

COMMENTS ON THE NOISE EQUIVALENT SOURCE IN THE LANGEVIN TECHNIQUE

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Abstract—The concept of the “noise equivalent source” in the Langevin equation description of fluctuations in nuclear reactors is reexamined within the framework of the Markov assumption to clarify the ambiguity in the recent literature concerning different interpretations of the Schottky formula. Particular attention is paid to the calculation of the cross power spectral density of two out-of-core detectors to elucidate the effect of the detection process on the observed power spectral densities.

1. INTRODUCTION

This work has been prompted by a recent paper by Difilippo (1988), in which it is stated that there are two different prescriptions for interpreting the Schottky formula, depending on whether or not the detector removes the neutron upon detection. It is implied, in fact, that only one of these interpretations is correct when the detected neutron is removed from the system. Apparently, this point has been a source of controversy among various groups (Difilippo, 1988; Mihalczko *et al.*, 1987). In this communication, we want to point out that the Langevin equation approach for studying fluctuations in linear physical systems is equivalent to the “Master” equation approach, provided the fluctuations can be assumed to be a Markov process. The power spectral density of the noise equivalent source is unambiguously determined by the transition probability rates characterizing the underlying Markov process, and thus leaves no room for different interpretations of the Schottky formula for different modes of detection. Finally, the same expression for the source power spectral density can be used under all conditions and at any level of detail such as diffusion, transport, multi-group, and/or multi-region descriptions. We hope to demonstrate this point by working out simple reactor models with different detector configurations.

We should mention at the outset that there is nothing new in our theoretical discussions that has not already been mentioned in the literature before the mid-seventies. In fact, the following theoretical discussions are based on a paper published by Lax (1960). However, the calculation of the auto- and cross-power spectral densities of the detection rates for out-of-core detectors in terms of random point processes may be novel. Also, in view of the controversy mentioned above, it seems worthwhile to restate some of the earlier work and apply it to several problems in neutronics.

2. A BRIEF DESCRIPTION OF THE MASTER EQUATION APPROACH

Let $\{\mathbf{X}(t)\}$ denote a vector Markov process characterized by $W(\mathbf{X}|\mathbf{X}', t)$, i.e. the transition probability per unit time from a state \mathbf{X}' at time t to another state \mathbf{X} . We assume that the process is homogeneous so that $W(\mathbf{X}|\mathbf{X}')$ is independent of time. The probability density $p(\mathbf{X}, t)$ satisfies the Master equation:

$$\frac{\partial p(\mathbf{X}, t)}{\partial t} = \int d\mathbf{X}' [W(\mathbf{X}|\mathbf{X}')p(\mathbf{X}', t) - W(\mathbf{X}'|\mathbf{X})p(\mathbf{X}, t)]. \quad (1)$$

The transition probability density $P(\mathbf{X}, t|\mathbf{X}_0, t_0)$ ($t > t_0$) also satisfies the same Master equation with the initial condition:

$$\lim_{t \rightarrow t_0^+} P(\mathbf{X}, t | \mathbf{X}_0, t_0) = \delta(\mathbf{X} - \mathbf{X}_0). \tag{2}$$

The equation for the mean is obtained from the Master equation as:

$$\frac{d\bar{\mathbf{X}}(t)}{dt} = \langle \underline{\eta}(\mathbf{X}(t)) \rangle, \tag{3}$$

where $\underline{\eta}(\mathbf{X})$ is called the drift vector and is defined by:

$$\underline{\eta}(\mathbf{X}) = \int d\mathbf{X}' (\mathbf{X}' - \mathbf{X}) W(\mathbf{X}' | \mathbf{X}). \tag{4}$$

For linear systems, such as zero power nuclear reactors without feedback, $\underline{\eta}(\mathbf{X})$ is a linear function of its argument, i.e. $\underline{\eta}(\mathbf{X}) \equiv \underline{\eta}_0 - \Lambda \mathbf{X}$, so that equation (3) reduces to:

$$\frac{d\bar{\mathbf{X}}(t)}{dt} + \Lambda \bar{\mathbf{X}}(t) = \underline{\eta}_0, \tag{5a}$$

where Λ may be identified as a relaxation matrix. The steady-state value of the mean, \mathbf{X}_0 , for linear systems is obtained from $\underline{\eta}_0 = \Lambda \mathbf{X}_0$. The mean of the fluctuations defined by $\underline{\alpha}(t) = \mathbf{X}(t) - \mathbf{X}_0$ during transients following an initial perturbation is thus given by:

$$\frac{d\bar{\underline{\alpha}}(t)}{dt} + \Lambda \bar{\underline{\alpha}}(t) = 0, \tag{5b}$$

with a solution

$$\bar{\underline{\alpha}}(t) = \exp[-\Lambda t] \bar{\underline{\alpha}}(0). \tag{6}$$

In nonlinear systems, equation (3) is not closed. Therefore, equation (5b) is only approximate and is obtained from equation (3) by expanding about an equilibrium state \mathbf{X}_0 , given by $\underline{\eta}(\mathbf{X}_0) = 0$, and ignoring higher order terms in $\underline{\alpha}(t)$ (linearization). The relaxation matrix in this case is defined as:

$$\Lambda = - \frac{\partial \underline{\eta}(\mathbf{X})}{\partial \mathbf{X}}.$$

