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ON THE *K*-THEORY OF CURVES OVER FINITE FIELDS

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Let X be a smooth projective curve over a finite field. The main result is that the odd-dimensional K-theory of the extension of X to the algebraic closure is the sum of two copies of the K-theory of the field. Two plausible conjectures are advanced which would suffice to compute the K-theory of X itself. These provisional computations are then related to the L-functions of X.

Introduction

Let X be a smooth projective curve over a finite field F. Let J denote the Jacobian variety of X. Write $\bar{F}$ for the algebraic closure of F and $\bar{X} = X \times_F \bar{F}$. The main result of this paper is

Theorem 1. *If* $n \geq 0$, *then* $K_{2n+1}(\bar{X}) = K_{2n+1}(\bar{F}) \oplus K_{2n+1}(\bar{F})$.

The theorem is proved in the first section. The techniques used are a shameless exploitation of the work of Quillen, Soulé, and Suslin. Quillen's results on higher K-theory [9] and finite fields [8] form a solid and indispensible foundation. Soulé's study [11] of the K-theory of varieties over finite fields and Suslin's study [12] of the torsion in higher K-theory raise on this foundation an imposing edifice, from which vantage point the way to Theorem 1 can be clearly seen.

The remainder of the paper is built on a more conjectural foundation. The computation of the K-theory of X is reduced to certain properties of function fields.

Conjecture A. Let Y be a geometrically integral variety over a field k. Let $G = \mathrm{Gal}(\bar{k}/k)$. Then

$$(K_i(\bar{k}(Y))/K_i(\bar{k}))^G \approx K_i(k(Y))/K_i(k) .$$

Conjecture B. The K-theoretic product yields an isomorphism

$$\bar{F}(X)^* \otimes K_{2n-1}(\bar{F}) \to K_{2n}(\bar{F}(X)) .$$

Conjecture A was known 'classically' for $i = 0$, 1 and has been proven for $i = 2$ by Colliot–Thélène [2] and Suslin [13].

Conjecture B was proven for $n = 1$ by Tate [14] and generalized by Suslin [13]. Furthermore, it will be seen that Conjecture B is equivalent to an isomorphism

$$K_{2n}(\bar{X}) \approx \mathrm{Tor}(J(\bar{F}), K_{2n-1}(\bar{F}))\,.$$

The main consequence of these conjectures is

Theorem 2. *If both Conjectures* A *and* B *hold, then*
(i) $K_{2n+1}(X) \approx K_{2n+1}(F) \oplus K_{2n+1}(F)$;
(ii) $K_{2n}(X) \approx \mathrm{Tor}(J(\bar{F}), K_{2n-1}(\bar{F}))^G$.

Theorem 1 and Theorem 2(i) will be proven in the first section of the paper. Theorem 2(ii) will be proven in the second section. The second section concludes with a study of the relationship between the K-theory of X and its L-function. The groups computed by Theorem 2 have the orders predicted by the Quillen–Lichtenbaum conjectures [6, 10].

Notation. Let A be an abelian group, n a positive integer, l a prime number. Write
- ${}_nA$ = n-torsion subgroup,
- A_{tor} = subgroup of all torsion elements,
- $A\{l\}$ = l-primary torsion subgroup.

1. Odd-dimensional K-groups

Let X be a smooth projective geometrically integral curve over a field F. Quillen [9] has constructed an exact localization sequence

$$\cdots \to K_n(X) \to K_n(F(X)) \to \coprod_{x \in X_0} K_{n-1}(F(x)) \to K_{n-1}(X) \to \cdots$$

where $F(X)$ is the function field of X and, for each closed point x, $F(x)$ is its residue field.

