\delta_i} &= (w_{2i} - w_{2i-1,2i} + w_{2i,2i+1} - w_{2i-1,2i+1})/2,
\end{aligned} \tag{3.2}$$

$i=1, ..., l-1,$

$$\begin{aligned}
y_{\delta_l} &= w_{2l-1,2l} + w_{2l},\\
y_{-\delta_l} &= w_{2l} - w_{2l-1,2l}.
\end{aligned}$$

Relative to our choice of Cartan subalgebra of $\mathfrak{g}_{(0)}$, the positive simple roots are

$$\gamma_i = (-1)^{l-i}\,\delta_i,$$

$i=1, ..., l$. Finally, put

$$\begin{aligned} E_i &= y_{\gamma_i}, \qquad F_i = y_{-\gamma_i}, \qquad i=1, ..., l-1, \\ H_i &= (-1)^{l-i}(w_{2i-1} + w_{2i+1}), \qquad i=1, ..., l-1, \qquad (3.3) \\ H_l &= 2w_{2l-1}. \end{aligned}$$

Then the set $\{E_i, F_i, H_i \mid i=1, ..., l\}$ is a set of Chevalley generators for $\mathfrak{g}_{(0)} \approx B_l$.

Now suppose $n=2l-1$. Let

$$\delta_i = (0, ..., 0, \underset{(i)}{1}, 1, 0, ..., 0), \qquad i=1, ..., l-1,$$

$$\delta_l = (0, ..., 0, -1, 1).$$

The root vectors y_{δ_i} are given as in the previous case for $i=1, ..., l-1$, and we set

$$\begin{aligned} y_{\delta_l} &= (-w_{2l-3,2l-2} + w_{2l-3,2l-1} + w_{2l-2} - w_{2l-2,2l-1})/2 \\ y_{-\delta_l} &= (w_{2l-3,2l-2} + w_{2l-3,2l-1} + w_{2l-2} + w_{2l-2,2l-1})/2. \end{aligned} \qquad (3.2')$$

As before, the positive simple roots are given by

$$\gamma_i = (-1)^{l-i}\, \delta_i, \qquad i=1, ..., l.$$

A set of Chevalley generators for $\mathfrak{g}_{(0)} \approx D_l$ is given by

$$\begin{aligned} E_i &= y_{\gamma_i}, \qquad F_i = y_{-\gamma_i}, \qquad i=1, ..., l, \\ H_i &= (-1)^{l-i}(w_{2i-1} + w_{2i+1}), \qquad i=1, ..., l-1, \\ H_l &= w_{2l-1} - w_{2l-3}. \end{aligned} \qquad (3.3')$$

Consider now the spaces $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$ as $\mathfrak{g}_{(0)}$-modules. For $\alpha, \beta \in \Phi$ with α even and β odd positive we have again the relation

$$[w_\alpha, w_\beta] = \delta_{\alpha+\beta}\varepsilon(\alpha, \beta)\, w_{\alpha+\beta} + \delta_{\beta-\alpha}\varepsilon(-\alpha, \beta)\, w_{\beta-\alpha}. \qquad (3.1')$$

(Note that if $\beta \pm \alpha \in \Phi$ then $\beta \pm \alpha$ must be positive.) We remark that the same relation holds with w_β replaced by $\bar{w}_\beta$, hence that $w_\beta \mapsto \bar{w}_\beta$ gives an isomorphism of $\mathfrak{g}_{(1)}$ with $\mathfrak{g}_{(3)}$ as $\mathfrak{g}_{(0)}$-modules. From (3.1′) it is evident that $\mathfrak{g}_{(1)}$ is an irreducible $\mathfrak{g}_{(0)}$-module with highest weight vector

$$v_{\text{high}} = w_0 + (-1)^l\, w_{0,1} \in \mathfrak{g}_{(1)}$$

and highest weight λ_1, where

$$\langle \lambda_1, H_i \rangle = \delta_{i1},$$

$i=1, \ldots, l$. We set

$$v_{\text{low}} = w_0 - (-1)^l w_{0,1} \in \mathfrak{g}_{(1)},$$
$$\bar{v}_{\text{high}} = \bar{w}_0 + (-1)^l \bar{w}_{0,1} \in \mathfrak{g}_{(3)},$$
$$E_0 = v_{\text{low}}/\sqrt{2\omega}, \qquad F_0 = \bar{v}_{\text{high}}/\sqrt{2\omega}, \qquad H_0 = (-1)^l w_1. \tag{3.4}$$

Now in $\tilde{\mathfrak{g}}(v) = (\sum_{i=0}^{i=3} \mathfrak{g}_{(i)} \otimes t^i \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$, we set

$$\alpha_i^\vee = H_i \otimes 1, \qquad e_i = E_i \otimes 1, \qquad f_i = F_i \otimes 1, \qquad i = 1, \ldots, l, \tag{3.5}$$

and

$$\alpha_0^\vee = H_0 \otimes 1 + c/2, \qquad e_0 = E_0 \otimes t, \qquad f_0 = F_0 \otimes t^{-1}. \tag{3.6}$$

Let $\mathfrak{h} = \operatorname{span}\{\alpha_0^\vee, \ldots, \alpha_l^\vee, d\}$, and define $\alpha_i \in \mathfrak{h}^*$, $i = 0, \ldots, l$ by

$$\langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}, \qquad \langle \alpha_i, d \rangle = \delta_{i0},$$

where $A = (a_{ij})$ is the Cartan matrix for $A^{(4)}(0, 2l)$ or $A^{(2)}(0, 2l-1)$. One checks that the hypotheses of Proposition 1 are satisfied; it follows that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(v)$ is isomorphic to $A^{(4)}(0, 2l)$ or $A^{(2)}(0, 2l-1)$. The element d defines the gradation

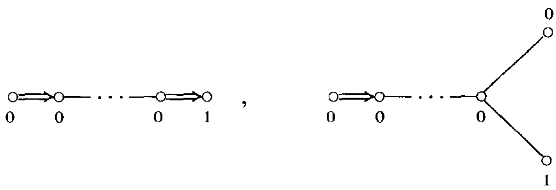
$$\deg e_0 = -\deg f_0 = 1,$$
$$\deg e_i = \deg f_i = 0, \qquad i = 1, \ldots, l.$$

The subalgebra $\mathfrak{a} \subset \mathfrak{g}$ generated by $\{x_{\pm\beta_i} \mid i = 1, \ldots, l\}$ is (of course) isomorphic to A_n; the corresponding subalgebra $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}(v) \subset \tilde{\mathfrak{g}}(v)$ is isomorphic to $A_{2l}^{(2)}$ or $A_{2l-1}^{(2)}$.

