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BAIRE IRRESOLVABLE SPACES AND LIFTING FOR A LAYERED IDEAL

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We show the consistency (modulo reasonable large cardinals) of the existence of a topological space of power $\aleph_1$ with no isolated points such that any real values function on it has a point of continuity. This is deduced from the following (by Kunen, Szymanski and Tall). We prove that if $2^\lambda = \lambda^+$, I is a λ-complete ideal on a regular λ which is layered, then the natural homomorphism from $\mathcal{P}(\lambda)$ to $\mathcal{P}(\lambda)/I$ (as Boolean algebras) can be lifted, i.e., there is a homomorphism h from $\mathcal{P}(\lambda)$ into itself with kernel I such that for every $A \subseteq \lambda$ we have $A \equiv h(A) (\mathrm{mod}\ I)$.

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real valued functions	μ-Woodin cardinal
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irresolvable spaces	λ-complete ideal
huge cardinal	lifting
layered ideal	

Katětov [2] asked in the 1940s:

Question. Is there a topological space without isolated points such that any real valued function has a point of continuity?

Malyhin [4] showed this could not happen if $V = L$, and showed its equivalence to the existence of irresolvable spaces satisfying the Baire category theorem.

Kunen, Szymanski and Tall [3] showed it is equivalent (consistency-wise) to the existence of a measurable cardinal. However, many mathematicians would not look at this as a satisfactory answer, as they are interested in smaller cardinals.

So Kunen, Szymanski and Tall rephrase the question.

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Question. Is there such a space of power $\aleph_1$? (and you can ask on $\aleph_2$, etc)

They proved that this is equivalent to the existence of an $\aleph_1$-complete ideal on ω_1 with lifting and know it follows from the existence of an $\aleph_1$-complete $\aleph_1$-dense ideal on ω_1.

Hence they could deduce the consequence from a result of Woodin using a hypothesis ZF+DC+ADR+"θ regular". (DC is the axiom of determinancy, see Moschovakis [8] for explanation, if you want to know what this means)

Franek studies this question in his Ph D. thesis (Toronto 83)

The author encounters and solves this problem during his visit in Toronto, April 1985, by showing:

(*) If I is a λ-complete layered ideal on $\mathcal{P}(\lambda)$, $2^\lambda = \lambda^+$, then $\mathcal{P}(\lambda)/I$ has lifting.

There are some proofs of consistency of the existence of such ideals By Foreman, Magidor and Shelah [1], starting with a huge cardinal we can, by forcing, get for a regular λ, that on λ^+ there is a layered normal ideal. By [6] supercompact cardinals suffice for showing the consistency of "GCH + on ω_1 there is a layered normal ideal" By [7] much weaker large cardinals suffice λ strongly inaccessible with $\{\kappa < \lambda . (*)_\kappa$ or κ is Woodin$\}$ is stationary, in $(*)_\kappa$ and Woodin cardinals (see [9]).

On previous applications of layered ideals on λ, $2^\lambda = \lambda^+$ see [1] (if $\diamondsuit_\lambda$ there is an ultrafilter D on λ which is not regular, moreover if $\lambda = \mu^+$, then $(\lambda)^\lambda / D = \lambda^+$) and [5] (if $\lambda = \lambda^{<\lambda}$, then $\mathcal{P}(\lambda)/I - \{0/I\}$ is the union of λ filters). Later Woodin gets an $\aleph_1$-dense ideal on ω_1 from huge I thank Toronto General Topology group for their hospitality and the question, a referee for many connections and a referee and Isaac Gorelic for shortening the proof of Fact 7 and Kitty Gubbay for typing it nicely and quickly.

1. Definition. For Boolean algebras $\mathcal{A}, \mathcal{B}$:

(1) $\mathcal{A} \subseteq \mathcal{B}$ means $\mathcal{A}$ is a subalgebra of $\mathcal{B}$

(2) $\mathcal{A} <\circ \mathcal{B}$ means $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and every maximal antichain of $\mathcal{A}$ is a maximal antichain of $\mathcal{B}$.

(3) $\mathcal{A} <\circ_\lambda \mathcal{B}$ means there are $\alpha < \lambda$ and $\mathcal{A}_\beta \subseteq \mathcal{A}$ for $\beta < \alpha$ such that

(i) $\mathcal{A} = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$, moreover for every finite $A \subseteq \mathcal{A}$ for some β, $A \subseteq \mathcal{A}_\beta$

(ii) $\mathcal{A}_\beta <\circ \mathcal{B}$ for each $\beta < \alpha$.

