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OPTIMAL TIME-FREE NODAL TRANSFERS BETWEEN ELLIPTICAL ORBITS†

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Abstract—This paper presents the necessary conditions for the minimum fuel, time-free transfer between two non-coplanar elliptical orbits. It is shown that the solution is obtained by solving a system of three non-linear equations for three unknowns. In the second part, we discuss the case where the impulses are applied along the line of nodes. In general, this nodal transfer is non optimal, but the characteristic velocity for the best nodal transfer, called the minimizing nodal transfer, is reasonably close to the one for the optimal transfer for it to be useful as a substitute for a practical transfer. Furthermore, when we continuously vary the relative position of the two terminal orbits, the two characteristic velocities, for the minimizing nodal transfer and the optimal transfer, exhibit the same trend in the sense that they pass through their maxima and minima at nearly the same relative position. This makes the set of explicit formulas for computing the minimizing nodal transfer, as presented in this paper, a useful tool for designing a minimum fuel transfer between several orbits.

1. INTRODUCTION

With the advent of the orbital transfer vehicle, there is a current research interest in the analysis of combined aerodynamic and propulsive maneuver as a technique for reduction of the fuel consumption in orbital transfer as compared to the pure propulsive maneuver. In the published literature, these comparisons are usually made in simple cases such as the transfer between non-coplanar circular orbits. This is because the computation of the optimal propulsive transfer between non-coplanar elliptical orbits in itself is a difficult problem.

In this paper we shall first present the necessary conditions for the minimum fuel, time-free, two-impulse transfer between two arbitrary ellipitical orbits. It is shown that basically the solution is obtained by solving a system of three non-linear equations for three unknowns. In the second part, we restrict the discussion of the optimal solution to the case of nodal transfers, that is the transfers where all the impulses are applied along the line of nodes. In general, they are non-optimal. They are possibly optimal only in the one-impulse or the two-impulse cases, except when the terminal orbits are coaxial the three-impulse optimal transfer also exists. When the terminal orbits are non-coaxial, it is easy to compute the one-impulse transfer when the orbits are intersecting, and also the best two-impulse transfer in the general case of non intersecting orbits. Then, this transfer, called the minimizing nodal transfer, is truly optimal only when some inequalities are verified, and also when one additional equality is satisfied in the two-impulse case.

In practical applications, the cost for the minimizing nodal transfer is close to the optimal solution. Hence, the best nodal transfer can be used as a practical transfer between non-coplanar elliptical orbits. It also serves as a reasonable upper bound for the fuel consumption in the search of the true optimal transfer.

2. NECESSARY CONDITIONS

Consider a two-impulse, time-free transfer between an initial orbit O_1 and a final orbit O_2. The two orbits are connected by a transfer orbit O through the applications of the impulses I_1 and I_2 (Fig. 1). Along the transfer orbit with center of attraction at point F, let MSTW be a rotating coordinate system with the origin M at the vehicle, the S-axis along the position vector positive outwards, the T-axis along the circumferential direction in the plane of O, positive toward the direction of motion and the W-axis completing a right-handed system. Then, if S, T, and W are the direction cosines of the optimal impulses, we have the identities

$$S_i^2 + T_i^2 + W_i^2 = 1, \quad i = 1, 2. \tag{1}$$

It is convenient to define the optimal direction of the impulse by its azimuth ψ and elevation ϕ. Then at the impulses

$$S_i = \sin\phi_i,\ T_i = \cos\phi_i \cos\psi_i,\ W_i = \cos\phi_i \sin\psi_i \tag{2}$$

$$i = 1, 2$$

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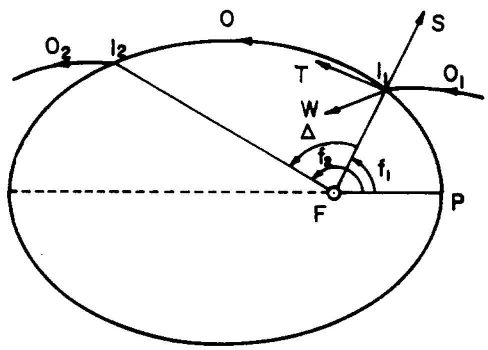

Fig. 1. Rotating coordinates and transfer geometry.