4. Vertex Operator Representation of $A^{(4)}(0, 2l)$, $A^{(2)}(0, 2l-1)$

From the equivalence theorem of Section 2 we know that to construct a representation of $\tilde{\mathfrak{g}}$ it is enough to find a space Ω with operators $Z_i(\beta)$, $i \in \mathbb{Z}$, $\beta \in \Phi$, satisfying the relations (2.5). In this section we do this explicitly for $A^{(4)}(0, 2l)$ and $A^{(2)}(0, 2l-1)$. Denote by Λ_i the fundamental weight $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$; the representations constructed are irreducible of highest weight Λ_l in case $n = 2l$, and the direct sum of the irreducible representations of highest weights Λ_{l-1} and Λ_l in case $n = 2l-1$. These representations have been constructed (in different fashion) by Feingold and Frenkel [FF].

As a byproduct we obtain constructions of the irreducible representations of (the subalgebras) $A_{2l}^{(2)}$, $A_{2l-1}^{(2)}$ of highest weights (respectively)



The $\tilde{\mathfrak{k}}'$-vacuum space Ω turns out to be the tensor product of an exterior algebra with a finite-dimensional representation space (which I call M_n) of a certain 2-group; the space M_n is the vacuum space for the representations of the affine subalgebras above, and also provides a construction of the spin representations of $\mathfrak{g}_{(0)}$ ($\approx B_l$ or D_l).

As a first step we define a finite group $\mathscr{P}_n$ with generators $p_0, ..., p_n, -1$ and relations

$$p_i^2 = 1,$$
$$p_i p_{i+1} = -p_{i+1} p_i,$$
$$p_i p_j = p_j p_i \qquad \text{when} \quad |i-j| > 1,$$
$$-1 \text{ central},$$

$i, j = 0, ..., n$. For $\beta \in \Phi$, say $\beta = \pm(\beta_i + \beta_{i+1} + \cdots + \beta_{i+k})$, set $p_\beta = p_i p_{i+1} \cdots p_{i+k}$. With our choice of root vectors for $\mathfrak{g}$ (Section 3) we have the following

LEMMA 4.1. *Suppose $\alpha, \beta, \alpha+\beta \in \Phi$. Then*

$$p_\alpha p_\beta = \varepsilon(\alpha, \beta)\, p_{\alpha+\beta},$$

unless both α and β are odd with α positive, in which case

$$p_\alpha p_\beta = -\varepsilon(\alpha, \beta)\, p_{\alpha+\beta}.$$

Proof. The proof is a straightforward exercise, on noting that

$$[x_{\beta_0}, x_{-(\beta_0 + \cdots + \beta_k)}] = -x_{-(\beta_1 + \cdots + \beta_k)},$$
$$[x_{-\beta_0}, x_{\beta_0 + \cdots + \beta_k}] = x_{\beta_1 + \cdots + \beta_k},$$

whereas for any $i, k > 0$,

$$[x_{\pm\beta_i}, x_{\mp(\beta_i + \cdots + \beta_{i+k})}] = x_{\mp(\beta_{i+1} + \cdots + \beta_{i+k})}. \quad \blacksquare$$

It is easy to identify the representations of $\mathscr{P}_n$. The order of $\mathscr{P}_n$ is 2^{n+2}. If

$n=2l$ then the elements ± 1 and $\pm p_0 p_2 p_4 \cdots p_n$ constitute their own conjugacy classes; every other class has two elements. There are then $2^{n+1}+2$ classes; as $2^{2l+2}=2^{2l}+2^{2l}+1+\cdots+1$ (2^{n+1}-many 1's), there are two irreducible, non-abelian representations of $\mathscr{P}_{2l}$: call them M_{2l} and M'_{2l}. If $n=2l-1$, there are $2^{n+1}+1$ classes; as $2^{2l+1}=2^{2l}+1+\cdots+1$ (2^{n+1}-many 1's), there is a unique irreducible non-abelian representation: call it M_{2l-1}. Note that $\dim M_{2l-1}=\dim M_{2l}=\dim M'_{2l}=2^l$.

Regard $\mathscr{P}_{n-1}\subset\mathscr{P}_n$ as the subgroup generated by $p_1, ..., p_n$. We have two splittings $\pm$: $\mathscr{P}_{2l}\to\mathscr{P}_{2l-1}$ given by

$$\begin{aligned} p_i &\mapsto p_i, \qquad i=1, ..., 2l,\\ p_0 &\mapsto \pm p_2 p_4 \cdots p_{2l}. \end{aligned} \tag{4.1}$$

The two 2^l-dimensional representations of $\mathscr{P}_{2l}$ are given by the composition $\mathscr{P}_{2l}\to\mathscr{P}_{2l-1}\to U(M_{2l-1})$. On the other hand, we have

$$\begin{aligned} M_{2l-1} &= \operatorname{Ind}_{\mathscr{P}_{2l-2}}^{\mathscr{P}_{2l-1}} M_{2l-2}\\ &= (1\otimes M_{2l-2})\oplus(p_0\otimes M_{2l-2})\\ &= M_{2l-2}\oplus M'_{2l-2} \end{aligned} \tag{4.2}$$

as $\mathscr{P}_{2l-2}$-space. This describes the representations M_n inductively.

