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A SURVEY OF THE THEORY OF HYPERCUBE GRAPHS

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Abstract—We present a comprehensive survey of the theory of hypercube graphs. Basic properties related to distance, coloring, domination and genus are reviewed. The properties of the n-cube defined by its subgraphs are considered next, including thickness, coarseness, Hamiltonian cycles and induced paths and cycles. Finally, various embedding and packing problems are discussed, including the determination of the cubical dimension of a given cubical graph.

1. INTRODUCTION

The *n-cube* or *n-dimensional hypercube* Q_n is defined recursively in terms of the cartesian product [1, p. 22] of two graphs as follows:

$$Q_1 = K_2$$
$$Q_n = K_2 \times Q_{n-1}. \tag{1}$$

Thus the n-cube, or more briefly the *cube*, Q_n may also be defined as a graph whose node set V_n consists of the 2^n n-dimensional boolean vectors, i.e., vectors with binary coordinates 0 or 1, where two nodes are adjacent whenever they differ in exactly one coordinate. Figure 1 shows the n-cubes for $n \leqslant 3$ with appropriate boolean vectors as node labels. Cube graphs have been much studied in graph theory. Interest in hypercubes has been increased by the recent advent of massively parallel computers whose structure is that of the hypercube [2, 3]. This not only provides potential applications for the existing theory, but also suggests some new aspects of cubes that deserve study.

We survey the graph-theoretic literature on n-cubes, and present a summary of the major known results. Some new problems, motivated in part by parallel computing considerations, are also presented. Emphasis is placed on the special properties and subgraphs of cubes, as well as the problems of embedding and packing graphs in cubes. Most of the notation used may be found in Harary [1]. A graph $G = (V, E)$ has $p = |V|$ nodes and $q = |E|$ edges, and is said to have *order* p and *size* q. Thus, the order of Q_n is 2^n and its size is $n\ 2^{n-1}$.

2. BASIC PROPERTIES

We begin by surveying some invariants of hypercubes related to the distance between two nodes. This distance is the number of coordinates in which the corresponding boolean vectors differ. The

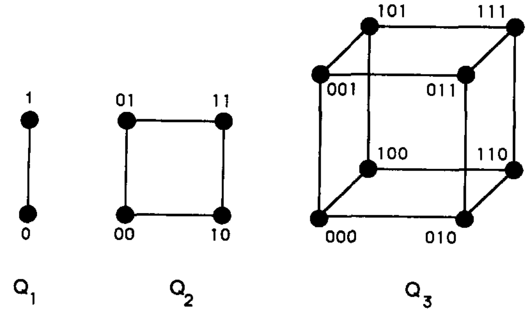


Fig. 1. n-cube graphs for $n = 1, 2, 3$.

diameter $d(G)$ of a graph G is the maximum distance between any pair of nodes; obviously $d(Q_n) = n$. The *total distance* of graph G with node set $V = \{v_1, v_2, \ldots, v_p\}$ is

$$\mathrm{td}(G) = \sum_{1 \leqslant i < j \leqslant p} d(v_i, v_j).$$

Clearly, $td(Q_n)$ may be calculated in the following way. Let Q_n be partitioned into two node-disjoint $(n-1)$-cubes Q_{n-1} and Q'_{n-1}. Let v_i be any node in Q_{n-1}, and let v'_i be its neighbor in Q'_{n-1}. We denote the total distance from v_i to all nodes of Q_n by

$$\mathrm{td}_i(Q_n) = \sum_j d(v_i, v_j).$$

Now the total distance from v_i to all nodes in Q_{n-1} is thus $\mathrm{td}_i(Q_{n-1})$ and the distance from v_i to any node v'_j in Q'_{n-1} is $d(v'_i, v'_j) + 1$, since the boolean vectors corresponding to v_i and v'_j must differ in exactly $d(v'_i, v'_j) + 1$ coordinates. Hence the total distance from v_i to all 2^{n-1} nodes of Q'_{n-1} is

$$\sum_j [d(v'_i, v'_j) + 1] = \mathrm{td}_i(Q_{n-1}) + 2^{n-1}.$$

This implies that

$$\mathrm{td}_i(Q_n) = 2\mathrm{td}_i(Q_{n-1}) + 2^{n-1},$$

from which it follows by induction that $\mathrm{td}_i(Q_n) = n2^{n-1}$. Now

$$\sum_i \mathrm{td}_i(Q_n) = 2^n \mathrm{td}_i(Q_n) = 2\mathrm{td}(Q_n)$$

as the summation counts the distance between every pair of nodes twice. Consequently,

$$\mathrm{td}(Q_n) = n2^{2n-2}. \tag{2}$$

A related invariant is the *average distance* $\bar{d}(G)$ which is defined as follows:

$$\bar{d}(G) = \frac{\mathrm{td}(G)}{\binom{p}{2}}.$$

Since $\mathrm{td}(Q_n) = n2^{2n-2}$ and $p = 2^n$, the average distance of the n-cube is given by

$$\bar{d}(Q_n) = \frac{n2^{n-1}}{2^n - 1} \tag{3}$$

which rapidly approaches $n/2$ as n increases.

The *(node) connectivity* $\kappa(G)$ is the minimum number of nodes whose removal results in a disconnected or trivial graph; the *edge connectivity* $\lambda(G)$ is defined similarly. We note that for a cube, $\kappa(Q_a) = \lambda(Q_n) = n$. This follows since for any graph G with minimum degree δ, we have $k \leqslant \lambda \leqslant \delta$, so $\kappa(Q_n) \leqslant \lambda(Q_n) \leqslant n$. It is also obvious that deleting any $n-1$ nodes or edges from Q_n will not disconnect it. Hence $\kappa(Q_n)$ and $\lambda(Q_n)$ are both n.

