SHAPE OPTIMAL DESIGN OF ELASTIC BODIES USING A MIXED VARIATIONAL FORMULATION*

Helder C. RODRIGUES

Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109, U.S.A.

This study is concerned with the development of a variational formulation and a procedure for the computational solution for the shape optimal design of a two-dimensional linear elastic body. The objective is to minimize the maximum value of the Von Mises equivalent stress in the body, subject to an isoperimetric constraint on the area. The optimality conditions for this problem are derived using a mixed variational formulation, where the equations defining the elastostatics problem are dealt with as additional constraints for the optimization. The results of the analysis are implemented via a finite element discretization. The discretized model is tested in two numerical examples, the shape optimization of a hole in a biaxially loaded sheet, and of the design of a fillet.

1. Introduction

This study is concerned with the development of a variational formulation and a procedure for the computational solution for the shape optimal design of a two-dimensional linear elastic body, using a mixed finite element discretization.

Shape optimal design is a problem that has interested many researchers in the last fifteen years. The subject has been surveyed in a number of review articles. The reader is referred to the recent paper by Haftka and Grandhi (see, e.g. [11]), and the papers cited therein for additional background information from the recent literature.

Zienkiewicz and Campbell (see, e.g. [12]), were among the first to approach this problem using a virtual displacement-based finite element model. Subsequently this method has been applied widely to problems in shape optimal design (see, e.g. [2, 3, 13–15]), but only with mixed success. The virtual displacement finite element method has two main disadvantages: (1) the increase of finite element error that results from mesh distortion during shape redesign, and (2) in some situations, a lack of sufficient precision in the prediction of stresses and strains at the boundary and internal nodes.

There are some methods one can consider to overcome these difficulties. Some investigators have applied the boundary element method (see, e.g. [16–18]). While the BEM has proved to be very useful and looks promising in certain applications on shape optimal design, for problems that require numerous evaluations of state variables in the domain (objective

function $= \max_{x \in \Omega} F(u, e)$, for example) the BEM loses some of its advantages, also at the current stage of development it lacks the generality provided by FEM in structural analysis. Within the FEM applications, the domain method (see, e.g. [19]), where sensitivity expressions are defined in terms of domain integrals rather than boundary integrals (thereby avoiding the evaluation of state variables at the boundary), provides for improved accuracy in the numerical calculation of sensitivities. Also, recently Haber proposed an Eulerian–Lagrangian formulation based on the mutual Reissner energy (see, e.g. [20]), where the shape-optimization problem can be formulated in an arbitrary initial domain as a means to overcome the difficulties inherited from shape redesign.

In this work another approach is considered. With the development of powerful automatic mesh generation and optimization techniques (see, e.g. [7]) the first of the cited disadvantages of FEM is avoided. Mixed finite element methods (see, e.g. [8]) that provide for accurate computation of stresses and strains at the element nodes appears to be a natural approach to resolve the other difficulty. These considerations are brought together in the developments reported here, to demonstrate a more effective approach to the overall treatment of shape optimal design.

2. Problem formulation

For the two-dimensional linear elastic body described as shown in Fig. 1 the objective is to determine the domain D such that the maximum value of the local measure $F(e, u)$ is minimized, i.e., to achieve

$$\underset{D}{\text{Min}} \ \underset{x \in D}{\text{Max}} \ F(e, u) \tag{1}$$

subject to the resource constraint
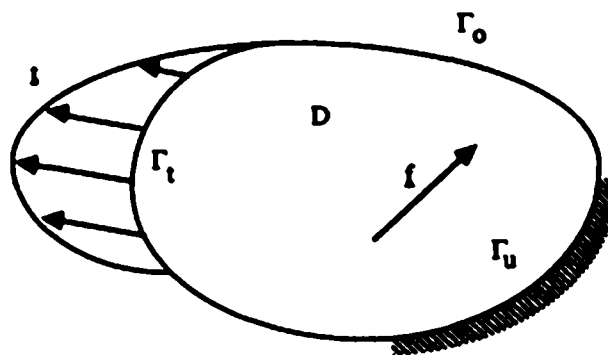
$$\int_{D} d\Omega - A \leq 0, \tag{2}$$

Fig. 1. Elastic body nomenclature.

and to the condition for equilibrium. The later requirement is defined via the equation of equilibrium

$$\nabla \cdot \sigma + f = 0, \qquad \sigma = \sigma^t \quad \text{on } D, \tag{3}$$

the strain-displacement relations (compatibility)

$$e = \tfrac{1}{2}(\nabla u + u\nabla) \quad \text{on } D, \tag{4}$$

the stress-strain relation

$$\sigma = E : e \quad \text{on } D, \tag{5}$$

the traction boundary condition

$$n \cdot \sigma = t \quad \text{on } \Gamma_t, \tag{6}$$

and the displacement boundary condition

$$u = 0 \quad \text{on } \Gamma_u. \tag{7}$$

The argument $F(e, u)$ in (1) is the objective (cost) function, which is assumed to be strictly positive (in the numerical examples presented below, F is the Von Mises equivalent stress written in terms of strains, for example).

The problem defined by (1) is not differentiable with respect to the design, so it is useful as a first step to restate it as a simple Min problem with respect to a bound β on the objective function $F(e, u)$ (see, e.g. [1]). Accordingly, the optimal design problem is restated:

$$\operatorname*{Min}_{D} \beta \tag{8}$$

subject to

$$F(e, u) - \beta \leq 0 \quad \text{on } D, \tag{9}$$

and to the same constraint equations (2)–(7).