We leave the group $\mathscr{P}_n$ for the moment and introduce an exterior algebra with operators. Let z_{2r+1}, $r<0$, be indeterminates, and let

$$L=\Lambda\{z_{2r+1}\mid r<0\}$$

be the exterior algebra on the z_i. Define operators Z_i, $i\in\mathbb{Z}$, on L by the conditions

$$\begin{aligned} Z_{2i}&=0, \qquad i\in\mathbb{Z},\\ Z_{2r+1}\cdot z&=z_{2r+1}\wedge z, \qquad z\in L, \qquad r<0, \end{aligned}$$

and for $r\geqslant 0$, $s_1<\cdots<s_j<0$,

$$Z_{2r+1}\cdot z_{2s_1+1}\wedge\cdots\wedge z_{2s_j+1} = \begin{cases} 0, \text{ if } 2r+1\neq-(2s_i+1) \text{ for all } i;\\ 2\omega(2r+1)(-1)^{r+k} z_{2s_1+1}\wedge\cdots\wedge\widehat{z_{2s_k+1}}\wedge\cdots\wedge z_{2s_j+1},\\ \qquad \text{if } 2r+1=-(2s_k+1) \text{ for some } k,\ 1\leqslant k\leqslant j. \end{cases}$$

This gives

$$\{Z_{2r+1}, Z_{2s+1}\}=2\omega(-1)^r(2r+1)\,\delta_{2r+1,-(2s+1)}, \qquad r, s\in\mathbb{Z}, \tag{4.3}$$

where as usual $\{a, b\}$ stands for the anticommutator $ab+ba$. We introduce formal variables ζ, ζ_1, ζ_2 and write $Z(\zeta)=\sum Z_i\zeta^i$. Then (4.3) may be written

$$\{Z(\zeta_1), Z(\zeta_2)\} = 2\omega \sum_{j\in\mathbb{Z}} (-1)^j(2j+1)(\zeta_1/\zeta_2)^{2j+1} \\ = (D\delta)(\omega\zeta_1/\zeta_2) - (D\delta)(-\omega\zeta_1/\zeta_2). \tag{4.4}$$

(Recall $(D\delta)(\zeta)=\sum_{i\in\mathbb{Z}} i\zeta^i$.)

We can now define the vacuum space Ω and its operators $Z_i(\beta)$. Set

$$\Omega = L\otimes M_n.$$

For an even root β set

$$Z_0(\beta) = (\tfrac{1}{4})(1\otimes p_\beta) \\ Z_i(\beta)=0, \qquad i>0. \tag{4.5}$$

For β odd positive set

$$Z_i(\beta) = (\tfrac{1}{4})(Z_i\otimes p_\beta), \qquad i\in\mathbb{Z}, \\ Z(-\beta, \zeta) = -Z(\beta, \omega\zeta), \tag{4.6}$$

where in general $Z(\gamma, \zeta)=\sum Z_i(\gamma)\,\zeta^i$, $\gamma\in\Phi$.

Recall from Section 2 the definitions of S-module and of the category $\mathscr{D}_k$. Note that since the automorphism ν of Section 3 fixes no points of $\mathfrak{t}$, the algebra $\mathfrak{b}$ is just $\mathbb{C}c\oplus\mathbb{C}d$. We give Ω a $\mathfrak{b}$-module structure by setting

$$\sigma(c)=1$$

$$\sigma(d)\cdot z_{i_1}\wedge\cdots\wedge z_{i_m} = \left(-\sum_{j=1}^{m} i_j\right)(z_{i_1}\wedge\cdots\wedge z_{i_m}).$$

LEMMA 4.2. *(Ω, Z) is in $\mathscr{D}_1$.*

Proof. We must verify relations (2.5) with $k=1$, $m=4$. Relations (i)–(iv) are immediate, (v) is empty since $\mathfrak{t}_{(0)}=(0)$, and (vi) is easily checked case by case, using (4.9) below. The generalized (anti-)commutation relations have the form ($\alpha, \beta\in\Phi$):

$$\prod_{p\in\mathbb{Z}/4\mathbb{Z}} (1-\omega^{-p}\zeta_1/\zeta_2)^{(\nu^p\alpha,\beta)/4}\, Z(\alpha,\zeta_1)\, Z(\beta,\zeta_2) \\ -(-1)^{d_\alpha d_\beta} \prod_{p\in\mathbb{Z}/4\mathbb{Z}} (1-\omega^{-p}\zeta_2/\zeta_1)^{(\nu^p\beta,\alpha)/4}\, Z(\beta,\zeta_2)\, Z(\alpha,\zeta_1) \\ = (\tfrac{1}{4})\sum{}'\, \eta(p,\alpha)\,\varepsilon(\nu^p\alpha,\beta)\, Z(\nu^p\alpha+\beta,\zeta_2)\,\delta(\omega^{-p}\zeta_1/\zeta_2) \\ + (\tfrac{1}{4})(x_{-\beta}, x_\beta)\sum{}''\, \eta(p,\alpha)(D\delta)(\omega\zeta_1/\zeta_2), \tag{4.7}$$

where $\sum'$ is over $p \in \mathbb{Z}/4\mathbb{Z}$ such that $\nu^p\alpha + \beta \in \Phi$, $\sum''$ is over $p \in \mathbb{Z}/4\mathbb{Z}$ such that $\nu^p\alpha + \beta = 0$, and $d_\gamma = \deg_2(x_\gamma)$ for $\gamma = \alpha, \beta$. Denote by $\imath(\alpha, \beta)$ the right hand side of (4.7), and by $l(\alpha, \beta)$ the left hand side. Set

$$f(\zeta) = (1 - \zeta^2)(1 + \zeta^2)^{-1};$$

since $\nu(\beta) = \nu^3(\beta) = -\beta$ and $\nu^2(\beta) = \beta$ for all $\beta \in \Phi$, we have

$$l(\alpha, \beta) = f(\zeta_1/\zeta_2)^{(\alpha,\beta)/4} Z(\alpha, \zeta_1) Z(\beta, \zeta_2) - (-1)^{d_\alpha d_\beta} f(\zeta_2/\zeta_1)^{(\alpha,\beta)/4} Z(\beta, \zeta_2) Z(\alpha, \zeta_1). \tag{4.8}$$

We need the following data:

$\eta(p, \beta)$: (4.9)