The correlation matrix $\Phi(t, t')$ is defined by $\Phi(t, t') = \langle \underline{\alpha}(t) \underline{\alpha}^T(t') \rangle$, with the convention $t > t'$. It can be shown that in linear systems:

$$\Phi(t, t') = \exp[-\Lambda(t-t')] \Phi(t', t'), \quad t > t'. \tag{7}$$

In a stationary process, $\Phi(t', t')$, which denotes the variance matrix at t' is independent of time, so that $\Phi(t, t')$ becomes a function of $\tau = t - t'$. Then equation (7) reduces to:

$$\Phi(\tau) = \exp[-\Lambda \tau] \Phi(0), \quad \tau > 0, \tag{8a}$$

$$\Phi(\tau) = \Phi(0) \exp[-\Lambda^T |\tau|], \quad \tau < 0, \tag{8b}$$

where Λ^T denotes the transpose of Λ .

The power spectral density (PSD) matrix $\mathbb{G}(\omega)$ for the case of a stationary process, is defined as the Fourier transform of $\Phi(\tau)$, and is obtained from equations (8) as:

$$\mathbb{G}(\omega) = [i\omega \mathbb{1} + \Lambda]^{-1} \Phi(0) + \Phi(0) [-i\omega \mathbb{1} + \Lambda^T]^{-1}. \tag{9}$$

The static correlation matrix $\Phi(0)$ can also be calculated in terms of the moments of the transition probability per unit time. We first consider the non-stationary case in which $\Phi(t, t) \equiv \sigma^2(t) = \langle \underline{\alpha}(t) \underline{\alpha}^T(t) \rangle$ depends on time. The following equation is found:

$$\frac{d\sigma^2}{dt} = \langle \mathbb{D}(\mathbf{X}(t)) \rangle + \langle \underline{\eta}(\mathbf{X}(t)) \underline{\alpha}^T(t) \rangle + \langle \underline{\alpha}(t) \underline{\eta}^T(\mathbf{X}(t)) \rangle, \tag{10}$$

where

$$\mathbb{D}(\mathbf{X}) \equiv \int d\mathbf{X}' (\mathbf{X}' - \mathbf{X})(\mathbf{X}' - \mathbf{X})^T W(\mathbf{X}' | \mathbf{X}) \quad (11)$$

and is referred to as the diffusion matrix. The process is not yet assumed to be linear in equation (10). When the system is linear, $\langle \mathbb{D}(\mathbf{X}) \rangle$ is a linear function of \mathbf{X} so that $\langle \mathbb{D}(\mathbf{X}(t)) \rangle = \mathbb{D}(\bar{\mathbf{X}}(t))$. Hence equation (10) reduces to:

$$\frac{d\sigma^2(t)}{dt} = \mathbb{D}(\bar{\mathbf{X}}(t)) - \Lambda \sigma^2(t) - \sigma^2(t) \Lambda^T, \quad (12)$$

which is coupled to the equation of the mean $\bar{\mathbf{X}}(t) = \mathbf{X}_0 + \bar{\alpha}(t)$ given in equation (5). In the case of a stationary process, $\bar{\mathbf{X}}(t) \equiv \mathbf{X}_0$ and σ^2 is independent of time, so that:

$$\mathbb{D}(\mathbf{X}_0) = \Lambda \Phi(0) + \Phi(0) \Lambda^T, \quad (13)$$

which is often referred to as the generalized Einstein relation, relating the diffusion matrix to the static correlation (or variance) matrix and the relaxation matrix Λ .

In conclusion, the inputs to the noise analysis in linear systems are Λ and \mathbb{D} , which can be obtained from the description of a given physical system in terms of its parameters, as the following examples will illustrate.

3. THEORETICAL FOUNDATIONS OF THE LANGEVIN EQUATION APPROACH

Let $\underline{\alpha}(t)$ be a particular realization of a vector Markov random process $\{\mathbf{X}(t)\}$ and operate on it by:

$$\left(\frac{d}{dt} + \Lambda \right) \underline{\alpha}(t) = \mathbf{S}(t), \quad (14)$$

to produce a new vector random process $\{\mathbf{S}(t)\}$. $\mathbf{S}(t)$ is often called the "noise equivalent source" and equation (14) is referred to as the "Langevin equation". The statistical properties of $\{\mathbf{S}(t)\}$ can be determined in terms of the properties of the original Markov process $\{\mathbf{X}(t)\}$. One can show that:

$$(i) \quad \langle \mathbf{S}(t) \rangle = 0, \quad (15a)$$

$$(ii) \quad \langle \mathbf{S}(t) \underline{\alpha}^T(t') \rangle = 0, \quad t > t' \quad (\text{causality}), \quad (15b)$$

$$(iii) \quad \langle \mathbf{S}(t) \mathbf{S}^T(t') \rangle = \mathbb{D}(\bar{\mathbf{X}}(t)) \delta(t-t') \quad (\text{white noise}), \quad (15c)$$