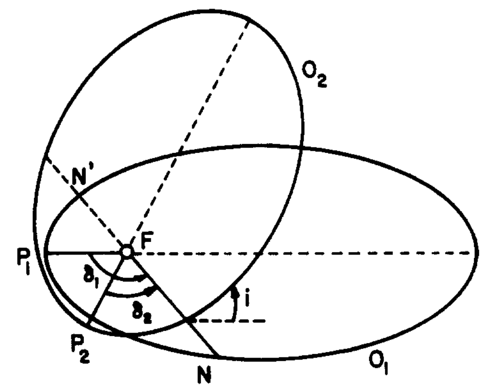

Fig. 2. Geometry of the terminal orbits.

Let f_i be the true anomalies of the points of application of the impulses, measured on the transfer orbit with eccentricity e. The transfer angle is

$$\Delta = f_2 - f_1, \tag{3}$$

From Lawden's theory of primer vector, it can be shown that for the transfer to be optimal, we have three necessary conditions, called the optimal switching conditions, relating the elements e, f_i, S_i, T_i and W_i[1–3].

$$(1+q_1)J_1 - q_1 T_1 = (1+q_2)J_2 - q_2 T_2 \tag{4}$$

$$(y_1 T_1 - q_1 S_1)(T_1 - J_1) - S_1 T_1 + y_1 W_1^2 + K_1 W_1 = 0 \tag{5}$$

$$(y_2 T_2 - q_2 S_2)(T_2 - J_2) - S_2 T_2 + y_2 W_2^2 + K_2 W_2 = 0. \tag{6}$$

In the following, we shall use the notation

$$x = e\cos f, \quad y = e\sin f, \quad q = 1 + e\cos f,$$
$$J_1 = (S_2 - S_1\cos\Delta)/\sin\Delta,$$
$$J_2 = (S_2\cos\Delta - S_1)/\sin\Delta,$$
$$K_1 = (q_2 W_2 - q_1 W_1\cos\Delta)/\sin\Delta,$$
$$K_2 = (q_2 W_2\cos\Delta - q_1 W_1)/\sin\Delta. \tag{7}$$

In addition to these three switching equations, some additional conditions and relations are required. First, for the impulse I_1, to precede the impulse I_2 in an optimal way, it is necessary that[3]

$$q_1 T_1 \geqslant q_2 T_2. \tag{8}$$

Furthermore, for the case where the transfer orbit is completely outside the attracting planet, we must verify the condition that the magnitude of the primer vector along all the three orbits involved does not exceed unity. Along the transfer orbit O, this condition is[3]

$$A_3 \geqslant \sqrt{A_1^2 + A_2^2} = 2e(S_1^2 + J_1^2) \tag{9}$$

where

$$A_1 = 2e(S_1 S_2 - J_1 J_2),$$
$$A_2 = 2e(S_1 J_2 + S_2 J_1),$$
$$A_3 = -A_1\cos f_1 - A_2\sin f_1 + 2[q_1(1 - 3S_1^2) - B^2]/(1 - \cos\Delta),$$
$$B^2 = q_1^2(T_1 - J_1)^2 + [y_1(T_1 - J_1) + S_1]^2 + [(T_1 - J_1)(y_1 T_1 - q_1 S_1) - S_1 T_1]^2/W_1^2. \tag{10}$$

The condition (9) always implies

$$S_1^2 + J_1^2 = S_2^2 + J_2^2 \leqslant \tfrac{1}{4}. \tag{11}$$

This restricts the elevation angle of the optimal impulse to be always less than 30° from the local horizontal plane.