β \ p	0	1	2	3
odd positive	1	−1	−1	1
odd negative	1	1	−1	−1
even	1	1	1	1

$$(x_{-\alpha}, x_\alpha) = (-1)^{h(\alpha)+1}/4, \tag{4.10}$$

for all positive roots α of $\mathfrak{g}$, where $h(\alpha)$ is the height of α.

$$p_\alpha^2 = (-1)^{h(\alpha)+1}, \tag{4.11}$$

for all $\alpha \in \Phi$. Recall from Section 3 our choice of form on $\mathfrak{g}$: $(a, b) = -(\frac{1}{4})\,\mathrm{str}(ab)$, $a, b \in \mathfrak{g}$. Relative to this form the positive simple roots of $\mathfrak{g}$ are $\beta_0 = -4h_0$, $\beta_i = 4h_i$, $i = 1, ..., n$, where $h_0, ..., h_n$ is the basis of $\mathfrak{t}$ given in Section 3. Hence if one of α, β is even, we have

$$(\alpha, \beta) = \begin{cases} 4 & \text{if } \alpha - \beta \in \Phi \\ -4 & \text{if } \alpha + \beta \in \Phi \\ 8 & \text{if } \alpha = \beta \\ -8 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

whereas if α and β are both odd we have

$$(\alpha, \beta) = \begin{cases} 4 & \text{if } \alpha + \beta \in \Phi \\ -4 & \text{if } \alpha - \beta \in \Phi \\ 0 & \text{if } \alpha = \pm\beta. \end{cases} \tag{4.13}$$

The following identities are also helpful:

$$
\begin{aligned}
f(\zeta_1/\zeta_2)^{-1}+f(\zeta_2/\zeta_1)^{-1}&=\delta(\zeta_1/\zeta_2)+\delta(-\zeta_1/\zeta_2)\\
f(\zeta_1/\zeta_2)+f(\zeta_2/\zeta_1)&=\delta(\omega\zeta_1/\zeta_2)+\delta(-\omega\zeta_1/\zeta_2)\\
f(\zeta_1/\zeta_2)^{-2}-f(\zeta_2/\zeta_1)^{-2}&=(D\delta)(\zeta_1/\zeta_2)+(D\delta)(-\zeta_1/\zeta_2)\\
f(\zeta_1/\zeta_2)^{2}-f(\zeta_2/\zeta_1)^{2}&=(D\delta)(\omega\zeta_1/\zeta_2)+(D\delta)(-\omega\zeta_1/\zeta_2).
\end{aligned}
\tag{4.14}
$$

We also use

LEMMA 4.3. *Let V be a vector space over $\mathbb{C}$, and let $f(\zeta_1/\zeta_2)=\sum v_{ij}\zeta_1^i\zeta_2^j\in V\{\zeta_1,\zeta_2\}$ be such that for some $n\in\mathbb{Z}$, either $v_{ij}=0$ whenever i or $j>n$, or $v_{ij}=0$ whenever i or $j>n$. Let $a\in\mathbb{C}$, $a\neq 0$. Then*

$$
\begin{aligned}
\delta(a\zeta_1/\zeta_2)\,f(\zeta_1,\zeta_2)&=\delta(a\zeta_1/\zeta_2)\,f(\zeta_1,a\zeta_1)\\
&=\delta(a\zeta_1/\zeta_2)\,f(a^{-1}\zeta_2,\zeta_2).
\end{aligned}
$$

(Lemma 4.3 appears in [LW] as Proposition 3.9.)

We introduce some notation: set

$$
Y(\beta,\zeta)=\begin{cases}1, & \beta\text{ even}\\ Z(\zeta), & \beta\text{ odd positive}\\ -Z(\omega\zeta), & \beta\text{ odd negative.}\end{cases}
$$

Then we may write uniformly for $\beta\in\Phi$:

$$Z(\beta,\zeta)=(\tfrac{1}{4})\,Y(\beta,\zeta)\otimes p_\beta.$$

Now we check the relations (4.7) case by case, starting with

1. $\alpha=\pm\beta$, α even.

From (4.5), (4.7)–(4.12) we have

$$
\begin{aligned}
l(\alpha,\alpha)&=f(\zeta_1/\zeta_2)^2\,Z(\alpha,\zeta_1)\,Z(\alpha,\zeta_2)-f(\zeta_2/\zeta_1)^2\,Z(\alpha,\zeta_2)\,Z(\alpha,\zeta_1)\\
&=(\tfrac{1}{16})(f(\zeta_1/\zeta_2)^2-f(\zeta_2/\zeta_1)^2)\,1\otimes p_\alpha^2\\
&=(-1)^{h(\alpha)+1}(\tfrac{1}{16})(f(\zeta_1/\zeta_2)^2-f(\zeta_2/\zeta_1)^2)\\
&=(-1)^{h(\alpha)+1}(\tfrac{1}{16})((D\delta)(-\omega\zeta_1/\zeta_2)+(D\delta)(\omega\zeta_1/\zeta_2))
\end{aligned}
$$

(by (4.14))

$$=\imath(\alpha,\alpha).$$

The case $\alpha=-\beta$, α even, is similar.

2. $\alpha \pm \beta \in \Phi$, α even.

Assume $\alpha + \beta \in \Phi$. We have

$$l(\alpha, \beta) = f(\zeta_1/\zeta_2)^{-1}(\tfrac{1}{4})(1 \otimes p_\alpha)(\tfrac{1}{4})(Y(\beta, \zeta_2) \otimes p_\beta)$$
$$- f(\zeta_2/\zeta_1)^{-1}(\tfrac{1}{4})(Y(\beta, \zeta_2) \otimes p_\beta)(\tfrac{1}{4})(1 \otimes p_\alpha)$$
$$= (\tfrac{1}{4})(f(\zeta_1/\zeta_2)^{-1} + f(\zeta_2/\zeta_1)^{-1})\, \varepsilon(\alpha, \beta)(\tfrac{1}{4})\, Y(\beta, \zeta_2) \otimes p_{\alpha+\beta}$$

(since $\varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha)$ when one of α, β is even, and by Lemma 4.1)

$$= (\tfrac{1}{4})(\delta(\zeta_1/\zeta_2) + \delta(-\zeta_1/\zeta_2))\, \varepsilon(\alpha, \beta)\, Z(\alpha + \beta, \zeta_2)$$

(by (4.14))

$$= \imath(\alpha, \beta).$$

The case $\alpha - \beta \in \Phi$, α even, is similar.