(4) for λ a cardinal, $\mathcal{P}(\lambda)$ is the Boolean algebra of subsets of λ

2. Definition. For I an ideal on λ (i.e., of $\mathcal{P}(\lambda)$) let $\mathcal{B}^I = \mathcal{P}(\lambda)/I$

3. Definition. If $\mathcal{B}$ is a Boolean algebra of power λ^+, λ regular, $\mathcal{B}$ is called *λ-layered* if for every (≡some) representation of $\mathcal{B}$ as $\bigcup_{\zeta < \lambda^+} \mathcal{B}_\zeta$, $\mathcal{B}_\zeta$ increasing continuous, $\|\mathcal{B}_\zeta\| \leq \lambda$, the set $\{\delta < \lambda^+ : \mathrm{cf}\, \delta = \lambda \Rightarrow \mathcal{B}_\zeta <\circ \mathcal{B}\}$ contains a closed unbounded set.

4. Definition. We say (for I an ideal on λ) that $\mathcal{B}^I \stackrel{\text{def}}{=} \mathcal{P}(\lambda)/I$ lifts if there is a homomorphism h from $\mathcal{B}^I$ into $\mathcal{P}(\lambda)$ such that for $A \subseteq \lambda$, we have $h(A/I) \in A/I$, i.e., $(h(A/I) - A) \cup (A - h(A/I)) \in I$

5. Theorem. *If λ is regular, $2^\lambda = \lambda^+$, I a λ-complete ideal on λ and $\mathcal{P}(\lambda)/I$ is λ-layered, then $\mathcal{P}(\lambda)/I$ lifts*

We first prove some facts.

6. Fact. *If λ is regular, $\mathcal{A}_i <\circ \mathcal{B}$ for $i < \alpha$, $\alpha < \lambda$, $\mathcal{A}_i$ increasing in i, then $\bigcup_{i<\alpha} \mathcal{A}_i <\circ_\lambda \mathcal{B}$.*

Proof. Immediate □

7. Fact. *If $\mathcal{A}_0 <\circ \mathcal{B}$, $x \in \mathcal{B}$, $\mathcal{A}_1 = \langle \mathcal{A}_0, x\rangle$ (the subalgebra of $\mathcal{B}$ generated by $\mathcal{A}_0, x$), then $\mathcal{A}_1 <\circ \mathcal{B}$*

Proof. Clearly $\mathcal{A}_1 \subseteq \mathcal{B}$ Let $\{a_i: i<j\}$ be a maximal antichain of $\mathcal{A}_1$.

W l o.g. $(\forall i)[a_i \leq x \vee a_i \leq 1-x]$; hence there is $c_i \in \mathcal{A}_0$ such that $a_i \in \{c_i \cap x, c_i - x\}$.

W.l.o g for some $j(1) \leq j$. $a_i = c_i \cap x$ for $i < j(1)$, $a_i = c_i - x$ for $j(1) \leq i < j$

Suppose $y_0 \in \mathcal{B}$, $y_0 > 0$; w l o g $y_0 \leq x$ Clearly $K_0 \stackrel{\text{def}}{=} \{b \in \mathcal{A}_0: b > 0$, and $\bigvee_{i<j} b \leq c_i$ or $\bigwedge_{i<j} b \cap c_i = 0\}$ is a dense subset of $\mathcal{A}_0$, hence there is $K \subseteq K_0$ which is a maximal antichain of $\mathcal{B}$. As $\mathcal{A}_0 <\circ \mathcal{B}$ for some $b \in K$, we have $b \cap y_0 \neq 0$. As we assumed $y_0 \leq x$, we have $b \cap x \neq 0$. Now $b \cap x \in \mathcal{A}_1$ cannot be disjoint to every a_i $(i<j)$ [as $\{a_i: i<j\}$ is a maximal antichain of $\mathcal{A}_1$] so there is $i<j$ such that $b \cap x \cap a_i \neq 0$, so necessarily $i < j(1)$ and $a_i = c_i \cap x$ So $b \cap c_i \neq 0$ hence (as $b \in K \subseteq K_0$) $b \leq c_i$, so $y_0 \cap a_i = y_0 \cap (x \cap c_i) = y_0 \cap c_i \geq y_0 \cap b > 0$. □

8. Fact. *If $\mathcal{A} <\circ_\lambda \mathcal{B}$, $A \subseteq \mathcal{B}$, $|A| < \lambda$, $\mathcal{A}' = \langle \mathcal{A}, A\rangle$, then $\mathcal{A}' <\circ_\lambda \mathcal{B}$.*