To find the solution for a transfer by two impulses between given terminal orbits O_1 and O_2, we need the relations connecting these orbits with the transfer orbit O and the expressions for the impulses. The geometry of the terminal orbits is defined by their eccentricities e_1 and e_2, semi-latus recta p_1 and p_2, the angle i between the orbital planes which intersect each other along the line of nodes FN, and the two angles δ_1 and δ_2 which are the true anomalies of this line of node in the initial and final plane respectively (Fig. 2). We acknowledge that in celestial mechanics if the plane of O_1 is used as the reference plane and the direction FP_1 to its perigee is used as the reference direction, then the angular orientation of the orbit O_2 is defined by the longitude of the ascending node Ω_2, the inclination i_2 and the argument of the perigee ω_2. Here we use i, δ_1 and δ_2 for the sake of symmetry in the resulting equations with the obvious relations $i = i_2$, $\delta_1 = \Omega_2$, $\delta_2 = 2\pi - \omega_2$. To solve the general problem of optimal two-impulse non-coplanar transfer between elliptical orbits, it has been suggested in[4] to use the unknowns: p for the semi-latus rectum of the transfer orbit, and θ_1 and θ_2 to be respectively, the true anomalies of the points of application of the impulse on the initial and the final orbit. All the pertinent elements can be expressed in terms of these variables and the given data e_i, p_i, δ_i and i.

Let r_i be the radial distance to the impulse I_i. Then

$$r_i = p_i/(1 + e_i\cos\theta_i). \tag{12}$$

Next, in the unit sphere, after the application of the impulse I_1, the plane O_1 is rotated to the transfer

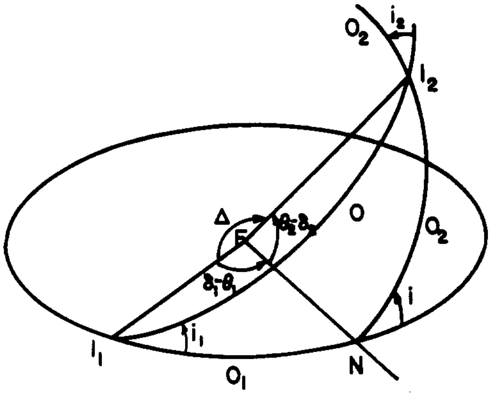

Fig. 3. Spherical geometry of the orbital planes.

plane O by the angle i_1 and after the application of the impulse I_2 the plane O is rotated to the final plane O_2 by the angle i_2 with the total plane change i as prescribed (Fig. 3). From spherical trigonometry based on the figure with the angles as labeled, we have

$$\cos\Delta = \cos(\delta_1-\theta_1)\cos(\delta_2-\theta_2) + \sin(\delta_1-\theta_1)\sin(\delta_2-\theta_2)\cos i, \tag{13}$$

$$\sin i_1 = -\frac{\sin(\delta_2-\theta_2)}{\sin\Delta}\sin i,$$

$$\sin i_2 = \frac{\sin(\delta_1-\theta_1)}{\sin\Delta}\sin i. \tag{14}$$

Hence, Δ and i_i are obtained explicitly in terms of θ_i. In the figure, the two rotations, i_1 and i_2 are in the same direction. This is the plane change of the rotating type. In the case where i_2 is in the opposite direction, we have a plane change of the reflecting type. The same equations apply if we take a negative value for i_2. The true anomalies f_i of the impulses I_i as measured in the transfer orbit are given by the equations

$$\tan f_1 = \cot\Delta - \frac{r_1(p-r_2)}{r_2(p-r_1)\sin\Delta},$$

$$\tan f_2 = -\cot\Delta + \frac{r_2(p-r_1)}{r_1(p-r_2)\sin\Delta}, \tag{15}$$

which involve p in addition to θ_1 and θ_2. Finally the eccentricity of the transfer orbit is given by