3. Necessary conditions for optimal solution

To obtain the necessary conditions for the shape-optimization problem defined, let us introduce the augmented functional

$$L = \beta + \Lambda\left(\int_D d\Omega - A\right) + \int_D \lambda(F(e, u) - \beta)\,d\Omega - \int_D v \cdot (\nabla \cdot \sigma + f)\,d\Omega$$

$$+ \int_D \varepsilon : (E : e - \sigma)\,d\Omega + \int_D \tau : (\tfrac{1}{2}(\nabla u + u\nabla) - e)\,d\Omega - \int_{\Gamma_u} p \cdot u\,d\Gamma \tag{10}$$

under conditions on the Lagrange multipliers,

$$\Lambda \geqslant 0, \qquad \lambda(x) \geqslant 0, \qquad x \in D, \tag{11}$$

and $\tau = \tau^t$, $\varepsilon = \varepsilon^t$.

Integrating by parts, applying boundary condition (6), and using the symmetry of σ this functional is reduced to

$$L = \beta + \Lambda\left(\int_D d\Omega - A\right) + \int_D \lambda(F(\varepsilon, u) - \beta)\, d\Omega + \int_D \tfrac{1}{2}(\nabla v + v\nabla) : \sigma\, d\Omega$$

$$- \int_D v \cdot f\, d\Omega - \int_{\Gamma_t} v \cdot t\, d\Gamma + \int_D \varepsilon : (E : \varepsilon - \sigma)\, d\Omega$$

$$+ \int_D \tau : (\tfrac{1}{2}(\nabla u + u\nabla) - \varepsilon)\, d\Omega - \int_{\Gamma_u} p \cdot u\, d\Gamma - \int_{\Gamma_u}(n \cdot \sigma) \cdot v\, d\Gamma. \tag{12}$$

The augmented functional (12) forms the Lagrangian for the shape optimization problem defined by (8), (9) and (2)–(7), from which the necessary conditions may be obtained by a formal procedure as stationarity conditions. Note that the multipliers τ, ε, and v represent stress, strain, and displacement for the adjoint problem, which is characterized by the necessary condition (23) given below.

The perturbed domain D_μ is defined *relative to the optimum design* D via the transformation T (see, e.g. [4–6] for applications of such transformation in a somewhat different context) i.e.,

$$D_\mu = T(D, \mu).$$

In this representation

$$D = T(D, 0).$$

Also, the curves $\partial\Gamma_u$ and $\partial\Gamma_t$ that bound the displacement and stress prescribed boundaries are fixed, i.e.,

$$T = I \quad \text{for } x \in \partial\Gamma_u \cup \partial\Gamma_t.$$

In the neighborhood of $\mu = 0$ and assuming sufficient regularity on T, the transformation can be expressed via

$$T(D, \mu) = T(D, 0) + \mu\, dT(D, \mu)/d\mu\big|_{\mu=0} + \cdots.$$

Then up to the first order

$$T(D, \mu) = (I + \mu\theta)(D),$$

where
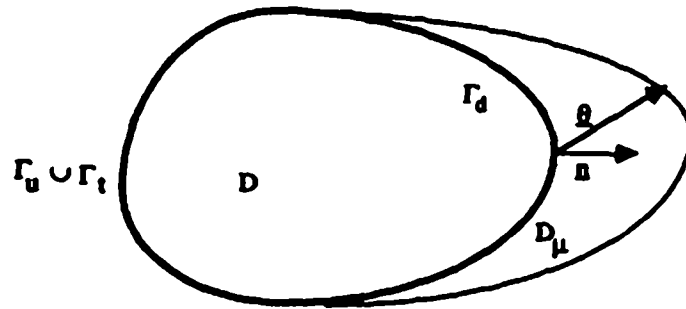
$$\theta(D) = dT(D, \mu)/d\mu\big|_{\mu=0},$$

see Fig. 2.

Fig. 2. Domain perturbation field θ.

All the expressions defined in the functional (12) are of the following types:

$$\int_D f(x)\, d\Omega \,, \tag{13}$$

$$\int_\Gamma b(x)\cdot n\, d\Gamma \,, \tag{14}$$

$$\int_\Gamma g(x)\, d\Gamma \,, \tag{15}$$

where the functions f, g, and the vector field b are defined on D_μ or its boundary. Using the previously introduced domain perturbation field θ, the first variations of these expressions with respect to μ are defined by (see, e.g. [5, 6])

$$d\left[\int_{D_\mu} f(x)\, d\Omega\right]\Big/d\mu\Big|_{\mu=0} = \int_D \partial f(x)/\partial\mu\Big|_{\mu=0}\, d\Omega + \int_\Gamma f(x)(\theta\cdot n)\, d\Gamma \,, \tag{16}$$

$$d\left[\int_{\Gamma_\mu} b(x)\cdot n\, d\Omega\right]\Big/d\mu\Big|_{\mu=0}\, d\Gamma = \int_\Gamma \partial b(x)/\partial\mu\Big|_{\mu=0}\cdot n + \nabla\cdot b(\theta\cdot n)\, d\Gamma \,, \tag{17}$$

$$d\left[\int_{\Gamma_\mu} g(x)\right]\, d\Gamma\Big/d\mu\Big|_{\mu=0} = \int_\Gamma \partial g(x)/\partial\mu\Big|_{\mu=0} + (g\nabla\cdot n + Hg)(\theta\cdot n)\, d\Gamma \,, \tag{18}$$

where H is the curvature of Γ in 2D problems and twice the mean curvature in 3D.

From the first variation with respect to μ of the augmented functional (12) based on (16)–(18), considering the conservative loading case in which the traction t in (6) depends on position only, and under the constraints (11) on the Lagrange multipliers, the necessary conditions for the optimum are obtained as presented below (see, e.g. [1–3]).