where $\mathbb{D}(\mathbf{X})$ was defined, in general, by equation (11). [Here we assume that the system is linear. The Langevin equation description of fluctuations in nonlinear systems is problematic (van Kampen, 1981; Akcasu, 1977)]. We emphasize that the above properties of $\mathbf{S}(t)$ in equations (15) are not introduced as *postulates* but rather *derived* from the Master equation (Lax, 1960; Akcasu, 1977). In the case of stationary processes $\mathbb{D}(\bar{\mathbf{X}}(t)) = \mathbb{D}(\mathbf{X}_0)$ and $\mathbb{D}(\mathbf{X}_0)$ in equation (15c) can be interpreted as the power spectral density \mathbb{G}_{SS} of the vector noise equivalent source, i.e.

$$\mathbb{G}_{SS} = \int d\mathbf{X} (\mathbf{X} - \mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)^T W(\mathbf{X} | \mathbf{X}_0). \quad (16)$$

Equation (16) is what is referred to in reactor noise analysis as the Schottky formula in matrix form. It is the 1-D form of this that was used by Cohn (1960) as the Schottky formula.

The important point here is that the PSD of the noise equivalent source is unambiguously defined in terms of the transition probability per unit time $W(\mathbf{X} | \mathbf{X}')$ of the underlying vector Markov process, and that there is no way to attach different interpretations to it. Furthermore, the Langevin equation and Master equation descriptions are completely equivalent.

In the Langevin equation description, one regards $\underline{\alpha}(t)$ as the response of a linear system with a transfer matrix:

$$\mathbb{T}(s) = [s\mathbb{I} + \Lambda]^{-1}, \quad (17)$$

to a random process $\mathbf{S}(t)$, so that one can take advantage of the known properties of stable linear circuits

when Λ is positive definite. For example, the PSD of the output process can immediately be written as $G(\omega) = T(i\omega)G_{SS}T^T(-i\omega)$ or more explicitly as :

$$G(\omega) = [i\omega\mathbb{1} + \Lambda]^{-1} G_{SS} [-i\omega\mathbb{1} + \Lambda^T]^{-1}, \tag{18}$$

which is identical to the form given in equation (9). In this example, the process is assumed to be stationary. Another example illustrating the utility of the Langevin equation method is to multiply equation (14) by $\underline{\alpha}^T(t')$ where $t > t'$ and average. Using the property in equation (15b), one finds :

$$\frac{\partial \Phi(t, t')}{\partial t} + \Lambda \Phi(t, t') = 0, \quad t > t', \tag{19}$$

the solution of which is given in equation (7).

As a last example, we demonstrate the calculation of $\sigma^2(t) = \langle \underline{\alpha}(t) \underline{\alpha}^T(t) \rangle$. Using the solution of equation (14) as :

$$\underline{\alpha}(t) = \int_0^\infty du \exp[-\Lambda u] \mathbf{S}(t-u) \tag{20}$$

and forming $\langle \underline{\alpha}(t) \underline{\alpha}^T(t) \rangle$, one finds :

$$\sigma^2(t) = \int_0^\infty du \exp[-\Lambda u] \mathbb{D}[\mathbf{X}(t-u)] \exp[-\Lambda^T u], \tag{21}$$

where we have used equation (15c) as $\langle \mathbf{S}(t-u) \mathbf{S}^T(t-v) \rangle = \mathbb{D}[\mathbf{X}(t-u)] \delta(v-u)$. This is precisely the solution of equation (12). In the stationary case $\mathbb{D}[\mathbf{X}(t-u)] = \mathbb{D}(\mathbf{X}_0)$, and hence $\sigma^2(t)$ becomes independent of time.

From a theoretical point of view, the Langevin equation approach lends itself better to generalizations. For example, one can obtain a non-Markovian description of the fluctuations through the so-called ‘‘Generalized Langevin Equation’’ derived microscopically from the Liouville equation using a projection operator technique. This generalized form has been the starting equation in the study of fluctuations of physical systems in statistical mechanics for the past 30 yr (Mori, 1965).

4. THE CONNECTION BETWEEN COUNT-RATES AND ACCUMULATED COUNTS

The statistical description of the detection process always involves the statistical properties of both the count rates $I(t)$ and the accumulated counts $D(t)$ in a time interval $(0, t)$:

$$D(t) = \int_0^t du I(u), \quad t \geq 0. \tag{22}$$

The variance $\sigma_{DD}^2(t)$ of the accumulated counts is related to the variance $C_1(t)$ of the detection rate by :

$$\sigma_{DD}^2(t) = \int_{-t}^t du (t-|u|) C_1(u), \quad t \geq 0, \tag{23}$$

where we have assumed that the count rate fluctuations are stationary. By differentiating once for values of $t > 0$, one finds :

$$\frac{d\sigma_{DD}^2(t)}{dt} = \int_{-t}^t du C_1(u), \quad t > 0. \tag{24}$$

Differentiation once more yields :

$$C_1(t) = \frac{1}{2} \frac{d^2 \sigma_{DD}^2(t)}{dt^2}, \quad t > 0, \tag{25}$$