$$e = \frac{r_2-r_1}{r_1\cos f_1 - r_2\cos f_2} \tag{16}$$

Let ΔV_i be the impulsive velocity changes with components on the STW system being u_i, v_i, and w_i. Then if μ is the gravitational constant, we have

$$u_1 = \sqrt{\mu}\left(\frac{e}{\sqrt{p}}\sin f_1 - \frac{e_1}{\sqrt{p_1}}\sin\theta_1\right),$$

$$v_1 = \frac{\sqrt{\mu}}{r_1}(\sqrt{p} - \sqrt{p_1}\cos i_1),$$

$$w_1 = \frac{\sqrt{\mu p_1}}{r_1}\sin i_1, \tag{17}$$

and

$$u_2 = \sqrt{\mu}\left(\frac{e_2}{\sqrt{p_2}}\sin\theta_2 - \frac{e}{\sqrt{p}}\sin f_2\right),$$

$$v_2 = \frac{\sqrt{\mu}}{r_2}(\sqrt{p_2}\cos i_2 - \sqrt{p}),$$

$$w_2 = \frac{\sqrt{\mu p_2}}{r_2}\sin i_2. \tag{18}$$

The magnitude of the respective characteristic velocity is

$$\Delta V_i = (u_i^2 + v_i^2 + w_i^2)^{1/2}. \tag{19}$$

We now summarize the problem. Two terminal orbits O_1 and O_2 are given. They are defined by the elements e_i, p_i, δ_i and the angle i between their planes. To find the optimum transfer orbit O connecting the given orbits, we use the three unknowns p, θ_1, and θ_2 which are the semi-latus rectum of the transfer orbit and the true anomalies of the impulses measured in the initial and the final orbit, respectively. The three non-linear equations to be solved are the switching eqns (4), (5) and (6). In these equations, all the elements Δ, $x_i = e\cos f_i$, $y_i = e\sin f_i$ can be expressed in terms of the unknowns p, θ_1, and θ_2, while the direction cosines of the thrust (and hence the thrust angles themselves) are obtained from

$$S_i = \frac{u_i}{\Delta V_i},\quad T_i = \frac{v_i}{\Delta V_i},\quad W_i = \frac{w_i}{\Delta V_i}. \tag{20}$$

From eqns (17) and (18), these direction cosines are also expressible in terms of the selected unknowns. Once the solution is obtained, the two conditions (8) and (9) must be verified to insure the optimality of the transfer.

3. THE OPTIMAL NODAL TRANSFER

We now impose the condition that the impulses are applied on the line of nodes. Then

$$\Delta = f_2 - f_1 = \pi. \tag{21}$$

From the definition (7), we obtain

$$x_1 + x_2 = 0,\quad y_1 + y_2 = 0,\quad q_1 + q_2 = 2. \tag{22}$$

From the same definition, we write

$$J_1 = \frac{(S_1+S_2)}{\sin\Delta} - S_1\cot\frac{\Delta}{2},$$

$$J_2 = S_2\cot\frac{\Delta}{2} - \frac{(S_1+S_2)}{\sin\Delta}, \tag{23}$$

When Δ tends to π, we have

$$\lim J_1 = \lim\frac{(S_1+S_2)}{\sin\Delta} = -\lim J_2 = J. \tag{24}$$

With this, we write the switching relation (4)

$$4J = q_1T_1 - q_2T_2. \tag{25}$$

We deduce that J is finite and as a consequence

$$S_1 + S_2 = 0. \tag{26}$$

Again, from the definition (7), we have singularity in K_1 and K_2. But now that J_1 and J_2 are finite, we see from eqns (5) and (6) that K_1 and K_2 are also finite. We write

$$K_1 = \frac{(q_1 W_1 + q_2 W_2)}{\sin\Delta} - q_1 W_1 \cot\frac{\Delta}{2},$$

$$K_2 = -\frac{(q_1 W_1 + q_2 W_2)}{\sin\Delta} + q_2 W_2 \cot\frac{\Delta}{2}. \tag{27}$$