The normalization

$$\int_D \lambda\, d\Omega = 1 \tag{19}$$

on the multiplier λ reflects stationarity of L with respect to β. Generalized Karush–Kuhn–Tucker conditions provide

$$\lambda(F(e, u) - \beta) = 0 ,$$

$$\lambda \geqslant 0 , \qquad (F(e, u) - \beta) \leqslant 0 , \qquad x \in D ,$$

(20)

$$\Lambda\left(\int_D d\Omega - A\right) = 0 ,$$

$$\Lambda \geqslant 0 , \qquad \left(\int_D d\Omega - A\right) \leqslant 0 .$$

(21)

From stationarity with respect to the adjoint state variables (τ, e, v, p) and the state variables (σ, e, u), respectively, the equilibrium and adjoint equilibrium equations for the optimal domain are stated as (this corresponds in a sense to the Hu–Washizu variational principle expressed in weak form (see, e.g. [9]))

$$\int_D \tfrac{1}{2}(\nabla \delta v + \delta v \nabla) : \sigma \, d\Omega - \int_D \delta v \cdot f \, d\Omega - \int_{\Gamma_t} \delta v \cdot t \, d\Gamma + \int_D \delta e : (E : e - \sigma) \, d\Omega$$

$$+ \int_D \delta\tau : (\tfrac{1}{2}(\nabla u + u\nabla) - e) \, d\Omega - \int_{\Gamma_u} \delta p \cdot u \, d\Gamma - \int_{\Gamma_u} (n \cdot \sigma) \cdot \delta v \, d\Gamma = 0 , \qquad (22)$$

$$\int_D \tfrac{1}{2}(\nabla v + v\nabla) : \delta\sigma \, d\Omega + \int_D \lambda(\partial F/\partial e : \delta e + \partial F/\partial u \cdot \delta u) \, d\Omega$$

$$+ \int_D \varepsilon : (E : \delta e - \delta\sigma) \, d\Omega + \int_D \tau : (\tfrac{1}{2}(\nabla \delta u + \delta u \nabla) - \delta e) \, d\Omega$$

$$- \int_{\Gamma_u} p \cdot \delta u \, d\Gamma - \int_{\Gamma_u} (n \cdot \delta\sigma) \cdot v \, d\Gamma = 0 . \qquad (23)$$

If the boundary conditions on the displacement fields u and v are satisfied a priori, (22) and (23) can equivalently be stated as

$$\int_D \tfrac{1}{2}(\nabla \delta v + \delta v \nabla) : \sigma \, d\Omega - \int_D \delta v \cdot f \, d\Omega - \int_{\Gamma_t} \delta v \cdot t \, d\Gamma$$

$$+ \int_D \delta e : (E : e - \sigma) \, d\Omega + \int_D \delta\tau : (\tfrac{1}{2}(\nabla u + u\nabla) - e) \, d\Omega = 0 ,$$

$$u = 0 \quad \text{on } \Gamma_u , \qquad (24)$$

$$\int_D \tfrac{1}{2}(\nabla v + v\nabla) : \delta\sigma \, d\Omega + \int_D \lambda(\partial F/\partial e : \delta e + \delta F/\partial u \cdot \delta u) \, d\Omega$$

$$+ \int_D \varepsilon : (E : \delta e - \delta\sigma) \, d\Omega + \int_D \tau : (\tfrac{1}{2}(\nabla \delta u + \delta u \nabla) - \delta e) \, d\Omega = 0 ,$$

$$v = 0 \quad \text{on } \Gamma_u , \qquad (25)$$

with p defined by

$$p = \tau \cdot n \quad \text{on } \Gamma_u .$$

The optimality condition, i.e., the necessary condition associated with variation of the domain is given by

$$\int_{\Gamma_d} (\varepsilon : \sigma - v \cdot f + \Lambda)(\theta \cdot n) \, d\Gamma - \int_{\Gamma_u} p \cdot (u\nabla) \cdot n(\theta \cdot n) \, d\Gamma$$

$$- \int_{\Gamma_u} \sigma : (v\nabla)(\theta \cdot n) \, d\Gamma - \int_{\Gamma_t} \{((t \cdot v)\nabla) \cdot n + Ht \cdot v\}(\theta \cdot n) \, d\Gamma = 0 . \tag{26}$$

If $T = I$ for $x \in \Gamma_u \cup \Gamma_t$ and assuming $f = 0$ on Γ_d, the optimality condition is simply

$$\int_{\Gamma_d} (\sigma : \varepsilon + \Lambda)(\theta \cdot n) \, d\Gamma = 0 . \tag{27}$$

According to this result, for the optimal domain *the mutual energy has constant value* on the design boundary Γ_d. Once again, θ represents the perturbation field of the domain defined on Γ_d (see Fig. 2).

4. Numerical formulation

4.1. Finite element model

The equilibrium equation (24) is discretized using four-node isoparametric finite elements to interpolate the stress, strain, and displacement fields (see Fig. 3).

For the 2D linear elasticity and satisfying a priori the symmetry condition for σ and ε, the finite element interpolations are expressed via[1]
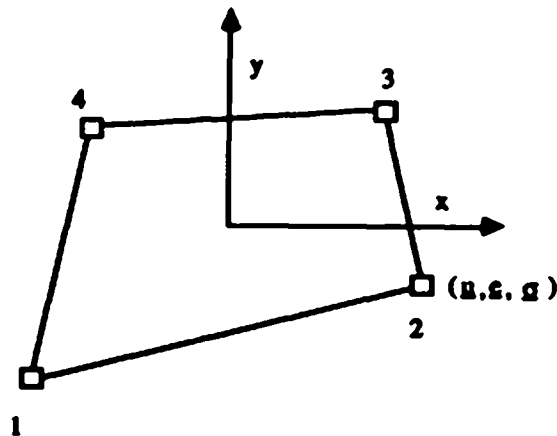

Fig. 3. Four-node finite element.