Let $\mathcal{K}_n$ denote the Zariski sheaf on X associated to the presheaf $U \to K_n(U)$. Then Quillen [9] has also constructed an acyclic resolution

$$0 \to \mathcal{K}_n \to \eta_* K_n(F(X)) \to \coprod_x i_* K_{n-1}(F(x)) \to 0\,.$$

Consequently, the localization sequence can be decomposed into two different flavors of shorter exact sequences:

(A_n) $\qquad 0 \to \Gamma(X, \mathcal{K}_n) \to K_n(F(X)) \to \coprod K_{n-1}(F(x)) \to H^1(X, \mathcal{K}_n) \to 0\,,$

(B_n) $\qquad 0 \to H^1(X, \mathcal{K}_{n+1}) \to K_n(X) \to \Gamma(X, \mathcal{K}_n) \to 0\,.$

For the remainder of this section, let F be a finite field with algebraic closure $\bar{F}$. Write $\bar{X}$ for the curve $X \times_{\mathrm{spec}\, F} \mathrm{Spec}\, \bar{F}$ obtained by base extension.

As a consequence of the work of Quillen and Soulé, sequences (A_n) and (B_n) simplify considerably. There is, however, a difference depending on whether n is odd or even. Let $n \geq 1$. Quillen's proof [8] that $K_{2n}(F) = 0$ implies

(A_{2n+1}) $\qquad \Gamma(X, \mathcal{K}_{2n+1}) = K_{2n+1}(F(X))\,.$

Now Soulé [11, Proposition 3] has shown that $H^1(X, \mathcal{K}_{n+1}) \approx K_n(F)$. Combined with (A_{2n+1}), this yields

(B_{2n+1}) $\qquad 0 \to K_{2n+1}(F) \to K_{2n+1}(X) \to K_{2n+1}(F(X)) \to 0\,.$

Using Soulé's result in the case of the even-dimensional K-groups, one has

(B_{2n}) $\qquad K_{2n}(X) = \Gamma(X, \mathcal{K}_{2n})\,.$

Therefore

(A_{2n}) $\qquad 0 \to K_{2n}(X) \to K_{2n}(F(X)) \to \coprod K_{2n-1}(F(x)) \to K_{2n-1}(F) \to 0\,.$

By passing to the direct limit over finite extensions of F, one obtains the same sequences for $\bar{X}$ over $\bar{F}$. For the sake of completeness, note also that

$$K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)\,,$$

$$K_1(X) = F^* \oplus F^*\,.$$

Lemma 1.1. *Let X be a curve over either a finite field or its algebraic closure. Let E be the function field of X. If $n \geq 2$, then $K_n(X)$ and $K_n(E)$ are torsion groups.*

Proof. Harder [5] showed that $K_n(X)$ is finite for X defined over a finite field and $n \geq 1$. The result follows for X over the algebraic closure by passage to the direct limit. Finally, the result follows for function fields from sequences (A_{2n}) and (B_{2n+1}). $\square$

Proposition 1.2. *Let X be a smooth projective curve over an algebraically closed field L. Let $E = \overline{L(X)}$ be the algebraic closure of its function field. Then there is an injection*

$$\Gamma(X_L, \mathcal{K}_n)_{\mathrm{tor}} \hookrightarrow \Gamma(X_E, \mathcal{K}_n)_{\mathrm{tor}}\,.$$

Proof. Write $E = \varinjlim A$ where $A \supset L(X)$ is a finite algebraic extension field. Since the exact sequences (A_n) and (B_n) are stable under base change and K-theory commutes with direct limits, it suffices to show that

$$\Gamma(X_L, \mathcal{K}_n)_{\mathrm{tor}} \hookrightarrow \Gamma(X_A, \mathcal{K}_n)_{\mathrm{tor}}\,.$$

For K-theory with any finite coefficients, Suslin [12] has constructed specialization maps

$$K_n(X_A; \mathbb{Z}/r) \to K_n(X_L; \mathbb{Z}/r)$$

which split off the K-theory of X_L. In particular,

$$K_n(X_L)_{\mathrm{tor}} \hookrightarrow K_n(X_A)_{\mathrm{tor}}\,.$$

Since the sequences (B_n) split compatibly with base change [11], the result follows. □

Theorem 1.3. *Let X be a smooth projective curve over a finite field F. Then*

(i) $K_{2n+1}(\bar{X}) = K_{2n+1}(\bar{F}) \oplus K_{2n+1}(\bar{F})$;

(ii) $K_{2n+1}(\bar{F}(X)) = K_{2n+1}(\bar{F})$.