Let

$$\lim K_1 = \lim \frac{(q_1 W_1 + q_2 W_2)}{\sin\Delta} = -\lim K_2 = K. \tag{28}$$

Since K is finite, we have at the limit when Δ tends to π,

$$q_1 W_1 + q_2 W_2 = 0. \tag{29}$$

By putting $K_1 = K = -K_2$ in eqns (5) and (6) and then eliminating K between these equations, we have a new relation free of singularities

$$y_1 = \frac{x_1 S_1 (1 - S_1^2)}{J(1 - S_1^2 - 2J^2)}. \tag{30}$$

Let

$$n = \frac{r_1}{r_2} = \frac{q_2}{q_1} = \frac{1 - x_1}{1 + x_1}, \tag{31}$$

be the ratio of the radii. In terms of the optimal thrust angles ϕ_i and ψ_i, we first have the condition (26) written as

$$\phi_1 = -\phi_2. \tag{32}$$

Next, we have the condition (29) which becomes

$$\sin\psi_1 = -n \sin\psi_2. \tag{33}$$

To put this equation in a more symmetric form, let

$$2\alpha = \psi_2 - \psi_1, \quad 2\beta = \psi_2 + \psi_1. \tag{34}$$

Then, equation (33) becomes

$$(1 - n)\tan\alpha = (1 + n)\tan\beta. \tag{35}$$

Finally, after some algebraic manipulation, we can transform eqn (30) into a more explicit form

$$y_1 \cos\beta[(n+1)^2 + 4n\cos^2\beta] = 4(n+1)^2 \tan\phi_1 \cos\alpha \tag{36}$$

In summary, for a two-impulse nodal transfer, the three necessary conditions for optimality are the eqns (32), (35) and (36). Besides, we must verify that the magnitude of the primer vector does not exceed unity on the three orbits O_1, O and O_2. This condition on the transfer orbit is given by the inequality (9).

4. THE MINIMIZING NODAL TRANSFER

There is an infinite number of nodal transfers connecting two orbits O_1 and O_2 with given elements e_i, p_i, δ_i and i. For distinction, we call the best of these nodal transfers the minimizing nodal transfer and when this nodal transfer also verifies all the Lawden's optimality conditions, we call it the optimal nodal transfer. The minimizing nodal transfer has been studied in[5], and it has been shown that it satisfies the two conditions (32) and (33). Since, in general, the third condition (36) is not satisfied, the minimizing nodal transfer is non optimal. But it is worth analyzing since the cost for this transfer is generally low and the transfer time is finite, of the order of half an orbital period and as such it can be used as a practical transfer between non coplanar elliptical orbits.

The line of nodes intersects the first orbit O_1 at the points N_1 and N_1' and the second orbit O_2 at the points N_2 and N_2', where N_i are defined by the true anomalies δ_i and N'_i by $\delta_i + \pi$. It can be shown that the optimal way in connecting these points by a transfer orbit is the same as for the transfer between two equivalent coaxial orbits O_1' and O_2', in the plane of O_1 and O_2 respectively, and having the points N_i and N_i' as apsidal points. Then, we have the following rule[2]: If $e_1 e_2 \cos\delta_1 \cos\delta_2 \geq 0$, the transfer is of the direct, or aligned coaxial type using a transfer orbit connecting the highest apogee with the perigee of the other orbit. If $e_1 e_2 \cos\delta_1 \cos\delta_2 < 0$, we have the inverse coaxial type and the transfer orbit is either between the apogees or between the perigees.