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[1] Repeated dummy indices $I, J = 1, \dots, 4$ are summed over their ranges.

$$u(x) = N_I u_I, \qquad e(x) = N_I e_I, \qquad \sigma(x) = N_I \sigma_I, \tag{28}$$

where N_I symbolizes the polynomial basis, bilinear for the four-node element, and the vectors $u_I = \{u_x, u_y\}_I$, $e_I = \{e_x, e_y, 2e_{xy}\}_I$, $\sigma_I = \{\sigma_x, \sigma_y, \sigma_{xy}\}_I$ represent the nodal values of the displacement, strain, and stress fields, respectively.

Using the displacement interpolation field, one can define

$$\begin{bmatrix} u_{x,x} & 0 \\ 0 & u_{y,y} \\ u_{x,y} & u_{y,x} \end{bmatrix} = B_I u_I, \tag{29}$$

with B_I given by,

$$B_I = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \\ N_{I,y} & N_{I,x} \end{bmatrix}. \tag{30}$$

Substituting the finite element interpolations (28), (29) in equation (24), the corresponding linear system of algebraic equations of equilibrium is

$$\sum_e \begin{bmatrix} 0_{IJ} & 0_{IJ} & B_{IJ} \\ 0_{IJ} & D_{IJ} & C_{IJ} \\ B_{IJ}^t & C_{IJ}^t & 0_{IJ} \end{bmatrix}^e \begin{bmatrix} u_J \\ e_J \\ \sigma_J \end{bmatrix}^e = \sum_e \begin{bmatrix} F_I \\ 0_I \\ 0_I \end{bmatrix}^e, \tag{31}$$

where the discrete gradient operator B_{IJ}, stress (strain) projection operator C_{IJ}, material properties matrix D_{IJ}, and load vector F_I are defined by

$$B_{IJ} = \int_{\Omega_e} B_I^t N_J \, d\Omega, \tag{32}$$

$$C_{IJ} = -\int_{\Omega_e} N_I I N_J \, d\Omega, \tag{33}$$

$$D_{IJ} = \int_{\Omega_e} N_I D N_J \, d\Omega, \tag{34}$$

$$F_I = \int_{\Omega_e} N_I f \, d\Omega + \int_{\Gamma_e} N_I t \, d\Gamma, \tag{35}$$

and 0_{IJ} represents a zero matrix.

The adjoint equilibrium equation (25) is discretized using the same interpolation functions, which implies that only the force term is changed while the coefficient matrix is the same as in the previous system:

$$\sum_e \begin{bmatrix} 0_{IJ} & 0_{IJ} & B_{IJ} \\ 0_{IJ} & D_{IJ} & C_{IJ} \\ B_{IJ}^t & C_{IJ}^t & 0_{IJ} \end{bmatrix}^e \begin{bmatrix} v_J \\ \varepsilon_J \\ \tau_J \end{bmatrix}^e = \sum_e \begin{bmatrix} 0_I \\ F_I^a \\ F_I^b \end{bmatrix}^e, \tag{36}$$

where the adjoint stress F_I^s and adjoint force F_I^s vectors are defined by

$$F_I^s = \int_{\Omega_e} \lambda N_I \, dF/d\varepsilon_I \, d\Omega \,, \tag{37}$$

$$F_I^s = \int_{\Omega_e} \lambda N_I \, dF/d u_I \, d\Omega \,. \tag{38}$$

Here λ is the Lagrange multiplier of the bound constraint (9); it is interpolated using bilinear shape functions within each element where this constraint is active. Recall that the function λ is required to satisfy the normalization condition (19).

4.2. Design-model discretization

The perturbation field $\theta(x)$, $x \in \Gamma_d$, is interpolated using linear boundary elements. The design variables d then become defined as the norm of the position vector of the respective interpolation nodes expressed with respect to a pre-defined origin O (see Fig. 4).

For the design element Γ_d^i, the perturbation field is expressed as $\theta = \vartheta^i(s)$, $s \in [0, L_i]$, where L_i is the element length. $\theta^i(s)$ is interpolated via

$$\theta_x^i(s) = N_1(s) \, \delta d_i \cos \beta_i + N_2(s) \, \delta d_{i+1} \cos \beta_{i+1} \,, \tag{39}$$

$$\theta_y^i(s) = N_1(s) \, \delta d_i \sin \beta_i + N_2(s) \, \delta d_{i+1} \sin \beta_{i+1} \,, \tag{40}$$

where

$$N_1(s) = (1 - s/L_i) \quad \text{and} \quad N_2(s) = s/L_i \,.$$

Since this discretization is needed in order to express the optimality condition (27), the design boundary and finite element meshes should coincide at this portion of the domain boundary. This implies that the design variables are related to the finite element nodes in Γ_d.