A *coloring* of a graph is an assignment of colors to its nodes so that no two adjacent nodes have the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors in any coloring of G. The fact that $\chi(Q_n) = 2$ follows from the theorem of König that a graph is bicolorable if and only if it has no odd cycles. Alternatively, let the *weight* of a node x of Q_n with boolean label $x_1 x_2 \ldots x_n$ be the integer

$$\sum_i x_i.$$

Then color the nodes of even weight with the first color and those of odd weight with the second color.

The concepts of domination and independence are closely related. A node or edge is said to *dominate* or *cover* the nodes or edges with which it is incident or adjacent. Four invariants of G can be derived from this definition. The minimum number of nodes that dominate the whole node set is the *node–node domination number* α_{00} of the graph, usually called more briefly the *domination* *number* of G. The *node–edge*, *edge–node* and *edge–edge domination numbers* can be similarly defined and are denoted by α_{01}, α_{10} and α_{11}, respectively. For example, $\alpha_{10}(G)$ is the minimum number of edges that dominate all nodes of G. Two nodes or two edges are *independent* if they are not dominated by each other. A set S of nodes or edges of G is independent if any pair of nodes or edges in S are independent in G. The maximum cardinality of such a set is the *node independence number* β_0 or *edge independence number* β_1. Gallai [4] proved that for any nontrivial graph G,

$$\alpha_{01} + \beta_0 = p = \alpha_{10} + \beta_1.$$

This result leads directly to general formulas for α_{10} and α_{01} in a cube, and hence for β_0 and β_1,

$$\alpha_{01}(Q_n) = \alpha_{10}(Q_n) = \beta_0(Q_n) = \beta_1(Q_n) = 2^{n-1}. \tag{4}$$

The following related result communicated to us by Q. F. Stout is known for $\alpha_{00}(Q_n)$. For the special case $n = 2^k - 1$, we have $\alpha_{00}(Q_n) = 2^n/n + 1$. The maximum number of node-disjoint copies of a star $S = K(1, 2^k - 1)$ that can be embedded into Q_n is 2^{n-k}. Note that the center of S dominates all other nodes of S. If the maximum number of node-disjoint copies of S is embedded in a hypercube, then their centers form a minimum dominating set. However, $\alpha_{00}(Q_n)$ is not known when $n \neq 2^k - 1$.

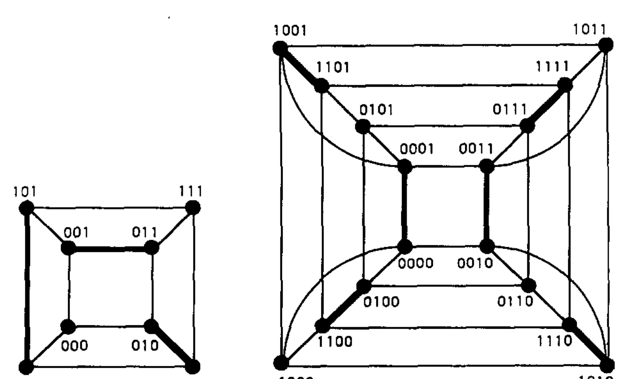
The exact determination of $\alpha_{11}(Q_n)$ is even more difficult. Obviously $\alpha_{11}(Q_2) = 2$. The heavy lines in Fig. 2 mark minimal sets of edges that dominate all edges of Q_3 and Q_4, proving that $\alpha_{11}(Q_3) = 3$ and $\alpha_{11}(Q_4) = 6$. Stout has also informed us of the following bounds for $n \geqslant 3$:

$$n2^n/(3n - 1) \leqslant \alpha_{11}(Q_n) \leqslant 3(2^{n-3}). \tag{5}$$

Equation (4) states that $\beta_1(Q_n) = 2^{n-1}$, so that the maximum number of independent edges in Q_n is one-half the number of nodes. In [5], the minimum number of edges in a maximal independent set, denoted by $\beta_1^-(G)$, was introduced. Clearly, for any graph G, $\alpha_{11}(G) = \beta_1^-(G)$. Thus Fig. 2(a) also illustrates the fact that $\beta_1^-(Q_3) = 3$.

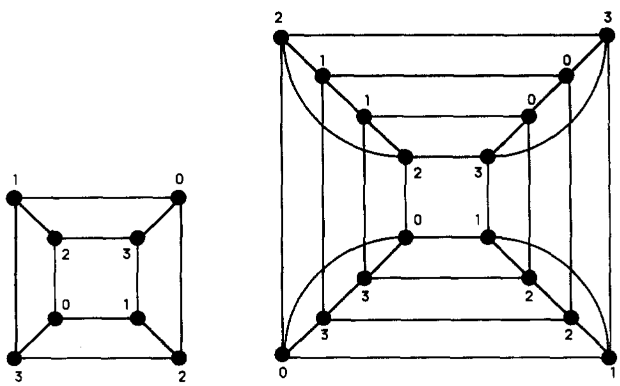
We can extend the definition of domination to arbitrary subsets V_1 and V_2 of $V(G)$. We say that V_1 dominates V_2 if for any $y \in V_2$, there exists $x \in V_1$ which dominates y. A *D-partition* of G is then defined as a partition of $V(G)$ into dominating sets. An invariant similar to the domination number was introduced in [6]. It is concerned with the order of the D-partition instead of the order of the dominating set. The *domatic number* $\tau(G)$ is the maximum order of the D-partitions of G. Figure 3 displays D-partitions of maximum order 4, where the nodes are labeled as residue classes mod 4. Clearly, D-partitions can be equivalently defined by a coloring procedure, called *domatic coloring*, where each node colored by a certain color is adjacent to nodes colored by all the other colors, and $\tau(G)$ is the maximum number of colors used. In [6], the theory of domination of a graph is reviewed in detail, and it is shown to be related to several fields of study. For example, a matching of a graph G corresponds to an independent dominating set of nodes in the line graph of G. One obvious observation is that $\tau(G) \leqslant \delta(G) + 1$. The following result on $\tau(Q_n)$ is due to Zelinka [7]: for all positive integers k,