Proof. The two parts are equivalent by sequence (B_{2n+1}). Write E for the algebraic closure of the function field $\bar{F}(X)$. There is a commutative diagram

$$\begin{array}{ccccc}
K_{2n+1}(\bar{F}) & \xrightarrow{\ \alpha\ } & K_{2n+1}(\bar{F}(X)) & \xrightarrow{\ \approx\ } & \Gamma(X_{\bar{F}}, \mathcal{K}_{2n+1}) \\
 & \swarrow_{\beta} & & & \Big\downarrow \text{base change} \\
K_{2n+1}(E) & & \xrightarrow{\qquad\qquad \gamma \qquad\qquad} & & \Gamma(X_E, \mathcal{K}_{2n+1})
\end{array}$$

where the isomorphism comes from (A_{2n+1}).

By Suslin [12], the composite $\beta\alpha$ is an isomorphism on torsion. Since $K_{2n+1}(\bar{F})$ is all torsion, α is an injection. By Proposition 1.2, the composite $\gamma\beta$ is also an injection on torsion. When $n \geq 1$, Lemma 1.1 shows that all of $K_{2n+1}(E)$ is torsion. So, β is an injection. Finally, since $\beta\alpha$ is also surjective on torsion, α must be surjective on torsion and hence surjective. Since the case $n = 0$ has been noted earlier, the theorem is proved. □

Corollary 1.4. *Let A be a smooth affine curve over a finite field F. If $n \geq 1$, then*

$$K_{2n+1}(\bar{A}) = K_{2n+1}(\bar{F})\,.$$

Proof. Let X be a smooth completion. The corollary follows from the localization sequence

$$\coprod_{x \in X - A} K_{2n+1}(\bar{F}(x)) \xrightarrow{N} K_{2n+1}(\bar{X}) \to K_{2n+1}(\bar{A}) \to 0$$

and the fact that the norm maps are surjections

$$N: K_{2n+1}(\bar{F}(x)) \twoheadrightarrow K_{2n+1}(\bar{F}) = H^1(\bar{X}, \mathcal{K}_{2n+2}) . \qquad \square$$

Corollary 1.5. *Assume Conjecture* A *holds when* $i = 2n + 1$, $Y = X$ *is a curve, and* $k = F$ *is a finite field. Then*

$$K_{2n+1}(X) \approx K_{2n+1}(F) \oplus K_{2n+1}(F) .$$

Proof. Using exact sequence (B_{2n+1}), one sees that the corollary is equivalent to showing that there is an isomorphism $K_{2n+1}(F(X)) \approx K_{2n+1}(F)$. By Theorem 1.3(ii), this isomorphism holds over E. The conjecture then implies that it also holds over F. $\square$

Remark. (i) Conjecture A is trivially true when $i = 0$, since both sides are zero. If $i = 1$, the conjecture follows from $K_1(F) = F^*$ and Hilbert's Theorem 90. The conjecture has been proven for $i = 2$ by Colliot–Thélène [2] under mild hypotheses and in general by Suslin [13]. Their proofs rely on the Merkurjev and Suslin [7] version of Hilbert's Theorem 90 for K_2. Conjecture A should, therefore, be related to a generalization of Hilbert's Theorem 90 to higher K-theory.

(ii) The conjecture must be made in the context of a quotient of K-groups. In general, the map $K_n(F) \to K_n(E)^G$ is not an isomorphism. For example, take $n = 2$, $F = \mathbb{Q}$, $E = \mathbb{Q}(i)$. Then $\{-1, -1\}$ is a nontrivial element of the kernel.

(iii) Corollary 1.5 would also follow if the conjecture were only known for the torsion subgroup of K_i. It is likely that it would suffice equally well to prove the conjecture for K-theory with finite coefficients.