The adjoint strain field, obtained by solving the algebraic system of equations (36) and needed to express the optimality condition, is very irregular along the design boundary nodes. Due to this fact the shape redesign process based on this optimality condition becomes quite
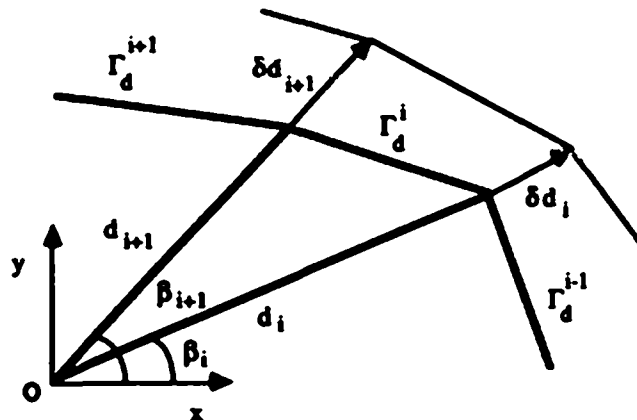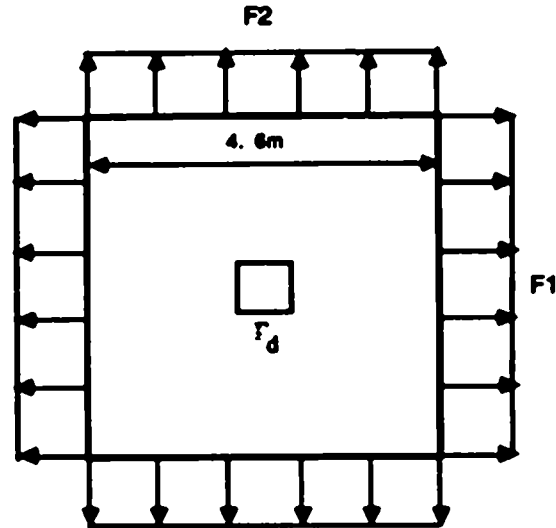

Fig. 4. Design variables d.

Fig. 5. Shape optimization of a hole in a square sheet. Plane stress, 110 finite elements, 11 design variables, objective function the Von Mises equivalent stress, maximum admissable area for the hole $0.16\,m^2$. Only one quarter of the sheet is modeled.
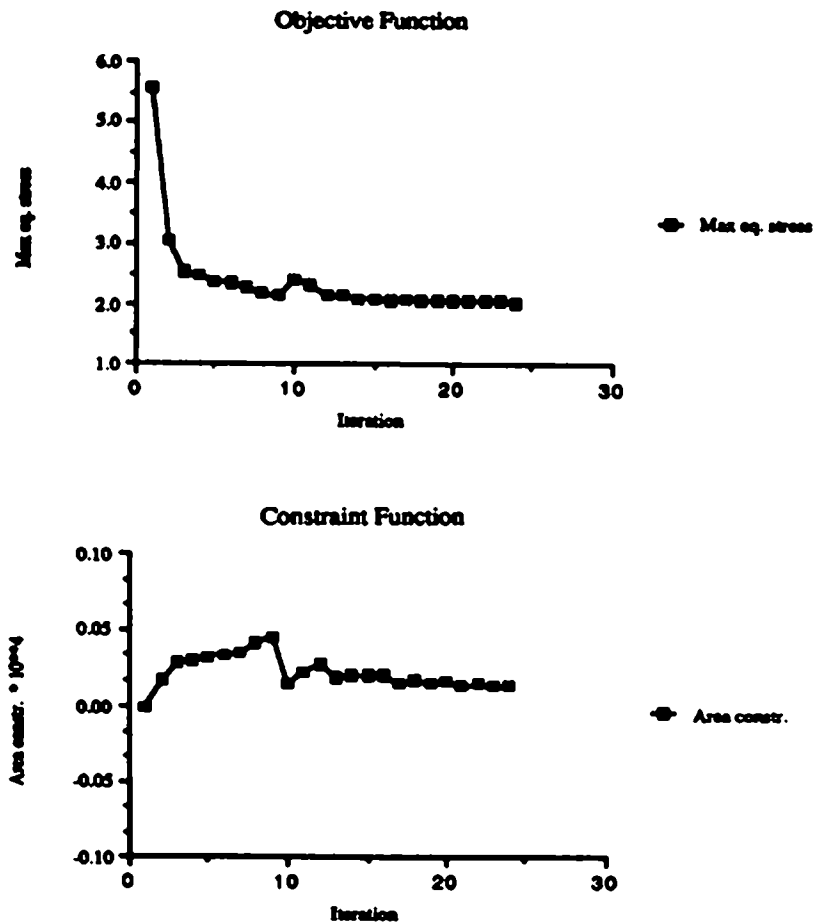

Fig. 6. Convergence of the iterative procedure for load $F1 = 1., F2 = 1.$

unstable. To avoid this problem the mutual energy term defined in (27) is only evaluated, in each element constituting the design boundary, at the respective middle node and it is assumed constant within each element. Following the previous comments and defining $U^I = \sigma : \varepsilon$, evaluated at the middle node of the ith design element with σ and ε obtained from (31) and (36), respectively, the optimality condition (27) can be approximated via

$$\sum_i (U^I + \Lambda) \int_{\Gamma_{d_i}} [(N_1(s) \, \delta d_i \cos \beta_i + N_2(s) \, \delta d_{i+1} \cos \beta_{i+1}) n_x$$