$$\tau(Q_{2^k-1}) = \tau(Q_{2^k}) = 2^k. \tag{6}$$



(a) (b)

Fig. 2. Minimal sets of edges (heavy lines) that dominate all edges of Q_3 and Q_4.



(a) (b)

Fig. 3. Domatic 4-colorings of Q_3 and Q_4.

Although the domatic numbers for cubes of other dimensions are not known, Zelinka [7] conjectured that if $n+1$ is not a power of 2, than $\tau(Q_n) = n$.

3. TOPOLOGICAL INVARIANTS

The *genus* $\gamma(G)$ is the minimum number of handles which must be added to a sphere so that G can be embedded in the resulting surface with no edges crossing. Thus a graph G is *planar* when $\gamma(G) = 0$, and is called *toroidal* if $\gamma(G) = 1$. The genus of a cube was first found by Ringel [8] and later was independently rediscovered by Beineke and Harary [9]. It is evident from Fig. 2 that $\gamma(Q_3) = 0$ and $\gamma(Q_4) = 1$. From Euler's characteristic equation for spherical polyhedra, we know that for a polyhedron of genus γ with V nodes, E edges and F faces,

$$V - E + F = 2 - 2\gamma. \tag{7}$$

If G is a connected graph of genus γ and has no triangles then, as mentioned in [9], the following inequality is implied by equation (7):

$$\gamma(G) \geqslant \frac{q}{4} - \frac{p-2}{2}.$$

Hence,

$$\gamma(Q_n) \geqslant \frac{n2^{n-1}}{4} - \frac{2^n - 2}{2} = (n-4)2^{n-3} + 1,$$

and we denote the latter expression by γ_n. Embeddings of Q_n in an orientable surface of genus γ_n are constructed in [8, 9], proving that $\gamma(Q_n) \leqslant \gamma_n$, therefore

$$\gamma(Q_n) = (n-4)2^{n-3} + 1. \tag{8}$$

The *crossing number* $\nu(G)$ of G is defined as the minimum number of pairwise intersections of its edges when G is drawn in the plane. The determination of the exact value of $\nu(G)$ is known to be NP-complete. It has been much studied for complete graphs and complete bigraphs but only upper bounds (which most likely give the exact values) are established [3, 10]. All that is known is

$$\nu(Q_3) = 0$$

as Q_3 is planar, and

$$\nu(Q_4) = 8$$

as illustrated in Fig. 2(b). It is easy to draw Q_5 in the plane with 56 crossings; hence

$$\nu(Q_5) \leqslant 56.$$

The determination of the exact value of $\nu(Q_n)$ for general n is a fiendishly intractable problem.

Two properties, thickness and coarseness, are defined by the planar subgraphs of a graph. The *thickness* $\theta(G)$ is the minimum number of planar subgraphs whose union is G. Since the maximal planar graph which has no triangles has $q = 2p - 4$ edges, it follows that

$$\theta(Q_n) \geqslant \left\lceil \frac{q}{2p-4} \right\rceil = \left\lceil \frac{n2^{n-1}}{2^{n+1}-4} \right\rceil = \left\lceil \frac{n}{4-2^{3-n}} \right\rceil .$$

Obviously, $\theta(Q_1) = \theta(Q_2) = \theta(Q_3) = 1$ as Q_3 is planar. But $\theta(Q_4) \neq 1$, so

$$\theta(Q_n) \geqslant \left\lceil \frac{n+1}{4} \right\rceil .$$

Kleinert [11] has shown by construction that Q_{4h-1} can be partitioned into h planar subgraphs. Hence for all n,

$$\theta(Q_n) \leqslant \left\lceil \frac{n+1}{4} \right\rceil .$$

From these two bounds, Kleinert concluded that

$$\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil \tag{9}$$

The *coarseness* $\xi(G)$, is defined as the maximum number of edge-disjoint nonplanar subgraphs of G. Hartman [12] has determined bounds for $\xi(Q_n)$. The upper bound was established by a homeomorphic embedding of $K_{3,3}$ into Q_4, also called a *cubical refinement of* $K_{3,3}$. The embedded version of $K_{3,3}$ in Q_4 has 14 edges as shown in Fig. 4(a). Since $K_{3,3}$ is not planar,