3. $\alpha \pm \beta \in \Phi$, α odd, β even.

The case $\alpha - \beta \in \Phi$ is a bit trickier, so we'll do that. Then much as in the last case, we find

$$l(\alpha, \beta) = (\tfrac{1}{4})(\delta(\omega\zeta_1/\zeta_2) + \delta(-\omega_1/\zeta_2))\, \varepsilon(-\alpha, \beta)(\tfrac{1}{4})\, Y(\alpha, \zeta_1) \otimes p_{-\alpha+\beta}.$$

Now if α is positive, Lemma 4.3 gives

$$l(\alpha, \beta) = (\tfrac{1}{4})(\delta(\omega\zeta_1/\zeta_2) - \delta(-\omega\zeta_1/\zeta_2))\, \varepsilon(-\alpha, \beta)(\tfrac{1}{4})\, Y(-\alpha, \zeta_2) \otimes p_{\beta-\alpha},$$

since $Y(\alpha, \zeta)$ is an odd function when α is odd; similarly if α is negative, Lemma 4.3 gives

$$l(\alpha, \beta) = (\tfrac{1}{4})(\delta(-\omega\zeta_1/\zeta_2) - \delta(\omega\zeta_1/\zeta_2))\, \varepsilon(-\alpha, \beta)(\tfrac{1}{4})\, Y(-\alpha, \zeta_2) \otimes p_{\beta-\alpha}.$$

In either case, (4.9) gives

$$l(\alpha, \beta) = (\eta(1, \alpha)\, \delta(\omega^{-1}\zeta_1/\zeta_2) + \eta(3, \alpha)\, \delta(\omega^{-3}\zeta_1/\zeta_2))\, \varepsilon(-\alpha, \beta)\, Z(\beta - \alpha, \zeta_2)$$
$$= \imath(\alpha, \beta).$$

There remain the cases in which both α and β are odd roots. When $\alpha \neq \pm\beta$ these are a bit more tedious than the foregoing. First we check the case

4. $\alpha = \beta$, α odd positive.

Since $(\alpha, \alpha) = 0$ for odd α, the "correction terms" are trivial and we are left with an ordinary anticommutation relation:

$$l(\alpha, \alpha) = Z(\alpha; \zeta_1)\, Z(\alpha, \zeta_2) + Z(\alpha, \zeta_2)\, Z(\alpha, \zeta_1)$$
$$= (\tfrac{1}{16})(Z(\zeta_1)\, Z(\zeta_2) + Z(\zeta_2)\, Z(\zeta_1)) \otimes p_\alpha^2$$
$$= (-1)^{h(\alpha)+1}(\tfrac{1}{16})((D\delta)(\omega\zeta_1/\zeta_2) - (D\delta)(-\omega\zeta_1/\zeta_2))$$

(by (4.4) and (4.11))

$$= \imath(\alpha, \alpha).$$

The remaining cases of the form $\alpha = \pm\beta$, α odd, are similar. The last case that we check is

5. $\alpha - \beta \in \Phi$, α odd positive, β odd:

We must have β positive as well, so $Y(\alpha, \zeta) = Y(\beta, \zeta) = Z(\zeta)$. Also $(\alpha, \beta) = -4$, hence

$$l(\alpha, \beta) = f(\zeta_1/\zeta_2)^{-1}(\tfrac{1}{16})(Z(\zeta_1) \otimes p_\alpha)(Z(\zeta_2) \otimes p_\beta) + f(\zeta_2/\zeta_1)^{-1}(\tfrac{1}{16})(Z(\zeta_2) \otimes p_\beta)(Z(\zeta_1) \otimes p_\alpha).$$

Since $-\alpha$ is negative, $p_\alpha p_\beta = \varepsilon(-\alpha, \beta)\, p_{-\alpha+\beta} = -\varepsilon(-\beta, \alpha)\, p_{-\alpha+\beta}$ by Lemma 4.1, hence $l(\alpha, \beta)$ equals

$$(\tfrac{1}{16})\, \varepsilon(-\alpha, \beta)(f(\zeta_1/\zeta_2)^{-1}\, Z(\zeta_1)\, Z(\zeta_2) - f(\zeta_2/\zeta_1)^{-1}\, Z(\zeta_2)\, Z(\zeta_1)) \otimes p_{\beta-\alpha}.$$

On the other hand,

$$\imath(\alpha, \beta) = (\tfrac{1}{16})\, \varepsilon(-\alpha, \beta)(\delta(\omega\zeta_1/\zeta_2) - \delta(-\omega\zeta_1/\zeta_2))(1 \otimes p_{\beta-\alpha}).$$

Hence we must show

$$f(\zeta_1/\zeta_2)^{-1}\, Z(\zeta_1)\, Z(\zeta_2) - f(\zeta_2/\zeta_1)^{-1}\, Z(\zeta_2)\, Z(\zeta_1) = 2\omega \sum_{j \in \mathbb{Z}} (-1)^j (\zeta_1/\zeta_2)^{2j+1}.$$

This holds if and only if

$$Z_{2r+1} Z_{2s+1} - Z_{2s+1} Z_{2r+1} + \sum_{k>0} Z_{2(r-k)+1} Z_{2(s+k)+1} - 2 \sum_{k>0} Z_{2(s-k)+1} Z_{2(r+k)+1} = 2\omega(-1)^r\, \delta_{2r+1,-(2s+1)}. \tag{4.16}$$

Suppose $r \geqslant s$; put $m = r - s$. Extract the first m terms of the first sum in (4.16): the remaining terms cancel those of the second sum. If $2r + 1 \neq -(2s+1)$, then the left hand side of (4.16) is a finite sum of anticommutators which are all zero, by (4.3). If $2r + 1 = -(2s+1)$, then $2r = m - 1 \geqslant 0$, and the left hand side of (4.16) equals

$$\{Z_{2r+1}, Z_{2s+1}\} + 2\sum_{j=1}^{j=r} \{Z_{2(r-j)+1}, Z_{2(j-r)-1}\}$$
$$= 2\omega(-1)^r \left(2r+1+2\sum(-1)^j(2(r-j)+1)\right)$$
$$= 2\omega(-1)^r,$$

by (4.3), as desired. The case $r \leqslant s$ is similar.

