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NOTE

On the Intersection of a Set of Direction Cones

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A new method is suggested to compute the intersection of a set of direction cones encountered in the problem of passing a convex polyhedron through a window. The time requirement of this method is $O(nm)$, where n is the number of vertices of the polyhedron and m is the number of vertices of the window. Besides this time improvement, the concept of parallel congruence, which the new method is crucially based on, is discussed. © 1989 Academic Press, Inc.

1. INTRODUCTION

The problem of passing a convex polyhedron through a window is described as: Given a convex polyhedron $P = (p_1, p_2, \ldots, p_n)$ and a convex polygon $W = (w_1, w_2, \ldots, w_m)$ on a plane h not intersecting P, find all the directions for translating (single translation) P through W.

Toussaint [1] proposed an $O(nm \log nm)$ algorithm to solve this problem. He first showed that P can pass W with a single translation in direction θ if, and only if, each vertex of P can be passed through W with a single translation in direction θ. Based on this observation, he posed the algorithm as follows: Consider vertex p_i of P. All directions for translating p_i from its initial position through window W are defined by all the vectors emanating at p_i and intersecting H in W. Therefore the cone determined by the half-lines from p_i through w_j, $j = 1, 2, \ldots, m$ specifies all such directions for p_i. This cone is named as *direction cone* of p_i and is denoted as CONE(p_i, W). Construct a 3D euclidean direction space D and translate all the cones CONE(p_i, W), $i = 1, 2, \ldots, n$ in D such that the p_i all overlap with the origin of D. Then the intersection of all the cones in D gives another cone which is the set of directions for simultaneous translation of all the p_i, and hence of P. Each cone can be computed in $O(m)$ time and thus all the cones can be found in $O(nm)$ time. All the cones can be translated to D in $O(nm)$ time. To compute the intersection of the cones, Toussaint views each cone as the intersection of m half spaces determined by the planes coplanar with the m faces of each cone. The interior half space contains the cone. Therefore the solution cone is the intersection of all the interior half spaces determined by all the cones in D. Since the intersection of k half spaces in 3D space can be computed in $O(k \log k)$ time (Preparata and Muller [3]), the overall time requirement is $O(mn \log mn)$.

The major computation in the algorithm cited above is to find the intersection of cones. As long as it is viewed as an intersection of mn half spaces, the time requirement cannot be improved since $O(k \log k)$ is the optimal to compute the intersection of k half spaces.

Finding the intersection of a set of half spaces is a general scheme allowing half spaces having arbitrary positions and orientations. In our case, however, those nm

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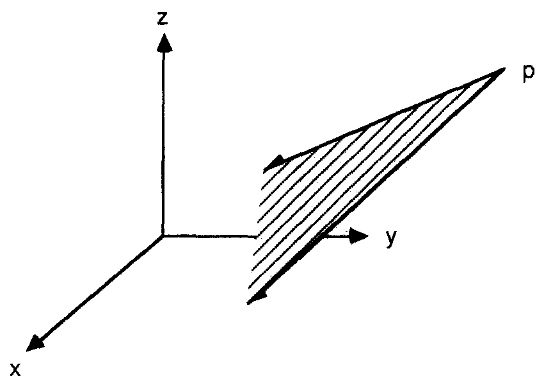

FIG. 1. An open triangle.

half spaces all overlap with a single point (the origin of D). With this constraint, it is natural to ask if a faster computational approach, rather than simply intersecting half spaces, to finding the solution cone is possible. The author's answer is yes. As we will see, by utilizing a concept called parallel congruence $O(nm)$ time is enough to compute the intersection of those n cones and hence the whole problem can be solved in $O(nm)$ time.

2. PRELIMINARIES

First let us define a geometric item called open triangle (or simply OT) as: an open triangle OT is an infinite planar region bounded by two rays emanating from a point p, called the origin of that OT, in 3D space. Figure 1 shows an example.