$$\xi(Q_n) \leqslant \left\lfloor \frac{n2^{n-2}}{7} \right\rfloor .$$

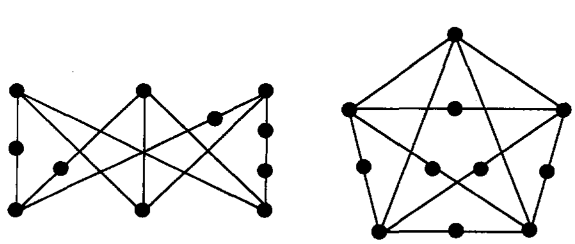
Hartman also proved the following inequality indicating how forming the iterated cartesian product $G^n = G \times G \times \cdots \times G$ of a graph G with p nodes increases the coarseness:

$$\xi(G^n) \geqslant np^{n-1}\xi(G).$$

The lower bound on $\xi(Q_n)$ follows from the previous inequality and the fact that Q_4 can be viewed as the edge-disjoint union of two isomorphic cubical refinements of K_5 of the kind shown in Fig. 4(b). Hence $\xi(Q_4) = 2$ and, in general,

$$\left\lfloor \frac{n2^{n-2}}{7} \right\rfloor \geqslant \xi(Q_n) \geqslant \left\lfloor \frac{n}{4} \right\rfloor 2^{n-3}. \tag{10}$$

It is easily shown and well known that every cube Q_n with $n \geqslant 2$ is hamiltonian; see Fig. 5. Of course Q_2 consists of a hamiltonian cycle and Q_3 has exactly one such cycle up to automorphism, but six of them when Q_3 has labeled nodes. Let h_n (or H_n) be the number of hamiltonian cycles in a labeled Q_n (or an unlabeled one). Gilbert [13] established most of the values in Table 1. For



(a) $K_{3,3}$ (b) K_5

Fig. 4. Cubical refinements of $K_{3,3}$ and K_5.

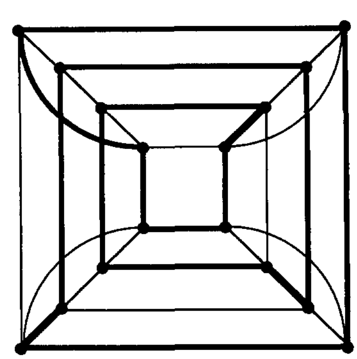


Fig. 5. Hamiltonian cycle (heavy lines) in Q_4.

Table 1. The numbers H_n and h_n of hamiltonian cycles in an unlabeled and in a labeled cube Q_n

n	H_n	h_n
2	1	1
3	1	6
4	9	1344
5	275,065	906,545,760

The cited value for h_5 is from A. Bell and P. Hallowell, "Crawling round a cube edge," *Computing* (*U.K.*), p. 9 (Feb. 1973). The value of H_5 was determined by D. Russell in June 1987; his article giving the method of solution will be published.

Table 2. The length s_n of snakes and c_n of coils in Q_n

n	s_n	c_n
1	1	2
2	2	4
3	4	6
4	7	8
5	13	14
6	26	26

$n \geqslant 6$, no exact values are known but Douglas [14] actually found the lower and upper bounds in inequality (11); see Dixon and Graham [15].

$$\left(\prod_{i=5}^{n-1} i^{2n-i-1}\right)(1344)^{2n-4} \cdot n \cdot 2^{[2^{n-2}-1-m]} \leqslant h(Q_n) \leqslant \lfloor n(n-1)/2 \rfloor^{2^{n-1}-2^{[n-1-\log_2 n]}} \tag{11}$$

By definition H is an *induced subgraph* of G if for any $u, v \in V(H)$, if u and v are adjacent in G then they are also adjacent in H. The *induced paths* and *induced cycles* of n-cubes are subgraphs of significance to coding theory and related areas [16–18]. An *n-snake* is a longest induced path in Q_n, and an *n-coil* is a longest induced cycle. Let s_n and c_n denote the lengths of the n-snakes and n-coils. Table 2, calculated with computer assistance by Davies [16], shows s_n and c_n for $n \leqslant 6$; the values for larger n are not known. As shown in Table 2, $c_{n+1} = 2s_n$ for $3 \leqslant n \leqslant 5$. Davies conjectured that this is true for all $n \geqslant 3$ and that an n-coil can be formed by joining two $(n-1)$-snakes at their ends. The following inequalities [18] give the known upper and lower bounds on c_n for $n \geqslant 6$:

$$\frac{7(2^n)}{4(n-1)} \leqslant c_n \leqslant 2^{n-1} - \frac{2n-12}{7n(n-1)^2+2} \tag{12}$$

4. MATRICES AND CHARACTERIZATIONS OF Q_n

The *adjacency matrix* $A = A(G)$ is the $p \times p$ matrix in which $\alpha_{ij} = 1$ if v_i is adjacent to v_j, and $\alpha_{ij} = 0$, otherwise. The *characteristic polynomial* $\phi(G)$ of G is defined as $\det(xI - A)$. The *spectrum* $S(G)$, is then the nondecreasing sequence, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p$ of eigenvalues of A [the roots of $\phi(G) = 0$]. For example, $S(K_2) = (+1, -1)$ and $S(K_p) = [p-1, (-1)^{p-1}]$. Two graphs G_1, G_2 are called *cospectral* if they have the same spectrum, i.e., the same characteristic polynomial. The smallest pair of cospectral graphs were given in Harary *et al.* [19]; they have just five nodes. Cvetkovic [20] and Schwenk [21] noted that the spectrum of the cartesian product of two graphs is the set-sum of their spectra:

$$S(G_1 \times G_2) = S_1 + S_2,$$

where

$$S_1 = S(G_1) = \{\lambda_1, \lambda_2, \ldots\}$$
$$S_2 = S(G_2) = \{u_1, \mu_2, \ldots\}$$

and

$$S_1 + S_2 = \{\lambda_i + \mu_j\}.$$

Applying this relationship to Q_n as in equation (1), we obtain:

$$\phi(Q_n) = \prod_{i=0}^{n} (x - n + 2i)^{\binom{n}{i}}. \tag{13}$$

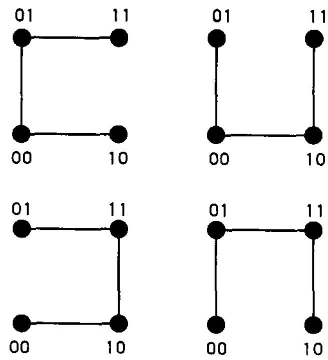


Fig. 6. The 4 spanning trees of Q_3.