$$+ (N_1(s) \, \delta d_i \sin \beta_i + N_2(s) \, \delta d_{i+1} \sin \beta_{i+1}) n_y] \, d\Gamma = 0 . \tag{41}$$
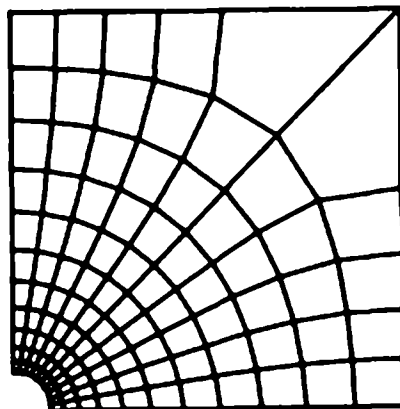
The design variables d are then computed iteratively, either based on a direct solution of the discrete optimality condition (41) or by employing well-known gradient-type algorithms. In this work the version of the linearization method due to Pshenichny (see e.g., [10]) was used.
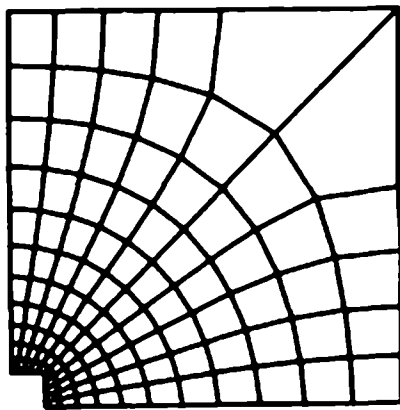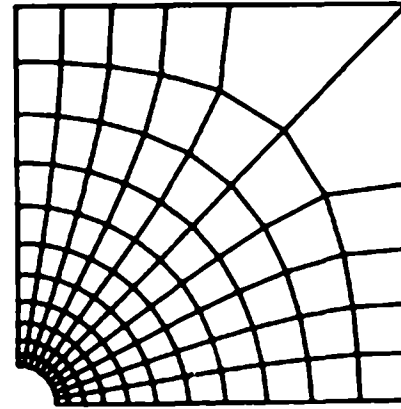

Iteration 1   Iteration 2
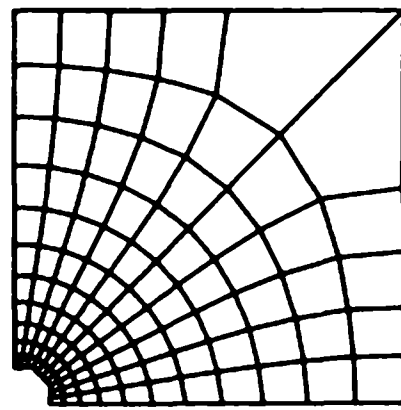
Iteration 5   Iteration 24

Fig. 7. Variation of the shape during iterative procedure.

5. Example problems

EXAMPLE 5.1. Results are presented for the computational prediction of the optimally shaped hole in a biaxially loaded sheet. Two load cases are treated. Details of the problem statement are provided in Fig. 5.

Iteration histories for the 'maximum equivalent stress' and the area constraint are provided for the different loads in Figs. 6 and 8. Evolutions of the hole design are shown in Figs. 7 and 9.

EXAMPLE 5.2. In the second example, for the two-dimensional structure supported and loaded as indicated in Fig. 10, the portion of boundary shown by heavy lines is fixed. Fillet design refers to the shape of the remaining part designated by Γ_d in the figure. Iteration histories for the 'maximum equivalent stress' and the area constraint are provided in Fig. 11. Evolution of the fillet shape is shown in Fig. 12.
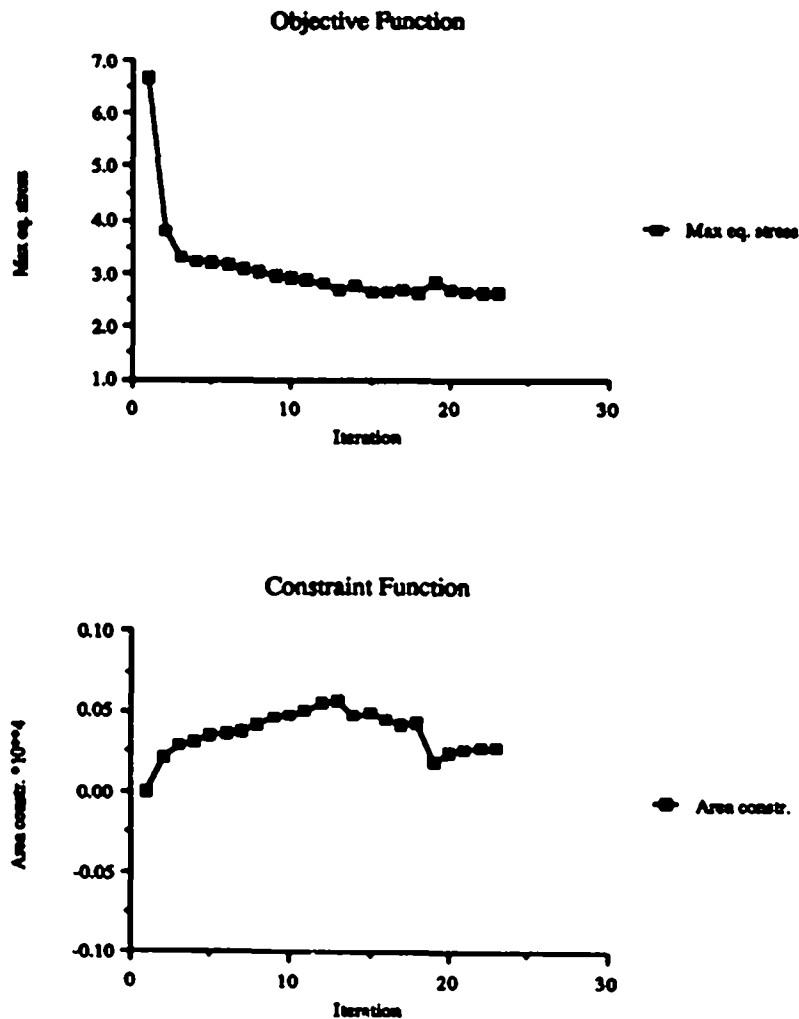
Fig. 8. Convergence of the iterative procedure for load $F1 = 1.$, $F2 = 1.5$.