which is often used to calculate $C_1(t)$ from the variance of the accumulated counts (Pluta, 1961). This relation, however, must be used with care since it yields $C_1(t)$ only for $t > 0$ [and also for $t < 0$ because $C_1(t) = C_1(-t)$] and its limit as $t \rightarrow 0$ fails to reproduce the correct behavior of $C_1(t)$ at $t = 0$ if $C_1(t)$ contains a dirac delta

function at $t = 0$. Indeed, if $C_1(t)$ is replaced in equation (23) by $C_1^R(t) + X\delta(t)$, where $C_1^R(t)$ is finite at $t = 0$, and X is a constant, one finds that the singular term does not contribute to equation (25). This implies that the limit of $C_1(t)$ as $t \rightarrow 0$, calculated from equation (25), yields only the regular part, $C_1^R(t)$, at $t = 0$. The singular part can be determined from equation (24), which becomes :

$$\frac{d\sigma_{DD}^2(t)}{dt} = X + \int_{-t}^t du C_1^R(u), \quad t > 0, \tag{26}$$

where $C_1(t) = X\delta(t) + C_1^R(t)$. Hence, by taking the limit as $t \rightarrow 0^+$, one obtains :

$$X = \lim_{t \rightarrow 0^+} \frac{d\sigma_{DD}^2(t)}{dt}. \tag{27}$$

This result also implies that the power spectral density of the count rate fluctuations is of the following form :

$$G_{ii}(\omega) = X + G_{ii}^R(\omega). \tag{28}$$

Hence, X can be interpreted as the power spectral density of the singular part of $C_1(t)$. [It seems that this term has been missed by Difilippo (1988) in his derivations ; cf. equation (9).] In reactor applications, the physical origin of the singular part of the PSD of the count rates is the probabilistic nature of the detection process. Consequently X is often called "detector noise". The flat spectrum in $G_{ii}(\omega)$ was observed in the power spectral density of the fluctuations in the flux of nuclear reactors by Nomura (1965).

5. APPLICATIONS

5.1. One-group, bare, point reactor model without delayed neutrons

5.1.1. In-Core Detector

Method 1. For this reactor model, the expression for the power spectral density of the count rate fluctuations will be obtained using the Master equation approach. The same problem will then be solved using the Langevin equation method, emphasizing the equivalence of the two approaches.

The state of the system $\mathbf{X}(t)$ can be written as :

$$\mathbf{X}(t) = \begin{bmatrix} N(t) \\ D(t) \end{bmatrix}, \tag{29}$$

where :

$N(t)$ = number of neutrons in the reactor at time t ,

$D(t)$ = number of neutrons detected in the time interval $(0, t)$.

The events resulting in a transition of the state of the system, as well as the corresponding transition probabilities per unit time, $W(N, D|N', D')$, are listed in tabular form below. It is these transition probability rates that define the power spectral densities.

Event	$W(N, D N', D')$
Capture	$r_c N' \delta(N, N' - 1) \delta(D, D')$
Detection	$r_D N' \delta(N, N' - 1) \delta(D, D' + 1)$
Fission	$r_f N' p_f(\nu) \delta(N, N' + \nu - 1) \delta(D, D')$
Source	$S_0 p_s(m) \delta(N, N' + m) \delta(D, D')$

where :

r_j = probability per unit time of event "j" ($j = C, D, f$),

S_0 = probability per unit time that a source event takes place,

$p_f(\nu)$ = probability of ν neutrons produced during a fission event,

$p_s(m)$ = probability of m neutrons produced during a source event

and δ is assumed to be the Kronecker delta function, unless otherwise specified.

The power spectral density, $G_{ii}(\omega)$ (where $i(t) = (dD(t)/dt) - (d\langle D(t) \rangle / dt)$ = detector count rate fluctuations), can be calculated by first determining the following quantities :

• Drift vector :

From equation (4), we find :

$$\begin{aligned} \underline{\eta}(N, D) &= \sum_{N', D'} \begin{bmatrix} N' - N \\ D' - D \end{bmatrix} W(N', D' | N, D) \\ &= - \begin{bmatrix} a & 0 \\ -r_D & 0 \end{bmatrix} \begin{bmatrix} N \\ D \end{bmatrix} + \begin{bmatrix} S_0 \bar{m} \\ 0 \end{bmatrix}, \end{aligned} \tag{30}$$

where $a = (r_c + r_D + r_f) - \bar{v}r_f = r_f - \bar{v}r_f$. We observe that $\underline{\eta}(\mathbf{X})$ is indeed a linear function of \mathbf{X} in the form $\underline{\eta} = \underline{\eta}_0 - \mathbb{\Lambda}\mathbf{X}$ with $\underline{\eta}_0 = [S_0 \bar{m}, 0]^T$ and

$$\mathbb{\Lambda} = \begin{bmatrix} a & 0 \\ -r_D & 0 \end{bmatrix}. \tag{31}$$

Note that $\mathbb{\Lambda}$ is not a positive definite matrix when the accumulated counts $D(t)$ are included in the state vector as one of its components.