For example,

$$\phi(Q_4) = (x-4)(x-2)^4 x^6 (x+2)^4 (x+4).$$

Another useful matrix associated with G is its *connection matrix* M. It can be obtained from A by replacing the zero ith diagonal entry of $-A$ by the degree of the ith node. The matrix-tree theorem of Kirchhoff [1] states that if G has the connection matrix M, then all cofactors of M are equal to the number of spanning trees of G which is denoted by $T(G)$. Schwenk [21] applied this theorem to get

$$T(Q_n) = 2^{-n} \prod_{i=1}^{n} (2i)^{\binom{n}{i}}. \tag{14}$$

For example, substituting 2 for n in equation (14), yields $T(Q_2) = 4$. This implies that Q_2 has four spanning trees as shown in Fig. 6.

The diverse applications of n-cube graphs have resulted in many ways of characterizing them. Some representative examples follow:

(i) Foldes [22] showed that every cube is a bipartite graph such that the number of shortest paths between any two nodes x, y is the factorial of the distance between them, i.e., $d(x, y)!$

(ii) Garey and Graham [23] gave a criterion similar to example (i), namely, that a cube is a bipartite graph such that the number of node-disjoint paths between any two nodes x, y of the graph is $d(x, y)$.

(iii) Laborde and Hebbare [24] found the following: let $\mathbf{C}_4$ be the class of connected graphs such that each pair of adjacent edges lies in exactly one 4-cycle. Then we can characterize the n-cube as follows: a graph G in $\mathbf{C}_4$ is an n-cube if and only if its minimum degree δ satisfies $p = 2^{\delta(G)}$.

(iv) The following result is due to van den Cruyce [25]. An induced subgraph H of G is *convex* [26] if for any two nodes of H, every geodesic joining them is in H. A convex subgraph is *proper* if it is not K_1, K_2, or G. The set of all pairwise nonisomorphic proper convex subgraphs of a graph G is denoted by $\mathbf{PC}(G)$. For any $n \geqslant 3$, $\mathbf{PC}(Q_n) = \{Q_2, \ldots, Q_{n-1}\}$. If G is a connected graph such that $\mathbf{PC}(G) = \{Q_2, \ldots, Q_{n-1}\}$ with $n \geqslant 3$ and $p = 2^n$, then G is isomorphic to Q_n.

5. EMBEDDING AND PACKING PROBLEMS

Embedding problems are concerned with finding mappings between two graphs that preserve certain topological properties. Various embedding problems of n-cubes have applications in coding theory [17], linguistics [27] and computer system design [28]. Following Ref. [1], a *homomorphism* h of G into G' can be considered as a function from $V(G)$ into $V(G')$ such that if u and v are adjacent in G, then $h(u)$ and $h(v)$ are adjacent in G'.

Several classes of embedding are discussed here. If there is an *isomorphic embedding* of G into Q_n, then G is isomorphic to a subgraph of Q_n. Graphs that can be isomorphically embedded in an n-cube are called *cubical*. A graph H has an *isometric embedding* into a graph G if and only if $H \subset G$ and for all $u, v \in V(H)$,

$$d_H(u, v) = d_G(u, v).$$

A *topological* or *homeomorphic embedding* can be derived from an isomorphic embedding by subdividing some edges of G so that there exists an isomorphic embedding from the edge-subdivided graph of G into Q_n. The composition of an edge-subdivision and an isomorphic embedding is then an homeomorphic embedding of G into Q_n. Harary [29] defined tcd(G) for an arbitrary graph G, not necessarily cubical, as the minimum n such that some subdivision H of G is contained in Q_n.

There is another class of embeddings where the range is restricted to subcubes instead of nodes, and which have a broader definition of distance. A subcube of G can be represented by a vector $X = x_1 \ldots x_i \ldots x_n$, where $x_i \in 0, 1, *$ and $*$ denotes a coordinate value that is either 0 or 1. For example, $X = 01{*}{*}$ represents the subcube of Q_4 with the node set $\{0100, 0101, 0110, 0111\}$. Given two subcubes

$$X = x_1 \ldots x_i \ldots x_n,$$

$$Y = y_1 \ldots y_i \ldots y_n,$$

the distance $D_i(X, Y)$ between X and Y along the ith dimension is 1 if $\{x_i, y_i\} = \{0, 1\}$; otherwise, it is 0. Then the *distance* between two subcubes X, Y is given by:

$$D(X, Y) = \sum_{i=1}^{n} D_i(X, Y) = \sum_{i=1}^{n} d(x_i, y_i).$$

We say that X and Y are *adjacent* if $D(X, Y) = 1$. A *squashed-cube embedding*, a concept due to Graham and Pollak [30], is a one-to-one homomorphism from $V(G)$ into a set of mutually disjoint subcubes which preserves distance as defined above. Figure 7 demonstrates a squashed-cube embedding of K_4 onto Q_3.