Iteration 1   Iteration 2
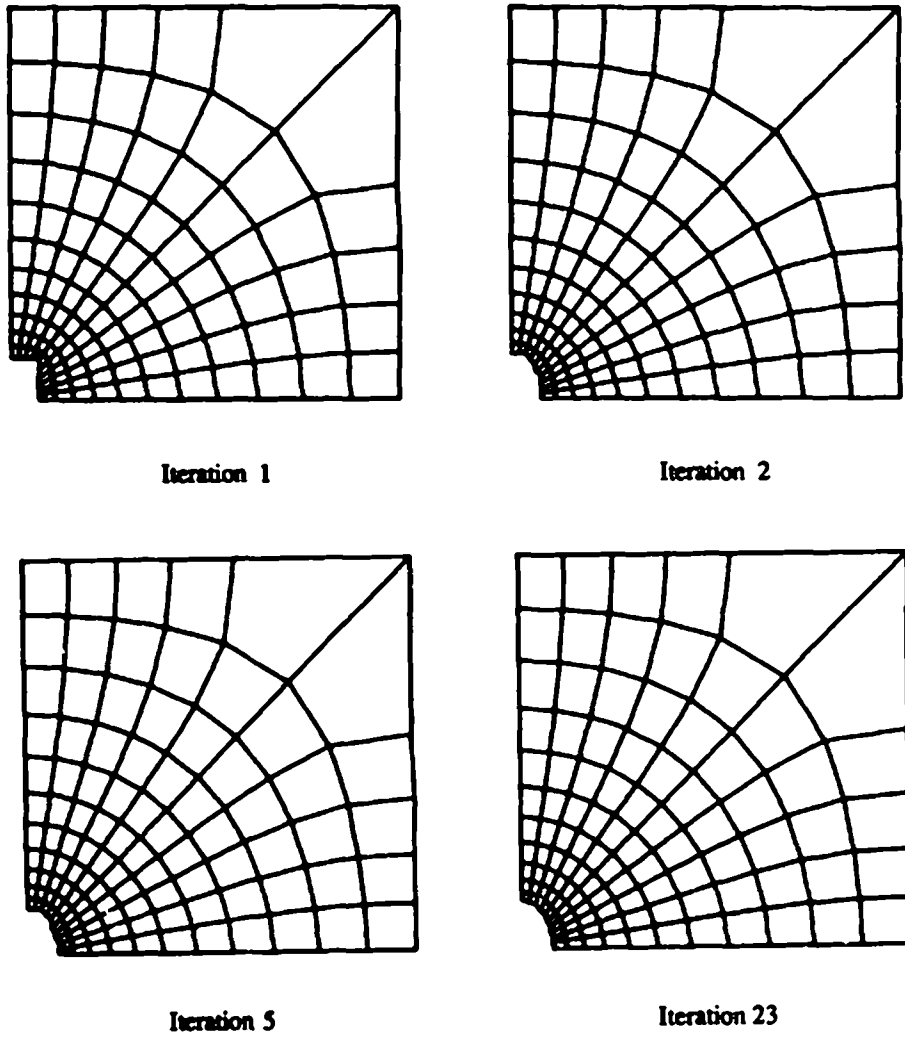
Iteration 5   Iteration 23

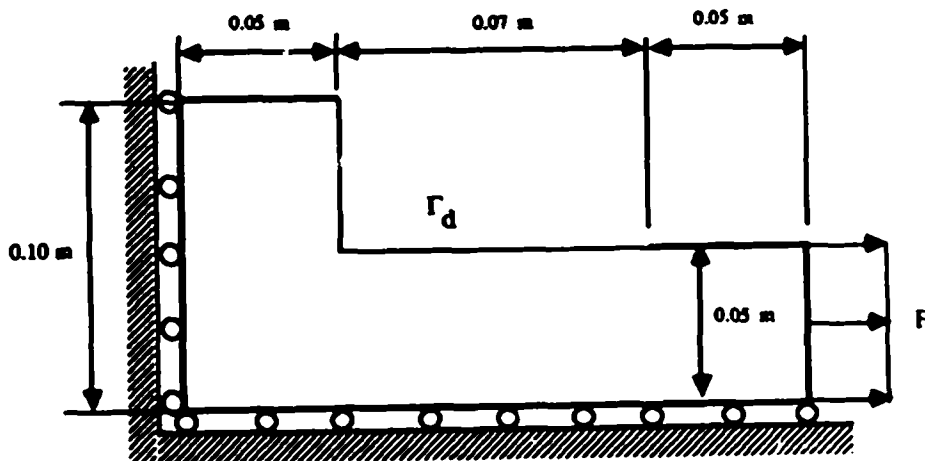Fig. 9. Variation of the shape during iterative procedure.