Each direction cone is an open prism bounded by m such OTs. The intersection of those OTs of a cone with plane H constitutes convex m-gon W. If a direction cone is translated to a new position, the intersection between H and the cone's m translated OTs is still a convex m-gon. The following theorem tells the relation between the two polygons. For the sake of discussion, suppose a convex k-gon is represented as $(e_1, e_2, \ldots, e_k)$, where each e_i is an edge of the polygon and a clockwisely sequential succeeding of $e_1, e_2, \ldots, e_k$ constitutes the polygon. The direction of an edge e_i is decided such that the polygon lies to its right.

THEOREM 1. *Given a direction cone bounded by m OTs, with the OTs' origin at a point p and their intersection with a plane H being a convex m-gon* $W = (w_1, w_2, \ldots, w_m)$. *If this cone is translated to a new position with the new origin at* p', *the intersection between H and m translated OTs is still a convex m-gon* $W' = (w_1', w_2', \ldots, w_m')$, *and each edge* w_i', $i = 1, 2, \ldots, m$ *is parallel to* w_i *and has the same direction as* w_i.

Proof. The convexity and having m edges of W' is conceivable. Consider an open triangle OT_i. Its intersection with H is edge w_i. The derivation of this edge can be thought of two steps: first the plane coplanar with OT_i intersects H and generates an infinite line l; then the two rays emanating from point p intersect l and delimits e_i. Suppose the origin of OT_i is moved to p' and OT_i becomes OT_i', as shown in Fig. 2. Since the cone is under pure translation, the normal of the plane of

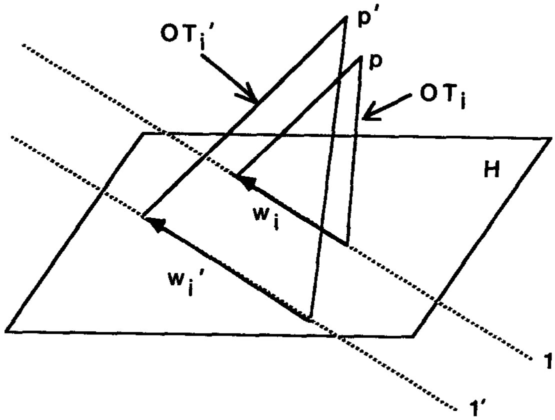

FIG. 2. Parallelism between edges *wi* and *wi'*.

OT_i' should be the same as that of OT_i and correspondingly the intersecting line l' of OT_i' plane with H must be parallel to l, only the length of edge w_i' may be different to that of w_i. The equality of the directions is obvious. Q.E.D.

Polygons W and W' actually bear an important characterization; parallel congruence, as we defined below.

DEFINITION. Two convex polygons $P = (p_1, p_2, \ldots, p_m)$ and $Q = (q_1, q_2, \ldots, q_k)$, $m \leq k$, are said parallel congruent to each other if and only if for each edge p_i there is an edge q_j such that p_i and q_j are parallel and have the same direction.

Figure 3 demonstrates two examples of parallel congruent convex polygons.

If two convex polygons are parallel congruent to each other, it is interesting to ask what would their intersection look like. Generally, the intersection of an arbitrary convex l-gon and an arbitrary m-gon is a convex polygon having up to $1 + m$ vertices.

THEOREM 2. *If a convex l-gon P and a convex m-gon Q are parallel congruent to each other, their intersection is a convex polygon with at most* $\mathrm{Max}\{l, m\}$ *vertices and it is parallel congruent to P and Q too.*

Proof. Without loss of generality, assume $l = m$. $P = (p_1, p_2, \ldots, p_m)$, $Q = (q_1, q_2, \ldots, q_m)$. Let the intersection of P and Q be $C = (c_1, c_2, \ldots, c_r)$. First we note since each c_i is either an edge of P (or Q) or a part of an edge of P (or Q), the

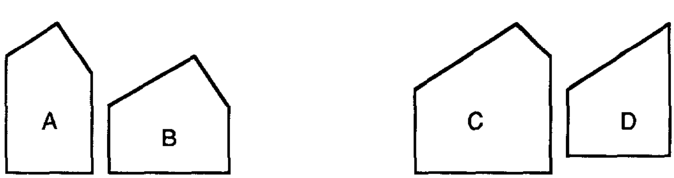

FIG. 3. *A* and *B*, *C* and *D* are parallel congruent to each other respectively.