• Diffusion matrix :

For this system, equation (11) reduces to :

$$\begin{aligned} \mathbb{D}(N, D) &= \sum_{N', D'} \begin{bmatrix} N' - N \\ D' - D \end{bmatrix} [N' - N, D' - D] W(N', D' | N, D) \\ &= \begin{bmatrix} aN + r_f \bar{N} \bar{v}(\bar{v} - 1) + S_0 \bar{m}^2 & -r_D N \\ -r_D N & r_D N \end{bmatrix}. \end{aligned} \tag{32}$$

We observe that $\mathbb{D}(N, D)$ is a linear function of N, D . It is, in fact, independent of D . The equations for the variances can now be found from equation (12). This gives :

$$\frac{d\sigma_{NN}^2}{dt} = -2a\sigma_{NN}^2 + r_f \bar{N} \bar{v}(\bar{v} - 1) + S_0 \bar{m}^2 + a\bar{N}, \tag{33}$$

$$\frac{d\sigma_{ND}^2}{dt} = -r_D \bar{N} - a\sigma_{ND}^2 + r_D \sigma_{NN}^2, \tag{34}$$

$$\frac{d\sigma_{DD}^2}{dt} = 2r_D \sigma_{ND}^2 + r_D \bar{N}. \tag{35}$$

When we consider a system that is stationary, $d\bar{N}/dt = 0$ and $d\sigma_{NN}^2/dt = 0$, and the steady-state neutron number density becomes $\bar{N} = N_0 = S_0 \bar{m}/a$. Thus, the variance of the accumulated counts $C_1(t)$ can be calculated from equations (25) and (27) where one finds :

$$C_1(t) = r_D N_0 \delta(t) + r_D^2 (\sigma_{NN}^2 - N_0) e^{-a|t|} \tag{36}$$

and the power spectral density becomes

$$G_{ii}(\omega) = r_D N_0 \left[1 + r_D \left(\frac{\sigma_{NN}^2}{N_0} - 1 \right) \frac{2a}{a^2 + \omega^2} \right], \tag{37}$$

where

$$\frac{\sigma_{NN}^2}{N_0} - 1 = \frac{r_f \bar{v}(\bar{v} - 1) + \frac{S_0}{N_0} \bar{m}(\bar{m} - 1)}{2a}, \tag{38}$$

which is the usual result for the variance-to-mean ratio. Equation (37) differs from equation (11) of Difilippo (1988) in that the constant term does not appear in the latter.

Method 2. The same reactor model is again considered as in Section 5.1.1, Method 1, but the power spectral

density of the count rate fluctuations is now obtained using the Langevin equation method. We immediately write equation (14), the stochastic equations for fluctuations in the state vector, as :

$$\frac{d\delta N(t)}{dt} + a\delta N(t) = s_N(t), \tag{39a}$$

$$\frac{d\delta D(t)}{dt} - r_D\delta N(t) = s_D(t), \tag{39b}$$

where

$$\underline{\alpha}(t) = \begin{bmatrix} \delta N(t) \\ \delta D(t) \end{bmatrix} = \begin{bmatrix} N(t) - N_0 \\ D(t) - \bar{D}(t) \end{bmatrix} \tag{40}$$

and

$$\mathbf{S}(t) = \begin{bmatrix} s_N(t) \\ s_D(t) \end{bmatrix} = \text{noise equivalent source.} \tag{41}$$

The fluctuation in the detector count rate is identical to $d\delta D(t)/dt$, and equation (39b) can be rewritten as :

$$i_D(t) = r_D\delta N(t) + s_D(t). \tag{42}$$

We can now write the correlation function as :

$$\langle i_D(t)i_D(t') \rangle = r_D^2 \langle \delta N(t)\delta N(t') \rangle + r_D \langle \delta N(t)s_D(t') \rangle + r_D \langle \delta N(t')s_D(t) \rangle + \langle s_D(t)s_D(t') \rangle. \tag{43}$$

Each of the terms in equation (43) are evaluated by employing the statistical properties of $\mathbf{S}(t)$ presented in equations (15). The power spectral density matrix for the noise equivalent source in equation (16) becomes :

$$\mathbb{G}_{SS} = \begin{bmatrix} G_{SS} & -r_D N_0 \\ -r_D N_0 & r_D N_0 \end{bmatrix}, \tag{44}$$

where G_{SS} is the PSD of the noise equivalent source, $s_N(t)$, appearing in the neutron balance equation [equation (39a)] and is found to be :

$$\begin{aligned} G_{SS} &= r_C N_0 + r_D N_0 + \overline{(v-1)^2 r_f N_0 + S_0 m^2} \\ &= a N_0 + \overline{v(v-1) r_f N_0 + S_0 m^2} \\ &= \overline{v(v-1) r_f N_0 + S_0 m(m+1)} \\ &= 2a N_0 + \overline{v(v-1) r_f N_0 + S_0 m(m-1)}. \end{aligned} \tag{45}$$

All the forms given for G_{SS} are identical and correct. There is no need to introduce different interpretations of the Schottky formula as done by Difilippo (1988) [see equations (13) and (14) of the cited reference]. The Schottky formula is defined, in general, by equation (16) in vector form and leaves no room for interpretation.