The remaining cases are similar to this last one, and we omit their verification here. This completes the proof of Lemma 4.2. ∎

From Lemma 4.2 and the equivalence theorem we obtain a $\mathfrak{g}$-module (V, π) in $\mathscr{C}_1$, with underlying space

$$V = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega.$$
$$= \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes (L \otimes M_n).$$

The space Ω is precisely the vacuum space Ω_V for the action of the Heisenberg algebra $\tilde{\mathfrak{t}}'$ on V. We remark that for n even, the $\mathscr{P}_n$-space M_n may be replaced by M'_n, yielding a $\tilde{\mathfrak{g}}$-module isomorphic to V. The action of $\tilde{\mathfrak{t}}$ on V is given by

$$\pi(c) = 1, \qquad \pi(d) = d \otimes 1 + 1 \otimes d,$$
$$\pi(h) = h \otimes 1, \qquad h \in \tilde{\mathfrak{t}}_- \otimes \tilde{\mathfrak{t}}_+,$$

while for each $\beta \in \Phi$ and $i \in \mathbb{Z}$, the element $(x_\beta)_{(0)} \otimes t^i$ acts as the ith coefficient of

$$E^-(-\beta, \zeta)\, E^+(-\beta, \zeta) \otimes Z(\beta, \zeta). \tag{4.17}$$

Recall from Section 3 that $\tilde{\mathfrak{g}}$ contains a subalgebra $\tilde{\mathfrak{a}}$ isomorphic to $A_n^{(2)}$; we study V as $\tilde{\mathfrak{a}}$-module. Let $\mathfrak{t}_1$ be the subalgebra of $\mathfrak{t}$ generated by $h' = \sum_{i=0}^{i=n}(-1)^i(n+1-i)\, h_i$, and let $\mathfrak{t}_2$ be the Cartan subalgebra of $\mathfrak{a}$ generated by $h_1, ..., h_n$. Denote by Φ_0 the set of roots of $\mathfrak{a}$ (equivalently, the set of even roots of $\mathfrak{g}$). Since $(h_i, h') = 0$ for $i = 1, ..., n$, the vacuum space of $\mathscr{S}(\tilde{\mathfrak{t}}_-)$ as $\tilde{\mathfrak{t}}'_2$-module is $\mathscr{S}(\tilde{\mathfrak{t}}_{1-})$, hence $\mathscr{S}(\tilde{\mathfrak{t}}_-) = \mathscr{S}(\tilde{\mathfrak{t}}_{1-}) \otimes \mathscr{S}(\tilde{\mathfrak{t}}_{2-})$ as $\tilde{\mathfrak{t}}'_2$-module. Similarly, the vertex operators $E^\pm(\beta, \zeta)$, $\beta \in \Phi_0$, act (with respect to this decomposition) on $\mathscr{S}(\tilde{\mathfrak{t}}_-)$ as $1 \otimes E^\pm(\pm\beta, \zeta)$. The algebra $\tilde{\mathfrak{a}}$ is spanned by $\tilde{\mathfrak{t}}_2$ and the coefficients of $X(\beta, \zeta)$, $\beta \in \Phi_0$. The latter act on

$$V = \mathscr{S}(\tilde{\mathfrak{t}}_{1-}) \otimes \mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes L \otimes M_n$$

as the coefficients of

$$1 \otimes E^-(\beta, \zeta)\, E^+(\beta, \zeta) \otimes 1 \otimes (\tfrac{1}{4})\, p_\beta. \tag{4.17'}$$

Consider then the $\tilde{\mathfrak{a}}$-module

$$W = \mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes M_n.$$

The $\tilde{\mathfrak{t}}'_2$-vacuum space of W is simply M_n; the Z-algebra associated with this $\tilde{\mathfrak{a}}$-module is just the group algebra on $\mathscr{P}_{n-1} = \langle p_1, ..., p_n \rangle$, which acts on M_n by restriction of the action of $\mathscr{P}_n = \langle p_0, ..., p_n \rangle$.

Recall that M_n remains irreducible under $\mathscr{P}_{n-1}$ via (4.1) when n is even, and that M_n decomposes into the sum $M_{n-1} \oplus M'_{n-1}$ when n is odd (4.2). By the equivalence theorem, W is irreducible under $\tilde{\mathfrak{a}}$ when n is even and decomposes into two irreducible $\tilde{\mathfrak{a}}$-modules when n is odd:

$$W = W_0 \oplus W_1 = \mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes (M_{n-1} \oplus M'_{n-1}).$$

By a theorem of Kac (see [K3], Prop. 9.3), we know that W (n even), W_0, W_1 (n odd) are highest weight modules; to identify these highest weights we consider the representation of $\mathfrak{a}_{(0)} = \mathfrak{g}_{(0)}$ afforded by the space M_n. Regard $\mathfrak{a}_{(0)} \subset \tilde{\mathfrak{a}}$ via the map $a \mapsto a \otimes 1$, $a \in \mathfrak{a}_{(0)}$. The algebras $\mathfrak{a}_{(0)}$ and $\tilde{\mathfrak{a}}$ share the Chevalley generators e_i, f_i, $i = 1, ..., n$; the remaining Chevalley generators of $\tilde{\mathfrak{a}}$ are