2. Even dimensional K-groups

In this section, we study the even-dimensional K-groups of a curve X over a finite field. First, Conjecture B will be used to compute the K-theory of $\bar{X}$. Then Conjecture A will be used to descend to X. Finally, the orders of these K-groups will be compared with special values of the L-functions of X. Throughout the section, write $Y = \bar{X}$ and $k = \bar{F}$.

Proposition 2.1. *Assume Conjecture* B *holds. Then*

$$K_{2n}(Y) \approx \mathrm{Tor}(J(k), K_{2n-1}(k)) .$$

Proof. To use the multiplicative structure of K-theory, it is necessary to study the effect on

$$(A_1) \qquad 0 \to k^* \to k(Y)^* \to \coprod \mathbb{Z} \to \mathrm{Pic}(Y) \to 0$$

of tensoring with $K_{2n-1}(k)$. There is also an exact sequence

$$(*) \qquad 0 \to J(k) \to \mathrm{Pic}(Y) \to \mathbb{Z} \to 0$$

where J is the Jacobian variety of Y. All the groups k^*, $J(k)$, and $K_{2n-1}(k)$ are divisible torsion groups. Therefore tensoring $(*)$ with $K_{2n-1}(k)$ yields the pair of isomorphisms

$$\mathrm{Pic}(Y) \otimes K_{2n-1}(k) \approx K_{2n-1}(k)\,,$$

$$\mathrm{Tor}(J(k), K_{2n-1}(k)) \simeq \mathrm{Tor}(\mathrm{Pic}(Y), K_{2n-1}(k))\,.$$

Next, introduce the group $D(Y) = k(Y)^*/k^*$ of principal divisors on Y. Then

$$k(Y)^* \otimes K_{2n-1}(k) \approx D(Y) \otimes K_{2n-1}(k)\,.$$

Finally, tensoring the short exact sequence

$$0 \to D(Y) \to \coprod \mathbb{Z} \to \mathrm{Pic}(Y) \to 0$$

with $K_{2n-1}(k)$ and using the above isomorphisms, one obtains a four-term exact sequence which can be compared with (A_{2n}):

$$\begin{array}{ccccccccc}
0 \to \mathrm{Tor}(J(k), K_{2n-1}(k)) & \to & k(Y)^* \otimes K_{2n-1}(k) & \to & \coprod K_{2n-1}(k) & \longrightarrow & K_{2n-1}(k) \to 0 \\
\downarrow & & \downarrow m & & \| & & \| \\
0 \to K_{2n}(Y) & \longrightarrow & K_{2n}(k(Y)) & \longrightarrow & \coprod K_{2n-1}(k(y)) & \to & K_{2n-1}(k) \to 0
\end{array}$$

The proposition follows. □

Remarks. (i) When $n = 1$, Tate [14] has proven Conjecture B. Many of the consequences to be drawn from this conjecture are based on his arguments in [14].

(ii) Since Quillen [8] has shown that $K_{2n-1}(k)(l) \approx \mathbb{Q}_l/\mathbb{Z}_l(n)$, one might look for a version of this conjecture computing torsion over any algebraically closed field k. Suslin [13] has proven such a generalization of Tate's result when $n = 1$.

Theorem 2.2. *If both Conjectures* A *and* B *hold, then*

$$K_{2n}(X) \approx \mathrm{Tor}(J(\bar{F}), K_{2n-1}(\bar{F}))^G\,.$$

Proof. Using Proposition 2.1, it suffices to show that $K_{2n}(X) \approx K_{2n}(\bar{X})^G$. Since $K_{2n}F = K_{2n}\bar{F} = 0$, it follows from Conjecture A that $K_{2n}(F(X)) = H^0(G, K_{2n}(\bar{F}(X)))$.