With the positions of the impulses settled, for the aligned coaxial type, we have either $\theta_1 = \delta_1$ and $\theta_2 = \delta_2 + \pi$ or $\theta_1 = \delta_1 + \pi$ and $\theta_2 = \delta_2$. For the inverse coaxial type we have either $\theta_1 = \delta_1$ and $\theta_2 = \delta_2$ or $\theta_1 = \delta_1 + \pi$ and $\theta_2 = \delta_2 + \pi$. With the values of θ_1 and θ_2 well defined, we have the radii r_1 and r_2 and also their ratio $n = r_1/r_2$. The semi-latus rectum of the transfer orbit can be easily evaluated as

$$p = \frac{2r_1 r_2}{r_1 + r_2}. \tag{37}$$

Before evaluating the characteristic velocities for the transfer, we notice that the rotation i_2 is of the reflecting type, and hence

$$i_1 - i_2 = i, \tag{38}$$

with a negative value for i_2.

Next we explicit the direction cosines of the impulses as given by eqn (20):

$$\sin\phi_1 = \frac{u_1}{\Delta V_1}, \quad \sin\phi_2 = \frac{u_2}{\Delta V_2},$$

$$\cos\phi_1 \cos\psi_1 = \frac{v_1}{\Delta V_1}, \quad \cos\phi_2 \cos\psi_2 = \frac{v_2}{\Delta V_2},$$

$$\cos\phi_1 \sin\psi_1 = \frac{w_1}{\Delta V_1}, \quad \cos\phi_2 \sin\psi_2 = \frac{w_2}{\Delta V_2}, \tag{39}$$

with u_i, v_i and w_i given by eqns (17) and (18). To evaluate the rotation angles i_1 and i_2 we first use eqn (39) to express the optimal relation (32) as

$$\frac{\sqrt{v_1^2 + w_1^2}}{\Delta V_1} = \frac{\sqrt{v_2^2 + w_2^2}}{\Delta V_2}, \quad \frac{u_1}{\Delta V_1} = -\frac{u_2}{\Delta V_2}. \tag{40}$$

On the other hand, we have from the optimal relation (33)

$$\frac{w_1}{w_2} = -n\frac{\Delta V_1}{\Delta V_2} = -n\frac{\sqrt{v_1^2 + w_1^2}}{\sqrt{v_2^2 + w_2^2}}. \quad (41)$$

With the aid of eqns (17) and (18) we explicit this equation as

$$\frac{\sin i_1}{\sin i_2} = -n\frac{[1 + (p/p_1) - 2\sqrt{(p/p_1)}\cos i_1]^{1/2}}{[1 + (p/p_2) - 2\sqrt{(p/p_2)}\cos i_2]^{1/2}}. \quad (42)$$

Upon solving eqns (38) and (42), we have the optimal rotations i_1 and i_2. From eqn (41), we immediately have the ratio of the characteristic velocities:

$$\frac{\Delta V_1}{\Delta V_2} = -\frac{1}{n^2}\sqrt{\frac{p_1 \sin i_1}{p_2 \sin i_2}} = m. \quad (43)$$

If we use $\sin f_2 = -\sin f_1$, and substitute eqns (17) and (18) into eqn (39) we have for the other elements

$$\tan\psi_1 = \frac{\sin i_1}{\sqrt{p/p_1} - \cos i_1},$$

$$\tan\psi_2 = -\frac{\sin i_2}{\sqrt{p/p_2} - \cos i_2}. \quad (44)$$

$$\frac{u_1}{\sqrt{\mu}} = -\frac{m}{(1+m)}\left(\frac{e_1}{\sqrt{p_1}}\sin\theta_1 + \frac{e_2}{\sqrt{p_2}}\sin\theta_2\right),$$

$$\frac{u_2}{\sqrt{\mu}} = \frac{1}{(1+m)}\left(\frac{e_1}{\sqrt{p_1}}\sin\theta_1 + \frac{e_2}{\sqrt{p_2}}\sin\theta_2\right). \quad (45)$$

$$\tan\phi_1 = -\frac{mr_1\left(\frac{e_1}{\sqrt{p_1}}\sin\theta_1 + \frac{e_2}{\sqrt{p_2}}\sin\theta_2\right)}{(1+m)\sqrt{p + p_1 - 2\sqrt{pp_1}\cos i_1}}. \quad (46)$$

The characteristic velocities ΔV_i are obtained by eqn (19). They are also explicit functions of the given data e_i, p_i, δ_i and the computed plane change angles i_1 and i_2 which, for each given i, depend on the three ratios n, p/p_1 and p/p_2.