$$e'_0 = [e_0, e_0]/2, \qquad f'_0 = -[f_0, f_0]/2.$$

Now e_i, $i = 1, ..., n$, acts on $1 \otimes M_n$ as follows. Recall

$$e_i = E_i \otimes 1 = y_{(-1)^{l-i}\delta_i},$$

where the $y_{\pm\delta_i}$ are given by (3.2) and (3.2′). For $\beta \in \Phi_0$ we have $w_\beta = 2(x_\beta)_{(0)}$; the element $(x_\beta)_{(0)}$, in turn, acts on $1 \otimes M_n$ as the constant coefficient of $E^-(-\beta, \zeta)\, E^+(-\beta, \zeta) \otimes (\frac{1}{4})\, p_\beta$. To this action the series $E^+(-\beta, \zeta)$ contributes only the identity, whence $(x_\beta)_{(0)}$ acts on $1 \otimes M_n$ as $1 \otimes (\frac{1}{4})\, p_\beta$. Following (3.2), (3.2′), define operators $Y_{\pm\delta_i}$ on M_n by

$$Y_{\delta_i} = (-p_{2i} - p_{2i-1,2i} + p_{2i,2i+1} + p_{2i-1,2i+1})/4$$

$$Y_{-\delta_i} = (p_{2i} - p_{2i-1,2i} + p_{2i,2i+1} - p_{2i-1,2i+1})/4,$$

$i = 1, ..., l-1,$

$$Y_{\delta_l} = (p_{2l-1,2l} + p_{2l})/2$$

$$Y_{-\delta_l} = (p_{2l} - p_{2l-1,2l})/2,$$

if $n = 2l$, and

$$Y_{\delta_l} = (-p_{2l-3,2l-2} + p_{2l-3,2l-1} + p_{2l-2} - p_{2l-2,2l-1})/4,$$

$$Y_{-\delta_l} = (p_{2l-3,2l-2} + p_{2l-3,2l-1} + p_{2l-2} + p_{2l-2,2l-1})/4,$$

if $n=2l-1$. Then the association

$$\begin{aligned} \rho(E_i) &= Y_{(-1)^{l-i}\delta_i}, \\ \rho(F_i) &= Y_{-(-1)^{l-i}\delta_i}, \qquad i=1, ..., l, \end{aligned} \tag{4.18}$$

defines a representation (M_n, ρ) of $\mathfrak{a}_{(0)}$. Note that

$$\rho(w_\beta) = (\tfrac{1}{2})\, p_\beta, \qquad \beta \in \Phi_0. \tag{4.19}$$

We claim this representation is irreducible when n is even, and the sum of two irreducible representations when n is odd. For simplicity consider the case $n=2l$. If ρ is not irreducible, then it is the sum of at least two highest weight modules: let u_1 and u_2 be two independent highest weight vectors. In the $\tilde{\mathfrak{a}}$-module W the vectors $1 \otimes u_i$, $i=1, 2$, are eigenvectors for $\alpha_1^\vee, ..., \alpha_l^\vee$, hence also for $\alpha_0^\vee$, since $\alpha_0^\vee, ..., \alpha_l^\vee$, c are linearly dependent. Moreover $1 \otimes u_1$ and $1 \otimes u_2$ are killed by e_0', which has positive operator degree. It follows that $1 \otimes u_1$ and $1 \otimes u_2$ are independent highest weight vectors in W, which contradicts the fact that W is irreducible (and so a highest weight module).

Recall that $\dim M_{2l} = 2^l$, $\dim M_{2l-1} = \dim M_{2l-1} + \dim M'_{2l-1} = 2^{l-1} + 2^{l-1}$. The only irreducible representations of B_l and D_l of dimensions 2^l and 2^{l-1}, respectively, are the spin (resp., half-spin) representations, i.e., those of highest weight λ_l (resp., λ_{l-1} or λ_l), where $\langle \lambda_i, H_j \rangle = \delta_{ij}$, $i, j = 1, ..., l$.

Denote by Λ_{l-1} and Λ_l the fundamental weights of $\tilde{\mathfrak{a}}$ (and by abuse of notation, of $\tilde{\mathfrak{g}}$ as well):

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \qquad i = l-1, l, \qquad j = 0, ..., l;$$

let $W(\Lambda_i)$ denote the corresponding irreducible representations of $\tilde{\mathfrak{a}}$. At this point we may conclude that ρ is irreducible of highest weight λ_l when n is even, and the sum of two irreducible representations of highest weights λ_{l-1} or λ_l, n odd. We find the $\tilde{\mathfrak{a}}$-module W in a similar situation. From the argument showing that ρ is irreducible, we have that if u is a highest weight vector in (M_n, ρ) then $1 \otimes u$ is a highest weight vector in W. From (3.3)–(3.6) we have

$$\begin{aligned} \alpha_0^\vee &= \frac{c}{2} - (\alpha_1^\vee + \cdots + \alpha_{l-1}^\vee + \alpha_l^\vee/2), && n \text{ even}, \\ \alpha_0^\vee &= \frac{c}{2} - (\alpha_1^\vee + \cdots + \alpha_{l-2}^\vee + \alpha_{l-1}^\vee/2 + \alpha_l^\vee/2), && n \text{ odd}. \end{aligned} \tag{4.20}$$

It follows that $\alpha_0^\vee \cdot 1 \otimes u = 0$, regardless of whether λ_{l-1} or λ_l occurs in

M_{2l-1}. Hence $W \approx W(\Lambda_l)$ if n is even, and $W \approx W(\Lambda_i) \oplus W(\Lambda_j)$, $i, j = l-1$ or l, if n is odd.