- $\langle \delta N(t)\delta N(t') \rangle$:

The power spectral density of $\delta N(t)$ can simply be written as :

$$G_{NN}(\omega) = \langle \delta N(\omega)\delta N^*(\omega) \rangle = \frac{G_{SS}}{a^2 + \omega^2}, \tag{46}$$

where the different forms of G_{SS} are given in equation (45).

- $\langle \delta N(t)s_D(t') \rangle$:

We can write $\delta N(t)$ as :

$$\delta N(t) = \int_0^\infty e^{-au} s_N(t-u) du, \tag{47}$$

so that

$$\begin{aligned}
\langle \delta N(t) s_D(t') \rangle &= \int_0^\infty e^{-au} \langle s_N(t-u) s_D(t') \rangle du \\
&= -r_D N_0 e^{-a(t-t')} \quad \text{for } t > t' \\
&= 0 \quad \text{for } t < t',
\end{aligned} \tag{48}$$

where we have used $\langle s_N(t-u) s_D(t') \rangle = -r_D N_0 \delta(t-t'-u)$ which follows from equation (44).

- $\langle \delta N(t') s_D(t) \rangle$:

In the same way:

$$\begin{aligned}
\langle \delta N(t') s_D(t) \rangle &= -r_D N_0 e^{-a(t-t')} \quad \text{for } t' > t \\
&= 0 \quad \text{otherwise.}
\end{aligned} \tag{49}$$

- $\langle s_D(t) s_D(t') \rangle = r_D N_0 \delta(t-t')$. (50)

The power spectral density for the count rate fluctuations can now be obtained from equation (43) using the above results:

$$\begin{aligned}
G_{ii}(\omega) &= r_D^2 G_{NN}(\omega) - r_D^2 N_0 \frac{1}{a-i\omega} - r_D^2 N_0 \frac{1}{a+i\omega} + r_D N_0 \\
&= r_D^2 \frac{G_{SS}}{a^2 + \omega^2} - \frac{2ar_D^2 N_0}{a^2 + \omega^2} + r_D N_0,
\end{aligned} \tag{51}$$

or finally

$$G_{ii}(\omega) = r_D N_0 \left[1 + r_D \frac{r_f v(v-1) + \frac{S_0}{N_0} m(m-1)}{a^2 + \omega^2} \right], \tag{52}$$

which is identical to equation (37). The cancellation of $2aN_0$ in the expression for G_{SS} in equation (45) by the terms arising from the negative cross correlation of $s_N(t)$ and $s_D(t)$ given by equations (48) and (49) [see also equation (51)] is to be noted. This cancellation corresponds to detection by removing a neutron.

We are now in a position to discuss the controversial point of interpreting the Schottky formula in two different ways. To do so, we first rewrite equation (52) as:

$$G_{ii}(\omega) = G_{ii}^D(\omega) + G_{ii}^N(\omega), \tag{53}$$

where $G_{ii}^D(\omega) = r_D N_0$ and accounts for the direct contribution of the detector noise $s_D(t)$ in equation (39b), to $G_{ii}(\omega)$. Although this term is not present in the expression for the APSD of the detection rate in equation (12) by Difilippo (1988), its absence is not relevant to the controversy. The second term contains the contributions of both the detector and neutronic noise, $s_D(t)$ and $s_N(t)$ appearing in equations (39), and is found in the present example to be:

$$G_{ii}^N(\omega) = \frac{r_D^2}{a^2 + \omega^2} G'_{SS}, \tag{54}$$

where the first factor is the magnitude of the transfer function from $s_N(\omega)$ to the detection rate $i_D(\omega)$. The factor G'_{SS} reduces to:

$$G'_{SS} = S_0 \overline{m(m-1)}, \tag{55}$$

in the absence of fission, as was the case in equation (11) by Difilippo. Equation (55) is not equal to the PSD of the original neutronic noise equivalent source, which was calculated in equation (45) to be:

$$G_{SS} = S_0 \overline{m(m+1)}, \tag{56}$$

in the absence of fission. [Equation (56) is the form appearing in equation (10) by Difilippo when a neutron is not removed by detection.] Two different interpretations of the Schottky formula are now introduced by

Difilippo (1988) to account for these two different forms in equations (55) and (56). However, there is no need for such an interpretation since the form for G'_{SS} is obtained directly as:

$$G'_{SS} = G_{SS} - 2aN_0,$$

when one accounts for the cross correlation in detector and neutronic noise which arises when a detector removes a neutron upon detection. Therefore, resorting to "recipes" for finding the correct form for G'_{SS} is not necessary and would prove confusing and difficult to do for every mode of detection. Since the Schottky formula in equation (16) yields the PSDs of the noise equivalent sources in the stochastic Langevin equations, it is not clear how to associate G'_{SS} with a noise equivalent source in these equations. The vector Langevin equation method, however, can be applied directly once the events and their corresponding transition probability rates are correctly identified for a given physical model of detection, as the following examples will illustrate.