5. NUMERICAL EXAMPLES

The minimizing nodal transfer is valuable only if it is not far from the true optimal transfer. It can be shown that if $C = \Delta V_1 + \Delta V_2$ is the total characteristic velocity for the minimizing nodal transfer and C' is the total characteristic velocity for the generalized Hohmann transfer between the equivalent coaxial orbits O_1' and O_2' with the same plane change i, as defined in Section 4, we have the relation

$$C^2 = C'^2 + \left(\frac{h_1}{p_1}e_1\sin\delta_1 - \frac{h_2}{p_2}e_2\sin\delta_2\right)^2. \quad (47)$$

Hence, not only that the transfer time of about half an orbital period is acceptable in practice, the cost for the minimizing nodal transfer under favorable conditions is reasonably close to the cost for an ideal transfer. It is easy to compute the minimizing nodal transfer. After evaluating its elements we can verify the third condition (36). It is trivially satisfied for the coaxial case, $y_1 = e\sin f_1 = 0$, $\tan\phi_1 = 0$. In general the condition is not satisfied and the minimizing nodal transfer is not the true optimal transfer between the given terminal orbits.

To assess the usefulness of the minimizing nodal transfer, we consider below two examples. The first one concerns the transfer between two equal orbits and the second example is the transfer from an elliptical orbit to a non coaxial circular orbit. We shall take $\mu = 1$ for the computation of the characteristic velocities.

By taking $e_1 = e_2 = 0.4$, $p_1 = p_2 = 1.68$, we have two equal terminal orbits. We assume that the initial orbit O_1 is in the equatorial plane. For the terminal orbit O_2 besides the prescribed values e_2 and p_2, we only require that its perigee is at 90° before the ascending node, that is $\delta_2 = 90°$, and that it is inclined at a certain angle, taken as $i = 30°$ for the computation. The fact that the longitude of the ascending node of the final orbit $\Omega_2 = \delta_1$ is not important is that the Earth is rotating about an axis orthogonal to the equatorial plane and hence with respect to Earth fixed axes, the line of nodes is automatically adjusted by the Earth's rotation.

Then, we shall use δ_1 as parameter and for each value of δ_1, from 0 to 180°, we compute the minimizing nodal transfer by the equations in Section 4, and analyze the variation of the total characteristic velocity.

Since $e_1 e_2 \cos\delta_1 \cos\delta_2 = 0$, we have the aligned coaxial type. Since $r_2 = p_2$, when $\delta_1 \leqslant 90°$, the first impulse is applied at $\theta_1 = \delta_1 + \pi$, and the second impulse at $\theta_2 = \delta_2$. For δ_1 between 90 and 180°, we shall use $\theta_1 = \delta_1$, and $\theta_2 = \delta_2 + \pi$. The total characteristic velocity for this transfer is plotted vs δ_1 as the upper curve in Fig. 4. In the same figure, we have plotted as the lower curve the total characteristic for the optimal two-impulse transfer, using the optimal switching relations in Section 2. The two curves exhibit symmetry with respect to the line $\delta_1 = 90°$. The usual deviation between these two curves is about 10% and reduces to 6% when they are near their minimum values which occur for a nearly same

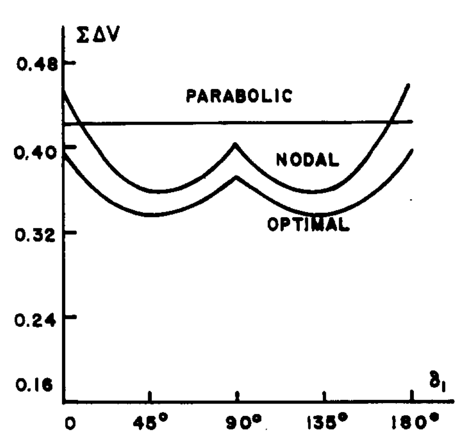

Fig. 4. Characteristic velocities for Case 1.