Fig. 10. Shape optimization of a fillet. Plane stress, 126 finite elements, 14 design variables, objective function the Von Mises equivalent stress, maximum admissable area for the fillet 1.135×10^{-2} m.
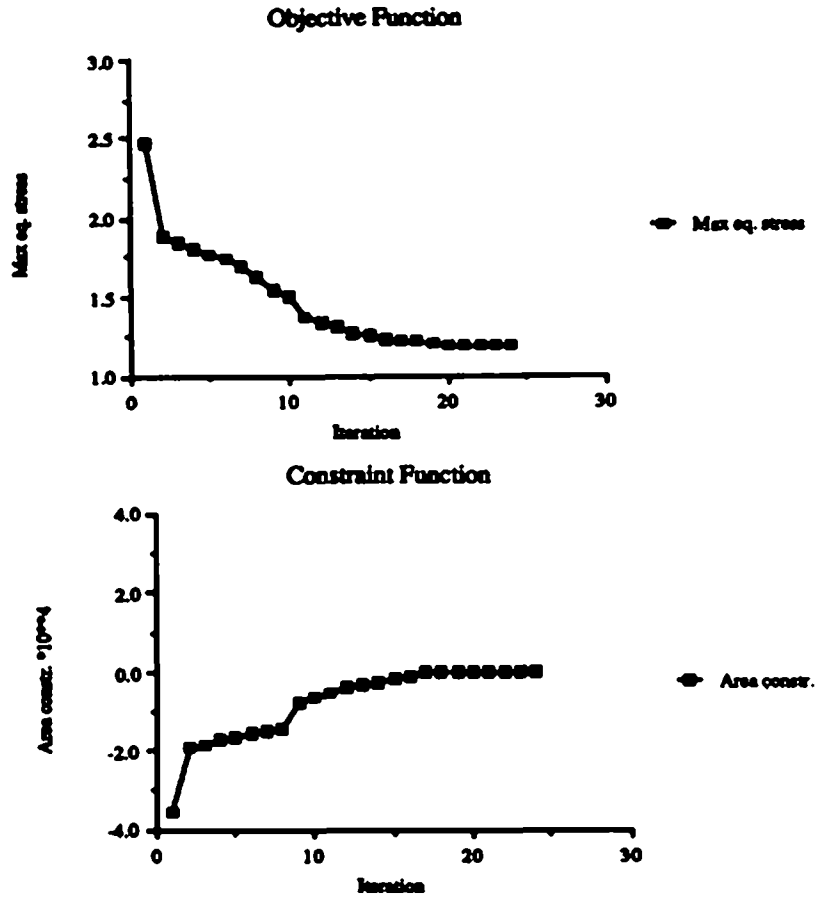
Objective Function

Constraint Function

Fig. 11. convergence of the iterative procedure, $F = 1$.

Iteration 1

Iteration 3
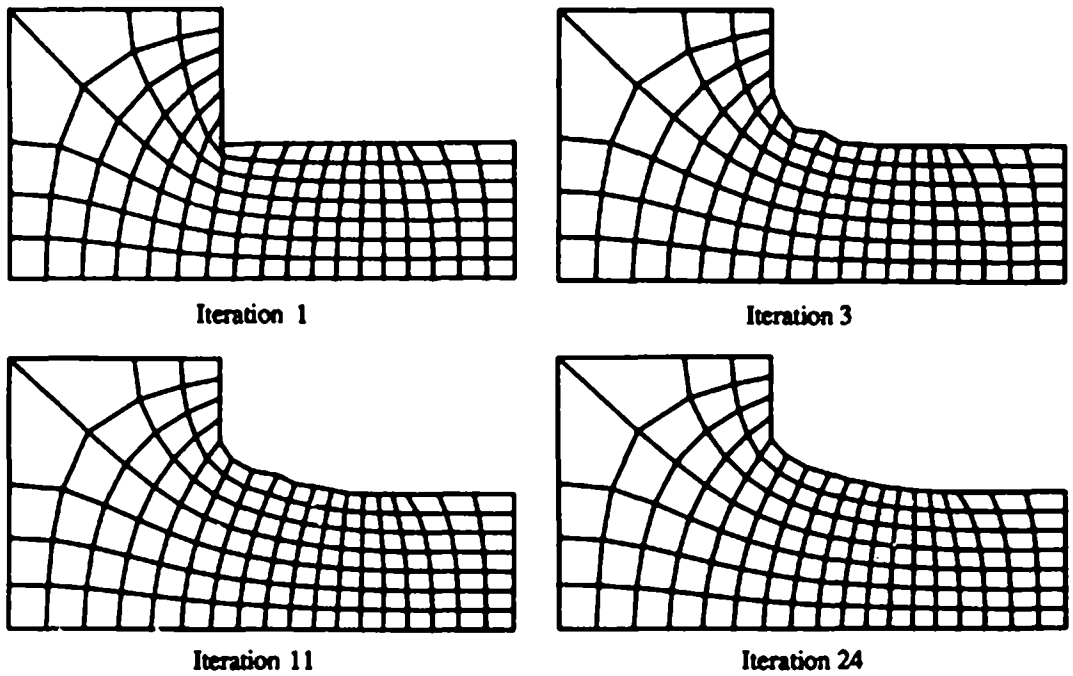

Iteration 11

Iteration 24

Fig. 12. Variation of the shape during iterative procedure.

6. Final remarks

The development presented introduces a quite general form of mixed formulation for the optimal shape design problem, and the associated optimality conditions are easily obtained without resorting to highly elaborate mathematical developments. Also the physical significance of the adjoint problem comes out to be clearly defined with this formulation.

In the examples presented, an elliptical automatic mesh generator assuring an orthogonal finite element mesh at the domain boundary (see, e.g. [7]) was used at each shape redesign. Although it might seem computationally to be a very expensive procedure, actually it guarantees a good accuracy for the discrete model with an increase on computational time of less than 5% of the actual time required for the finite element analysis.

At the same time from the numerical examples executed the procedure appears to be quite stable and problems commonly encountered in shape optimization arising from the development of instabilities in the design boundary definition were largely avoided. As is to be expected, this improvement is accomplished at the expense of the increase in cost of computation as compared to the simple displacement formulation.

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