5.1.2. Single Out-of-Core Detector

When the detector is outside of the core, one must first obtain the statistical properties of the leakage current, such as its power spectral density, because detectors are driven by the neutrons that escape the core. Since the core is assumed to be bare (vacuum), we can no longer use diffusion theory outside the core where the detector is assumed to be located. The statistical properties of the fluctuations in the leakage rate can easily be obtained from the treatment of in-core detection by simply replacing r_D by r_L , the probability per unit time that a neutron will leak out of the core. Hence, one can now write directly from equation (52):

$$G_{i,t}(\omega) = r_L N_0 \left[1 + r_L \frac{r_f v(v-1) + \frac{S_0}{N_0} m(m-1)}{a^2 + \omega^2} \right]. \quad (57)$$

A neutron that leaks out will either be detected instantaneously (ignoring transit times in free streaming) with probability p , or will be lost permanently. Therefore, the detection must be treated as a part of the leakage event. We shall see that the situation is different when the out-of-core detector is located in a reflector. Before we obtain the statistics of the count rates for a single out-of-core detector directly from the statistical properties of the leakage current, we shall first present a method which is analogous to the one we used in the case of the in-core detector.

Method 1. We redefine the leakage event in the core as one in which the number of neutrons in the core decreases by one and the number of accumulated counts in the out-of-core detector increases by ε with probability $P(\varepsilon)$. Here ε will have values 0 and 1 and $P(\varepsilon = 1) = p =$ detection probability. The similarity between the treatment of this extended leakage event and a source event is to be noted. We therefore write the transition probability per unit time associated with the leakage event as:

$$W(N, D | N', D') = r_L N' P(\varepsilon) \delta(N, N' - 1) \delta(D, D' + \varepsilon). \quad (58)$$

The transition probabilities of the other events remain unchanged. The components of the drift vector now become:

$$\eta_N = \text{same as in equation (30)}, \quad (59a)$$

where a in this case becomes $a = (r_C + r_L + r_f) - \bar{v}r_f$.

$$\begin{aligned} \eta_D &= \sum_{N', D'} (D' - D) W(N', D' | N, D) = r_L N \sum_{\varepsilon} P(\varepsilon) \varepsilon \\ &= r_L N [p \cdot 1 + (1-p) \cdot 0] \\ &= pr_L N. \end{aligned} \quad (59b)$$

The elements of the diffusion matrix can be found as:

$$D_{NN} = \text{same as in equation (32)}, \quad (60a)$$

$$D_{ND} = \sum_{N', D'} (N' - N)(D' - D) W(N', D' | N, D)$$

$$\begin{aligned}
&= -r_L N \sum_{\varepsilon} P(\varepsilon) \varepsilon \\
&= -pr_L N \\
&= D_{DN}, \tag{60b}
\end{aligned}$$

$$\begin{aligned}
D_{DD} &= \sum_{N', D'} (D' - D)^2 W(N', D' | N, D) \\
&= r_L N \sum_{\varepsilon} P(\varepsilon) \varepsilon^2 \\
&= pr_L N. \tag{60c}
\end{aligned}$$

The PSD of the detection rate can be written immediately by replacing r_L in equation (57) by pr_L and hence:

$$G_{i_{\text{ob}}}(\omega) = pr_L N_0 \left[1 + pr_L \frac{r_L v(v-1) + \frac{S_0}{N_0} m(m-1)}{a^2 + \omega^2} \right], \tag{61}$$

for an out-of-core detector.

Method 2. This method is simpler and yet more physical. It relates the PSD of the detection rate to that of the leakage rate. We can visualize the leakage current as a sequence of random points on the time axis and express the instantaneous value of the leakage current $I_L(t)$ as:

$$I_L(t) = \sum_k \delta(t - t_k), \tag{62}$$

such that $\langle I_L(t) \rangle = r_L N_0$ is the mean count rate. Equation (62) is nothing more than an expression for the instantaneous number density of points on the time axis. The autocovariance function of $I_L(t)$ is given [see equation (57)] by:

$$\langle i_L(t) i_L(t') \rangle = r_L N_0 \left[\delta(\tau) + r_L \frac{r_L v(v-1) + \frac{S_0}{N_0} m(m-1)}{2a} e^{-a|\tau|} \right], \tag{63}$$

where $\tau = t - t'$. The detection rate $I_D(t)$ can also be expressed as the sum of delta functions by:

$$I_D(t) = \sum_k \varepsilon_k \delta(t - t_k), \tag{64}$$

where the random points t_k again represent the instants of time when a neutron escapes the core. The ε_k s, with probability $P(\varepsilon_k)$, indicate whether a neutron that escapes the core at $t = t_k$ is detected ($\varepsilon_k = 1$) or lost ($\varepsilon_k = 0$).