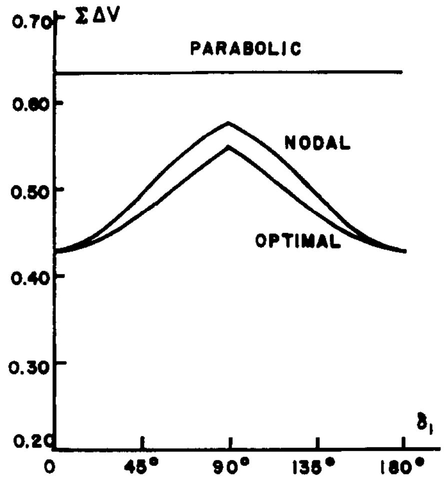

Fig. 5. Characteristic velocities for Case 2.

value of δ_1. Hence, for a transfer from O_1 to O_2 when $\delta_1 = \Omega_2$ is not specified, we can use the minimizing nodal transfer to quickly determine the approximate optimum value Ω_2^* before calculating the transfer. As a check of the optimality of the solution we have plotted in Fig. 4 the constant cost for a transfer via parabolic orbits which is

$$\Delta V_P = \sum \sqrt{\frac{\mu}{p_i}} [\sqrt{2(1+e_i)} - (1+e_i)]. \quad (48)$$

In the present case, $\Delta V_P = 0.4217$. It is seen that the cost for the minimizing nodal transfer is sufficiently close to the optimal value for it to be a useful transfer.

In the second example, we use the same initial orbit O_1 in the equatorial plane, with the same plane change $i = 30°$. This time we take the final orbit as circular with $e_2 = 0$, $p_2 = 1$. Again we use the longitude of the ascending node $\delta_1 = \Omega_2$ as a parameter varying from 0 to 180°, while δ_2 is of course arbitrary. This is a practical configuration usually encountered when we transfer from an elliptical orbit in the equatorial plane to an inclined circular orbit at low altitude. Again, the transfer is of the aligned coaxial type and since O_2 is completely inside O_1, we use the same selection for θ_1 and θ_2 as in the first case. The characteristic velocities are plotted in Fig. 5 and we have a deviation of about 6%. When $\delta_1 = 0$ and 180°, the two orbits are coaxial and the minimizing nodal transfer is also the optimal transfer. In this case, the cost for the parabolic transfer as computed from eqn (48) is $\Delta V_P = 0.6251$ and is non optimal.

6. CONCLUSION

In this paper, we have presented the necessary conditions for the minimum fuel, time-free transfer between two non-coplanar elliptical orbits. It is shown that the solution is obtained by solving a system of three non-linear equations for three unknowns. It is then the best two-impulse transfer but if some additional inequalities are not satisfied, the three-impulse transfer may be optimal.

In the second part we discuss the case where the impulses are applied along the line of nodes. In general, this nodal transfer is non optimal, but the best nodal transfer, called the minimizing nodal transfer is reasonably close to the optimal transfer to be useful as its substitute for a preliminary evaluation of the fuel consumption. Furthermore, when we continuously vary the relative position of the terminal orbits, the variations of the two total characteristic velocities exhibit the same trend in the sense that they pass through their minima at nearly the same relative position. This makes the set of explicit formulae for computing the minimizing nodal transfer, as presented in this paper, a useful tool for designing a minimum fuel transfer between several orbits. For lack of space we plan to discuss the special case of one impulse transfer and the non optimality of the non-coaxial three-impulse nodal transfer in a separate paper.

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