We now show that when $n = 2l-1$, both λ_{l-1} and λ_l occur as highest weights in M_n. From this it follows that M_n is the spin representation of D_l, and that

$$W \approx W(\Lambda_{l-1}) \oplus W(\Lambda_l).$$

From (3.3′) and (4.19), we have

$$\begin{aligned} \rho(H_i) &= (-1)^{l-i}(p_{2i-1} + p_{2i+1})/2, \qquad i = 1, ..., l-1, \\ \rho(H_l) &= (p_{2l-1} - p_{2l-3})/2. \end{aligned} \tag{4.21}$$

Also, note that (3.4), (3.6), and (4.19) give

$$\alpha_0^\vee |_{M_n} = ((-1)^l p_1 + 1)/2. \tag{4.22}$$

As $\mathscr{P}_{n-1}$-module, $M_n = M_{n-1} \oplus p_0 M_{n-1}$. Without loss of generality, we may assume that M_{n-1} has highest weight λ_l: let u be a highest weight vector, and let $u' = p_0 u$. Then from (4.21) and (4.22) we have

$$\begin{aligned} p_1 u &= (-1)^{l-1} u, \qquad & p_1 u' &= (-1)^l u', \\ p_{2i-1} u &= (-1)^{l-i} u, \qquad & p^{2i-1} u' &= (-1)^{l-i} u', \end{aligned}$$

$i = 2, ..., l$. Then from (4.21) again, it follows that u' has weight $\lambda_l - \lambda_1$. Now if M'_{n-1} $(= p_0 M_{n-1})$ were also of highest weight λ_l, then λ_1 would be a sum of roots of $\mathfrak{a}_{(0)}$, which is false. Hence M'_{n-1} must be of highest weight λ_{l-1}.

We are now able to identify the $\tilde{\mathfrak{g}}$-module V. First, write $L = L_0 \oplus L_1$ for the decomposition of the exterior algebra into elements of total even and odd degree, respectively. Denote by $\mathscr{Z}$ the subalgebra of $\mathrm{End}(\Omega)$ generated by the $Z_i(\beta)$, $\beta \in \Phi$, $i \in \mathbb{Z}$. It is immediate from (4.5) and (4.6) that if n is even, the $\mathscr{Z}$-module Ω is irreducible, and that if n is odd,

$$\Omega = \Omega_0 \oplus \Omega_1,$$

where

$$\begin{aligned} \Omega_0 &= L_0 \otimes M_{n-1} + L_1 \otimes M'_{n-1}, \\ \Omega_1 &= L_0 \otimes M'_{n-1} + L_1 \otimes M_{n-1} \end{aligned} \tag{4.23}$$

are irreducible $\mathscr{Z}$-modules. From the equivalence theorem we have that

$$\begin{aligned} V &= \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega, \qquad & n \text{ even}, \\ V &= (\mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega_0) \oplus (\mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega_1), \qquad & n \text{ odd} \end{aligned}$$

are decompositions of V into irreducible $\tilde{\mathfrak{g}}$-modules. Denote by $V(\Lambda_i)$ the irreducible $\tilde{\mathfrak{g}}$-modules of highest weight Λ_i, $i=l-1, l$. Let n be even, $u \in M_n$ a highest weight vector for $\mathfrak{g}_{(0)}$. Put $v = 1 \otimes 1 \otimes u \in V = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes L \otimes M_n$. Clearly v generates V as $\tilde{\mathfrak{g}}$-module; we have $e_0 \cdot v = 0$ since e_0 has positive operator degree, and $e_i \cdot v = 0$, $i = 1, ..., l$, since u is a highest weight vector for $\mathfrak{g}_{(0)}$. From (3.5), (3.6), and (4.20) it follows that v has highest Λ_l, whence

$$V \approx V(\Lambda_l).$$

Similarly if n is odd, we have

$$V \approx V(\Lambda_{l-1}) \oplus V(\Lambda_l).$$

We summarize the results of this section as follows:

THEOREM 3. *Let M_n (and M'_n, if n is even) be the representations of the 2-group $\mathscr{P}_n$ given by* (4.1) *and* (4.2). *Write $n = 2l$ or $2l-1$.*

(i) *The representation (M_n, ρ) of $\mathfrak{g}_{(0)} = B_l$ or D_l, given by* (4.18), *is isomorphic to the spin representation.*

(ii) *For $n = 2l$, the $\tilde{\mathfrak{a}}$ $(=A_{2l}^{(2)})$-module*

$$W = \mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes M_n$$

is irreducible, of highest weight Λ_l. For $n = 2l-1$, the $\tilde{\mathfrak{a}}$ $(=A_{2l-1}^{(2)})$-module W is the sum of two irreducible modules, one of highest weight Λ_{l-1} and one of highest weight Λ_l:

$$W = (\mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes M_{n-1}) \oplus (\mathscr{S}(\tilde{\mathfrak{t}}_{2-}) \otimes M'_{n-1}).$$

The action of $\tilde{\mathfrak{a}}$ on W is given by (4.5) *and* (4.17′).

(iii) *For $n = 2l$, the $\tilde{\mathfrak{g}}$ $(=A^{(4)}(0, 2l))$-module*

$$V = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega = \mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes L \otimes M_n$$

is irreducible, of highest weight Λ_l. For $n = 2l-1$, the $\tilde{\mathfrak{g}}$ $(=A^{(2)}(0, 2l-1))$-module V is the sum of two irreducible modules, one of highest weight Λ_{l-1}, the other of highest weight Λ_l:

$$V = (\mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega_0) \oplus (\mathscr{S}(\tilde{\mathfrak{t}}_-) \otimes \Omega_1),$$

where Ω_0 and Ω_1 are given by (4.23). *The action of $\tilde{\mathfrak{g}}$ on V is given by* (4.5), (4.6), *and* (4.17).

Theorem 3(ii) is subsumed by Corollary 3.2 of [FLM].

REFERENCES

[FF] A. FEINGOLD AND I. B. FRENKEL, Classical affine algebras, *Adv. in Math.* **56** (1985), 117–172.

[FLM] I. B. FRENKEL, J. LEPOWSKY, AND A. MEURMAN, An E_8-approach to F_1, *Contemp. Math.* **45** (1985), 99–120.

[G] G. GOLITZIN, "Representations of Affine Superalgebras," Thesis, Yale University, 1985.

[K1] V. G. KAC, Lie superalgebras, *Adv. in Math.* **26** (1977), 8–96.

[K2] V. G. KAC, Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function, and the very strange formula, *Adv. in Math.* **30** (1978), 85–136.

[K3] V. G. KAC, "Infinite-Dimensional Lie Algebras," Birkhäuser Boston, Cambridge, MA, 1983.

[LW] J. LEPOWSKY AND R. L. WILSON, The structure of standard modules I: Universal algebras and the Rogers–Ramanujan identities, *Invent. Math.* **77** (1984), 199–290.