Write $C = K_{2n}(\bar{F}(X))/K_{2n}(\bar{X})$. There are exact sequences

$$0 \to H^0(G, K_{2n}(\bar{X})) \to K_{2n}(F(X)) \to H^0(G, C) \to H^1(G, K_{2n}(\bar{X})) \to \cdots$$

and

$$0 \to H^0(G, C) \to \coprod_x K_{2n-1}F(x) \to K_{2n-1}F \to 0$$

obtained by taking the Galois cohomology of (A_{2n}) over $\bar{F}$. Then the result follows by comparing these sequences with (A_{2n}) over F. □

In order to get a more complete description of $K_{2n}(X)$, it is useful to keep track of the action of $G = \mathrm{Gal}(\bar{F}/F)$. Recall [3, 6, 11, 14] the definitions of the standard l-adic Galois modules

$$T_l = \varprojlim {}_{l^n}J(\bar{F}),$$

$$V_l = T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l = \varinjlim {}_{l^n}J(\bar{F}).$$

$$\mathbb{Z}_l(1) = \varprojlim \mu_{l^n}(\bar{F}),$$

$$\mathbb{Z}_l(n) = \mathbb{Z}_l(1) \otimes \cdots \otimes \mathbb{Z}_l(1) \quad (n \text{ copies}).$$

For any l-adic Galois module M, let $M(n) = M \otimes \mathbb{Z}_l(n)$. Also, write $W_l = \mathbb{Q}_l/\mathbb{Z}_l$ so that

$$W_l(1) = \mathbb{Q}_l/\mathbb{Z}_l(1) = \bar{F}^*\{l\}.$$

Lemma 2.3. $\mathrm{Tor}(J(\bar{F}), K_{2n-1}(\bar{F}))\ \{l\} = V_l(n)$.

Proof. Quillen's computation [8] of the K-theory of finite fields says that $K_{2n-1}(\bar{F})\{l\} = W_l(n)$. Since $\mathrm{Tor}(J(\bar{F}), W_l) = V_l$, the result follows. □

Proposition 2.4. *Assume both Conjectures* A *and* B *hold. Let X be a smooth projective curve over a finite field F with $q = p^s$ elements. Let $l \neq p$ be prime. Let f denote the Frobenius endomorphism of T_l. Then*

$$K_{2n}(X)\{l\} = T_l/(1 - fq^n)T_l.$$

Proof. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & T_l & \to & T_l \otimes \mathbb{Q}_l & \to & V_l & \to 0 \\ & \downarrow{\scriptstyle 1-fq^n} & & \downarrow{\scriptstyle 1-fq^n} & & \downarrow{\scriptstyle 1-fq^n} & \\ 0 \to & T_l & \to & T_l \otimes \mathbb{Q}_l & \to & V_l & \to 0 \end{array}$$

By Deligne's proof [3] of the Weil conjectures, the middle vertical arrow is an isomorphism. So

$$
\begin{aligned}
T_l/(1-fq^n)T_l &= \operatorname{Ker}(1-fq^n : V_l \to V_l) \\
&= \operatorname{Ker}(1-\mathrm{Frob}: V_l(n) \to V_l(n)) \\
&= H^0(G, V_l(n)) = H^0(G, \operatorname{Tor}(J, K_{2n-1})) \\
&= H^0(G, K_{2n}(\bar{X})\{l\}) \\
&= K_{2n}(X)\{l\} . \qquad \square
\end{aligned}
$$

Let ϕ be the action of the geometric Frobenius [3] on $H^1(\bar{X}, \mathbb{Q}_l)$. The L-function of X is defined as

$$L(X, s) = P(X, q^{-s})$$

where

$$P(X, t) = \det(1 - \phi t) .$$

For any pair of rational numbers a, b, write $a \sim b$ to mean a/b is a power of p.

Corollary 2.5. *Assume both Conjectures* A *and* B *hold. Then* $L(X, n+1) \sim \# K_{2n}(X)$.

Proof. By the functional equation, $L(X, n+1) \sim L(X, -n)$. It is a standard fact that

$$P(X, t) = \det(1 - ft \mid T_l) .$$

So, $L(X, -n) = \det(1 - fq^n \mid T_l)$. The l-part of the L-function is therefore given by $\# T_l/(1 - q^n f) = \# K_{2n}(X)\{l\}$. $\square$

Remark. This is precisely the relation between K-theory and L-functions predicted by the Quillen–Lichtenbaum conjectures [6, 10].

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