Proof. The family of finite subsets of A, $\{A_\gamma: \gamma < \gamma^0\}$ has cardinality $<\lambda$. Let $\mathcal{A} = \bigcup_{\zeta<\xi} \mathcal{A}_\zeta$, $\xi < \lambda$ exemplify Definition 1(3). Now $\{\langle A_\gamma, \mathcal{A}_\zeta\rangle: \gamma < \gamma^0, \zeta < \xi\}$ exemplifies $\mathcal{A}' <\circ_\lambda \mathcal{B}$ (clearly every finite subset of $\mathcal{A}'$ is included in one of them, and $\langle A_\gamma, \mathcal{A}_\zeta\rangle <\circ \mathcal{B}$ by Fact 7 (by induction on $|A_\gamma|$)). □

9. Conclusion. *If $|\mathcal{B}| = \lambda^+$ (λ regular) and $\mathcal{B}$ is λ-layered, then we can find $\mathcal{B}_\zeta$ for $\zeta < \lambda^+$ such that $\mathcal{B} = \bigcup_{\zeta<\lambda^+} \mathcal{B}_\zeta$, $\|\mathcal{B}_0\| = 2$, $\mathcal{B}_\zeta$ increasing continuous, $\mathcal{B}_\zeta <\circ_\lambda \mathcal{B}$ and $\mathcal{B}_{\zeta+1} = \langle \mathcal{B}_\zeta, x_\zeta\rangle$.*

Proof. Let $\langle \mathcal{B}_\zeta < \lambda^+\rangle$ be such that $\mathcal{B} = \bigcup_{\zeta<\lambda^+} \mathcal{B}_\zeta$, $\|\mathcal{B}_\zeta\| \leq \lambda$, $\mathcal{B}_\zeta$ increasing continuous. As $\mathcal{B}$ is λ-layered we know that for some closed unbounded $C \subseteq \lambda^+$, $(\forall \zeta \in C)[\mathrm{cf}\,\zeta = \lambda \Rightarrow \mathcal{B}_\zeta <\circ \mathcal{B}]$. By thinning the sequence $\langle \mathcal{B}_\zeta: \zeta < \lambda^+\rangle$ we can assume $\mathcal{B}_0 <\circ \mathcal{B}$, $\mathcal{B}_{\zeta+1} <\circ \mathcal{B}$, and $\mathrm{cf}\,\zeta = \lambda \Rightarrow \mathcal{B}_\zeta <\circ \mathcal{B}$. So by Fact 6 $(\forall \zeta)\mathcal{B}_\zeta <\circ_\lambda \mathcal{B}$.

W.l.o.g. $\mathcal{B}_0=\{0,1\}$. let $\|\mathcal{B}_{\zeta+1}\|=\{x^{\zeta}_i: i<\lambda\}$, let $\mathcal{B}'_\xi$ be defined by $\mathcal{B}'_0=\mathcal{B}_0$. $\mathcal{B}_{\lambda\zeta+j}=\langle\mathcal{B}_\zeta, \{x^\zeta_i\ i<j\}\rangle$ for $j<\lambda$ ($\lambda\zeta$ ordinal multiplication) Clearly $\mathcal{B}'_\xi$ is increasing continuous, $\bigcup_{\xi<\lambda}\mathcal{B}'_\xi=\mathcal{B}$, $\mathcal{B}'_0=\{0,1\}$. $\mathcal{B}'_{\xi+1}=\langle\mathcal{B}'_\xi, x_\xi\rangle$ for appropriate x_ξ. Why $\mathcal{B}'_\xi<\circ_\lambda\mathcal{B}$? By Fact 8 □

Proof of Theorem 5. By Conclusion 9, $\mathcal{B}'$ can be represented as $\bigcup_{\zeta<\lambda^+}\mathcal{B}^I_\zeta$, $\mathcal{B}^I_\zeta$ increasing continuous (in ζ) $\|\mathcal{B}^I_0\|=2$, $\mathcal{B}^I_{\zeta+1}=\langle\mathcal{B}^I_\zeta, x_\zeta\rangle$, $B^I_\zeta<\circ_\lambda\mathcal{B}$. We define by induction on $\zeta<\lambda^+$ a homomorphism h_ζ from $\mathcal{B}^I_\zeta$ into $\mathcal{P}(\lambda)$ such that $h_\zeta(A/I)\in A/I$, i.e. $(h_\zeta(A/I))/I=A/I$ and such that h_ζ is increasing continuous (in ζ) This suffices, as then $\bigcup_{\zeta<\lambda^+}h_\zeta$ is a lifting as desired

For $\zeta=0$, ζ limit we have no problem.

For $\zeta+1$. h_ζ defined

Let $x_\zeta=A_\zeta/I$. It suffices to find $A'_\zeta\in\mathcal{B}(\lambda)$ such that.