Several interesting problems arise from the various kinds of embeddings. Let f be any embedding function of one of the four types, i.e., isomorphic, isometric, homeomorphic, squashed-cube.

Problem 1

Characterize the graphs which can be embedded in Q_n by f.

Problem 2

What is the smallest n for which f embeds G in Q_n?

Problem 3

If f embeds G in Q_n, then for $k \geq n$, what is the maximum number of node- or edge-disjoint copies of G that can be embedded in Q_k?

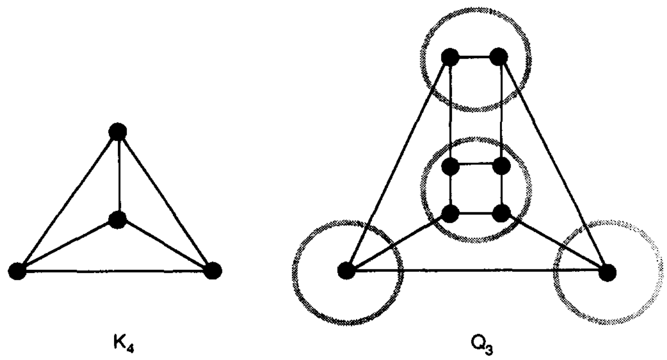


Fig. 7. A squashed-cube embedding of K_4 onto Q_3.

Problem 4

If f embeds G in Q_n, then in how many ways can this be done?

We now briefly discuss each of these problems. Hartman [31] and Winkler [32] prove that we can always find a homeomorphic or squashed-cube embedding from K_{n+1} into Q_n. Hence there are homeomorphic and squashed-cube embeddings from any connected graph of order $n + 1$ into Q_n.

Djokovic [33] characterized the graphs that are isometrically embeddable in Q_n as follows. Obviously, such graphs must be bipartite. Let $L(v_1, v_2)$ be the set of all nodes x of G such that $d(v_1, x) < d(v_2, x)$ with L standing for "less than". For sufficiently large n, a connected bipartite graph G has an isometric embedding into Q_n if and only if for every edge (v_1, v_2) of G and all $x, y, z \in L(v_1, v_2)$, $d(x, y) + d(y, z) = d(x, z)$ implies $y \in C(v_1, v_2)$. It is interesting to note that a cubical graph need not have an isometric embedding into any Q_n. Figure 8 shows a example of a cubical graph $P(3, 3, 3, 3)$ which is isomorphically embeddable in Q_5 as indicated by the labeling. However, it is not isometrically embeddable since $d(v, v') = 3$, whereas $d(f(v), f(v')) = 1$.

Although characterizations have been found of the graphs for which an isometric, homeomorphic, or squashed-cube embedding into Q_n exists, no criterion for cubical graphs is known as yet. The following results pertaining to Problem 1 have been found for isomorphic embeddings:

(1) If a graph is cubical then it is bipartite, but the converse is not true [27]. The smallest counterexample is $K_{2,3}$.
(2) All trees are cubical [27]. The proof by induction is trivial.
(3) Two-dimensional meshes and hexagonal graphs are cubical [34].
(4) "One-legged caterpillars" span the hypercube [35].

Havel and Moravek [34] found a criterion for a graph G to be cubical which is based on a technique called c-valuation for labelling the edges of G. A *c-valuation* of a bipartite graph G is a labeling of $E(G)$ such that: (i) for each cycle in G, all distinct edge labels occur an even number of times; (ii) for each path in G, there exists at least one edge label which occurs an odd number of times. The dimension of a c-valuation is the number of edge labels used. It is shown in [34] that a graph G is cubical and $G \subset Q_k$ if and only if there exists a c-valuation of G of dimension k. Intuitively, the labels on the edges are coordinated with the directions of the edges in a k-cube embedding of G.

The use of c-valuations for deciding the embeddability of hexagonal graphs and two-dimensional meshes is demonstrated in Fig. 9. We also observe at once that the graph product operation preserves isomorphic embeddability, that is, if G_1 and G_2 are cubical, then $G_1 \times G_2$ is cubical. An *n-dimensional mesh* or *grid* is an n-fold cartesian product of paths. It follows from the foregoing results that every n-dimensional mesh is cubical. Since many problems in scientific computation are defined on n-dimensional meshes, this important result implies that hypercube-structured computers are perfectly suited to such problems.

Garey and Graham [23] proposed another way of tackling Problem 1 in terms of graphs that are not cubical. A graph G is *cube-critical* if it is not cubical and every proper subgraph H of G is cubical. (Thus, such a graph is minimal noncubical.) Figure 10 shows several examples of cube-critical graphs. Thus, a graph is cubical if and only if it contains no cube-critical subgraph. Garey and Graham [23] and Gorbatov [36] have given procedures for constructing cube-critical graphs from other cube-critical graphs.