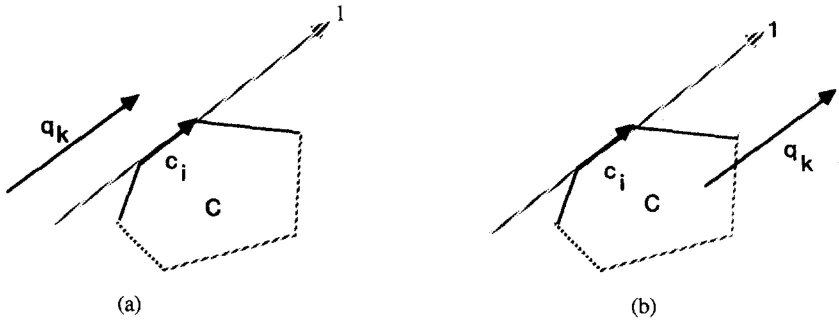

FIG. 4. Mutual exclusion property.

direction of c_i must be the same as that of the constituting edge of P (or Q). Second, due to the convexity each edge of P and Q can constitute at most one edge of C. Now consider Fig. 4. Suppose c_i is constituted by edge p_j and edge q_k is parallel congruent to p_j. The extension ray line l of c_i partitions the plane into two half planes. If q_k is to the left of l, as in (a), it will never intersect any edges of P due to convexity, neither can it be an edge of C. Conversely, if q_k is to the right of l, as in (b), either P and Q do not intersect at all, or the whole polygon C is to the right of q_k because of convexity. In both cases c_i and hence p_j cannot be an edge of C, contradicting our assumption. (In the degenerate case when q_k colides with l, at most one edge of C can be constituted by P and Q.) This mutual exclusion between p_j and q_k implies that each pair of parallel congruent edges of P and Q can constitute at most one edge of C. Therefore $r \leq \text{Max}\{l, m\}$. The proof of the parallel congruence between C and P, Q is trivial. Q.E.D.

3. THE ALGORITHM

With all the preliminaries discussed above, we are now ready to present the algorithm. The main idea is, instead of intersecting the defining half spaces of those cones, the m parallel congruent polygons that are generated by the intersections between plane H and the cones are intersected first and then the resultant convex polygon is mapped back to the origin of D. Suppose polyhedron P has n vertices, window W has m vertices and lies on plane H.

PROCEDURE **Find All** (P, Q, W).
% find the solution cone of a polyhedron P and a window W%
Step 1. For each vertex p_i of P constitute its direction cone CONE(p_i, W);
Step 2. Translate all the n direction cones to a particular vertex point, say p_1;
Step 3. Intersect each translated cone with plane H and let array $HP[1:n]$ store those resultant convex m-gons;
Step 4. $IW \leftarrow HP[1]$
For $i = 2$ to n do
$IW \leftarrow IW$ intersecting $HP[i]$;
Step 5. Draw rays emanating from p_1 and going through each vertex of IW, constitute the solution cone from these rays,
Step 6. return the solution cone
end **Find All**.

Complexity Analysis. Step 1 runs $O(nm)$ time. Step 2 also runs $O(nm)$ time, since for each direction cone we have to translate its m OTs. Step 3 is $O(nm)$. In step 4, $n - 1$ intersections of two parallel congruent convex polygons IW and $HP[i]$, $i = 2, 3, \ldots, n$ are performed. The intermediate intersection polygon IW has at most m edges. Each of these intersection takes at most $O(2m) = O(m)$ time (refer to Preparata and Shamos [2]), resulting in an $O(nm)$ time upper bound for step 4. Step 5 runs at most $O(m)$ time. Overall the algorithm **Find All** takes $O(nm)$ time.

4. CONCLUSION

In this paper, a new approach for finding the intersection of direction cones encountered in the problem of passing a convex polyhedron through a convex window is proposed. Besides the time improvement from $O(nm \log nm)$, as originally given by Toussaint [1], to $O(nm)$, the concept of parallel congruence introduces a new subset of convex polygons. The mutual exclusion property among parallel congruent polygons is not only useful in finding intersection of convex polygons, but might also helpful, as the author expects, in solving some other computational geometry problems.

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