(i) $A'_\zeta/I=A_\zeta/I$,

(ii) $y\in\mathcal{B}_\zeta\ y\cap x_\zeta=0\Rightarrow A'_\zeta\cap h_\zeta(y)=\emptyset$ (the empty set).

(iii) $y\in\mathcal{B}_\zeta. y\leq x_\zeta\Rightarrow(\lambda-A'_\zeta)\cap h_\zeta(y)=\emptyset$

Let

$$Q^+_\zeta=\{y\in B_\zeta\ y\leq x_\zeta\ (\text{in } B_\zeta)\},\qquad Q^-_\zeta=\{y\in\mathcal{B}_\zeta\cdot y\cap x_\zeta=0\ (\text{in } B_\zeta)\}$$

Let

$$A^0_\zeta=A_\zeta.\qquad A^1_\zeta=A^0_\zeta-\bigcup\{h(y).\ y\in Q^-_\zeta\},\qquad A^2_\zeta=A^1_\zeta\cup\bigcup\{h(y)\ \ y\in Q^+_\zeta\}$$

Now A^2_ζ satisfies (iii) trivially. It satisfies (ii) as $y\in Q^-_\zeta\wedge z\in Q^-_\zeta\Rightarrow y\cap z=0\Rightarrow h(y)\cap h(z)=\emptyset$, hence $z\in Q^-_\zeta\Rightarrow h(z)\cap A^2_\zeta=h(z)\cap A^1_\zeta$ which is $\emptyset$ by A^1_ζ's definition. What about (i)? We shall prove:

(a) $A^0_\zeta/I=A^1_\zeta/I$.

(b) $A^1_\zeta/I=A^2_\zeta/I$.

This suffices, and the two proofs are the same so we prove (a)

To prove (a) it suffices to prove:

$$A^0_\zeta\cap\bigcup\{h(y): y\in Q^-_\zeta\}\in I$$

As $\mathcal{B}_\zeta<\circ_\lambda\mathcal{B}$, there are $\alpha_\zeta<\lambda$ and $\mathcal{B}_{\zeta\gamma}$, $\gamma<\alpha_\zeta$, such that $\mathcal{B}_\zeta=\bigcup_{\gamma<\alpha_\zeta}\mathcal{B}_{\zeta\gamma}$, $\mathcal{B}_{\zeta\gamma}<\circ\mathcal{B}$ As I is λ-complete it suffices to prove for each γ

$$A^0_\zeta\cap\bigcup\{h(y): y\in Q^-_\zeta\cap\mathcal{B}_{\zeta\gamma}\}\in I.$$

Call this set Y, suppose $Y\notin I$, so $Y/I\in\mathcal{B}$, $Y/I>0_{\mathcal{B}}$ in $\mathcal{B}$. As $\mathcal{B}_{\zeta\gamma}<\circ\mathcal{B}$ there is $t\in\mathcal{B}_{\zeta,\gamma}$, $t>0$, such that $(\forall s)[s\in\mathcal{B}_{\zeta,\gamma}\wedge 0<s\leq t\Rightarrow s\cap Y/I\neq 0$ in $\mathcal{B}]$. As $Y/I\leq A^0_\zeta/I=x_\zeta$, clearly $(\forall s\in\mathcal{B}_{\zeta,\gamma})(0<s\leq t\Rightarrow s\cap x_\zeta\neq 0)$ hence $(\forall s\in\mathcal{B}_{\zeta\gamma})$ $(0<s\leq t\Rightarrow s\notin Q^-_\zeta)$ hence (as Q^-_ζ is downward closed) $z\in Q^-_\zeta\Rightarrow t\cap z=0$ hence (by the induction hypothesis) $h(t)\cap h(z)=\emptyset$ for $z\in Q^-_\zeta\cap\mathcal{B}_{\zeta\gamma}$ hence $h(t)\cap\bigcup\{h(z): z\in Q^-_\zeta\cap\mathcal{B}_{\zeta\gamma}\}=\emptyset$, but Y is included in the latter set so $h(t)\cap Y=\emptyset$.

But remember $t \cap Y/I \neq 0$ in $\mathscr{B}$ hence

$$h(t) \cap Y \notin I, \text{ hence } h(t) \cap Y \neq \emptyset, \text{ a contradiction.}$$

Similarly (b) holds, so A_ζ^2 satisfies (i), (ii), (iii). We extend $h_\zeta: \mathscr{B}_\zeta \to \mathscr{P}(\lambda)$ to $h_{\zeta+1}: B_{\zeta+1} \to \mathscr{P}(\lambda)$ by $h_{\zeta+1}(x_\zeta) = A_\zeta^2$, so h_ζ is defined and $\bigcup_{\zeta<\lambda^+} h_\zeta$ is as required □

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