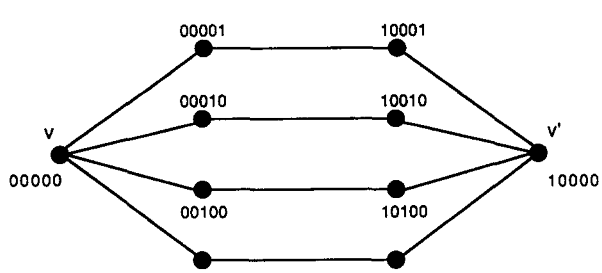
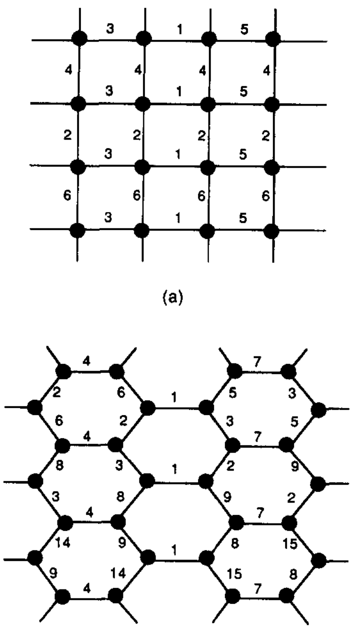


Fig. 8. A cubical graph with no isometric embedding.



(a)

(b)

Fig. 9. c-Valuations of a 2-dimensional mesh and a hexagonal graph.

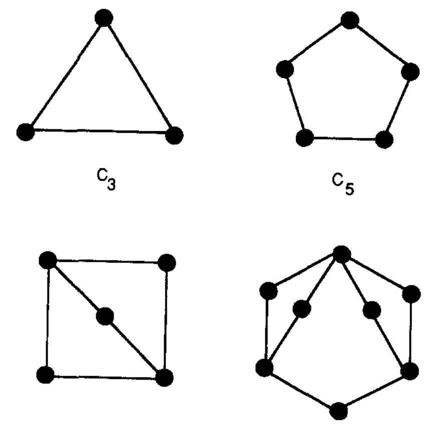


Fig. 10. Four cube-critical graphs.

Next we consider Problem 2. The minimum n required for an isomorphic embedding from G into Q_n is defined as the *cubical dimension* of G and is denoted by $\mathrm{cd}(G)$. The following simple result relates the cubical dimension of the cartesian product $G_1 \times G_2$ to those of G_1 and G_2; we omit its straightforward proof. If $\mathrm{cd}(G_1) = k_1$ and $\mathrm{cd}(G_2) = k_2$, then $\mathrm{cd}(G_1 \times G_2) = k_1 + k_2$. An immediate corollary of this result gives the cubical dimension of the mesh $G_{r,s} = P_r \times P_s$ (and extends at once to higher dimensional meshes):

$$\mathrm{cd}(G_{r,s}) = \lceil \log_2 r \rceil + \lceil \log_2 s \rceil. \tag{15}$$

Havel and Liebl [35, 37–39] and Nebesky [40, 41] obtained several results concerning Problem 2 in terms of c-valuation. The more important of these results are now summarized: if $T_n^{(2)}$ is the complete binary tree of height n, then

$$\mathrm{cd}(T_n^{(2)}) = n + 2 \tag{16}$$

To prove equation (16), we note that a c-valuation of order $n + 2$ exists for $T_n^{(2)}$. Assume that there is an isomorphic embedding F from $T_n^{(2)}$ into Q_{n+1}, and that a node v in Q_{n+1} is the image of the root of $T_n^{(2)}$. Then there are 2^n nodes D_0 in Q_{n+1} which are at an odd distance from v, with the same number at an even distance from v. Since $T_n^{(2)}$ has $2^{n+1} - 1$ nodes, only one node v^* in Q_{n+1} is not in the image of the tree. The number of nodes whose distance from the root $F^{-1}(v)$ of $T_n^{(2)}$ is odd is $2 + 2^3 + 2^5 + \dots$, which is $(2^{n+2} - 2)/3$ when n is odd and $(2^{n+1} - 2)/3$ when n is even. No matter whether v^* is assigned to D_0 or not, the number of nodes which are at odd distance from v cannot be 2^n. Hence, $T_n^{(2)}$ cannot be embedded in Q_{n+1}, so its cubical dimension is $n + 2$.

Figure 11 shows a c-valuation of $T_4^{(2)}$. The cubical dimension of a complete k-ary tree $T_n^{(k)}$ for $k \geqslant 3$, is still unknown. However, Havel and Liebl [39] have found some results about the cubical dimension of a complete k-ary tree with two levels, i.e., $T_2^{(k)}$.

By a similar method, that is, by determining an n-dimensional c-valuation and showing that no lower-dimensional c-valuation exists, the following cubical dimensions were calculated. Nebesky [40] defined $D_n^{\#}$, for $n \geqslant 1$, as the tree obtained by joining the roots of two disjoint copies of $T_n^{(2)}$ with a new edge and proved that

$$\mathrm{cd}(\mathrm{D}_n^{\#}) = n + 2. \tag{17}$$

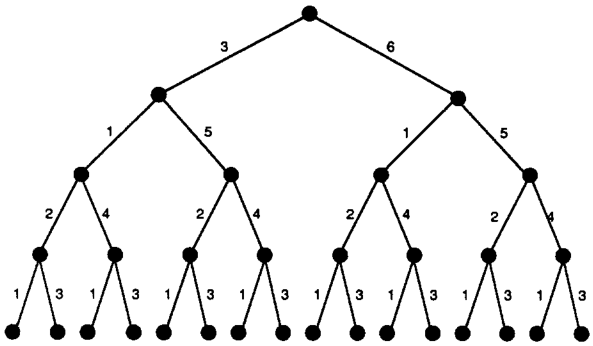


Fig. 11. A c-valuation of the binary tree $T_r^{(2)}$.

Figure 12 shows $D_3^{\#}$. Similarly, Havel [39] defined $B_n^{(k)}$ as the tree obtained from $T_n^{(2)}$ by adding one edge to the root and splitting each node into k nodes. He then determined the cubical dimension of $B_n^{(k)}$ to be

$$\mathrm{cd}(B_n^{(k)}) = n + \lfloor \log_2 k \rfloor. \tag{18}$$

A k-star is a star with k endnodes, i.e., $K_{1,k}$. A *k-quasistar* denoted $S(t_1, t_2, \ldots, t_k)$ is a starlike tree that is homeomorphic to a k-star in which each edge e_i has been expanded to a path of length t_i. A bipartite graph is *equitable* if it can be colored by two colors in such a way that there are equal numbers of nodes of each color. Nebesky [41] found that if S is an equitable 3-quasistar with $|V(S)| = 2^n$ for some $n \geqslant 3$, then S is a spanning tree of Q_n, so $\mathrm{cd}(S) = n$.

Figure 13 shows the only two spanning 3-quasistars with 8 nodes. This result has been generalized to a characterization of those starlike trees that span a hypercube by Harary and Lewinter [42], namely, the equitable ones.

Problem 3 is an instance of the general graph packing problem in Q_n. First we consider the problem of packing subcubes into Q_n. If $N(Q_m \subset Q_n)$ is the number of distinct (labeled) m-cubes in Q_n for $m \leqslant n$, then

$$N(Q_m \subset Q_n) = 2^{n-m}\binom{n}{m}. \tag{19}$$

More specifically, the *node-disjoint* (or *edge-disjoint*) *packing*, denoted $\mathrm{pac}_0(Q_m \subset Q_n)$ [or $\mathrm{pac}_1(Q_m \subset Q_n)$], is the maximum number of m-dimensional subcubes that can be embedded in Q_n without overlapping nodes (or edges). For example, $\mathrm{pac}_0(Q_2 \subset Q_4) = 4$ as illustrated in Fig. 14, and $\mathrm{pac}_1(Q_2 \subset Q_4) = 8$.

An interesting related concept is that of mispacking. The *node-disjoint* (or *edge-disjoint*) *mispacking*, denoted as $\mathrm{mispac}_0(Q_m \subset Q_n)$ [or $\mathrm{mispac}_1(Q_m \subset Q_n)$], is the minimum number of copies of Q_m in a maximal node-disjoint set embeddable into Q_n. For example,

$$\mathrm{mispac}_0(Q_2 \subset Q_4) = \mathrm{mispac}_1(Q_2 \subset Q_4) = 3;$$

see Fig. 15. The well-known problem of finding a matching for a graph G is exactly the problem of determining a node-disjoint packing of edges into G. The maximum matching size is $\mathrm{pac}_0(Q_1 \subset Q_n)$ and the minimum maximal matching size is $\mathrm{mispac}_0(Q_1 \subset Q_n)$. Obviously,

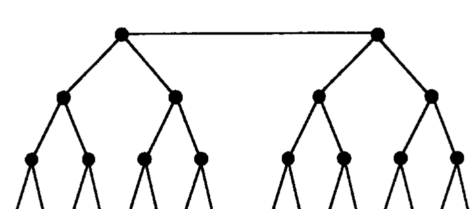


Fig. 12. The tree $D_3^{\#}$ of cubical dimension 5.

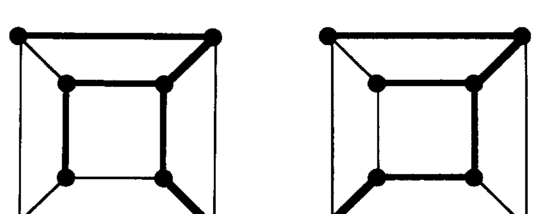


Fig. 13. Two spanning 3-quasistars (heavy lines) of Q_3.

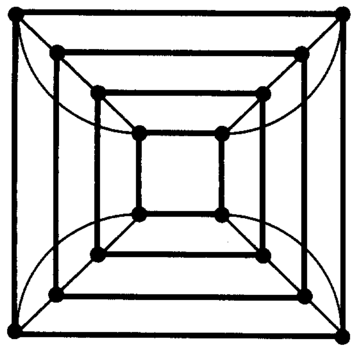


Fig. 14. A node-disjoint packing of Q_2 onto Q_4.

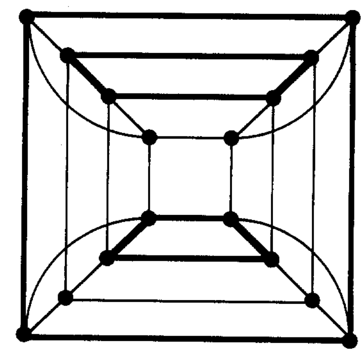


Fig. 15. Node- and edge-disjoint mispacking of Q_2 into Q_4.

$\mathrm{pac}_0(Q_1 \subset Q_n) = \beta_1(Q_n)$ and $\mathrm{mispac}_0(Q_1 \subset Q_n) = \beta_1^-(Q_n)$; Fig. 2 shows minimum maximal matchings for Q_3 and Q_4. Forcade [43] has determined the limit of the ratio of $\mathrm{mispac}_0(Q_1 \subset Q_n)$ to $|V(Q_n)|$ when n approaches infinity, viz.,

$$\lim_{n\to\infty} \frac{\mathrm{mispac}_0(Q_1 \subset Q_n)}{2^n} = \frac{1}{3}. \tag{20}$$

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