Now the calculation of the power spectral density of the detection rate is reduced to simple manipulations. The mean detection rate of equation (64) is just $\langle I_D \rangle = pr_L N_0$, where we use $\langle \varepsilon_k \rangle = p$. The autocorrelation function follows as:

$$\langle I_D(t) I_D(t') \rangle = \sum_{k, k'} \langle \varepsilon_k \varepsilon_{k'} \delta(t - t_k) \delta(t' - t_{k'}) \rangle. \tag{65}$$

The ensemble average on the right-hand-side can be written as the product of the averages $\langle \varepsilon_k \varepsilon_{k'} \rangle$ and $\langle \delta(t - t_k) \delta(t' - t_{k'}) \rangle$ because the random variables $\{\varepsilon_k\}$ and the point process $\{t_k\}$ are independent of each other. Hence, separating the diagonal term with $k = k'$ and using $\langle \varepsilon_k^2 \rangle = p$ and $\langle \varepsilon_k \varepsilon_{k'} \rangle = p^2$ for $k \neq k'$, we find:

$$\langle I_D(t) I_D(t') \rangle = p \delta(t - t') r_L N_0 + p^2 \left\langle \sum_{\substack{k, k' \\ k \neq k'}} \delta(t - t_k) \delta(t' - t_{k'}) \right\rangle. \tag{66}$$

Adding and subtracting when $k = k'$ in the second term, one obtains an interesting result:

$$\langle I_D(t)I_D(t') \rangle = p(1-p)\delta(t-t')r_L N_0 + p^2 \langle I_L(t)I_L(t') \rangle, \quad (67)$$

which relates the autocorrelation functions of the detection rate and the leakage current. The presence of the term with $\delta(t-t')$ and its coefficient $p(1-p)$ is particularly to be noted. When $p = 1$, this term vanishes, as it must, because in this case $I_D(t) \equiv I_L(t)$.

The remaining task is to calculate the autocovariance and PSD of the detection rate by first subtracting the mean values of $I_L(t)$ and $I_D(t)$, and working with fluctuations $i_L(t) = I_L(t) - r_L N_0$ and $i_D(t) = I_D(t) - p r_L N_0$. One finds:

$$\langle i_D(t)i_D(t+\tau) \rangle = p(1-p)\delta(\tau)r_L N_0 + p^2 \langle i_L(t)i_L(t+\tau) \rangle. \quad (68)$$

The PSD calculated from equation (68) is identical to equation (61).

5.1.3. Two Out-of-Core Detectors

This method is easily extended to calculate the cross power spectral density (CPSD) of two detectors by writing the detection rates in detectors 1 and 2 as:

$$\begin{aligned} I_{D1}(t) &= \sum_k \delta(t-t_k) \varepsilon_k^{(1)}, \\ I_{D2}(t) &= \sum_k \delta(t-t_k) \varepsilon_k^{(2)}. \end{aligned} \quad (69)$$

Using $\langle \varepsilon_k^{(1)} \rangle = p_1$ and $\langle \varepsilon_k^{(2)} \rangle = p_2$,

$$\begin{aligned} \langle \varepsilon_k^{(i)} \varepsilon_{k'}^{(j)} \rangle &= p_i p_j \quad k \neq k', \\ \langle \varepsilon_k^{(i)} \varepsilon_k^{(j)} \rangle &= p_j \quad i = j \\ &= 0 \quad i \neq j. \end{aligned} \quad (70)$$

The last equality indicates that a particular neutron emitted at t_k cannot be detected in both detectors. With these observations and with the procedure described above, one finds:

$$\langle i_{D1}(t)i_{D2}(t+\tau) \rangle = p_1 p_2 \langle i_L(t)i_L(t+\tau) \rangle - p_1 p_2 r_L N_0 \delta(\tau) \quad (71)$$

and

$$G_{i_{D1}i_{D2}}(\omega) = p_1 p_2 r_L^2 \frac{r_L N_0 v(v-1) + S_0 m(m-1)}{a^2 + \omega^2}. \quad (72)$$

It is noted that, in this case, the detector noise term is absent. These results have been observed experimentally by Nomura (1965).

5.2. One-group, reflected, point reactor model without delayed neutrons with a detector in the reflector

The core is now surrounded by a reflecting medium which contains the neutron detector. Once again we seek the expression for the power spectral density of the detector count rate fluctuations. The situation is different from the out-of-core detector case in the bare core model. A neutron, upon leaking out of the core into the reflector, diffuses and may be captured, reflected or detected with a finite probability per unit time r_j , where j once again denotes the particular removal event. Its detection or escape is no longer instantaneous. In this respect, the present model physically illustrates the opposite extreme of instantaneous detection of neutrons leaking out of the core.

We now write the state of the system as:

$$\mathbf{X}(t) = \begin{bmatrix} N_C(t) \\ N_R(t) \\ D(t) \end{bmatrix}, \quad (73)$$

where

$N_C(t)$ = number of neutrons in the core at time t ,

$N_R(t)$ = number of neutrons in the reflector at time t ,

$D(t)$ = number of neutrons detected in the interval $(0, t)$.

The transition probabilities per unit time corresponding to all possible transitional events in this reactor model are tabulated below.

Event	$W(N_C, N_R, D N'_C, N'_R, D')$
Core	
Capture	$r_c N'_C \delta(N_C, N'_C - 1) \delta(N_R, N'_R) \delta(D, D')$
Fission	$r_f N'_C p_f(v) \delta(N_C, N'_C + v - 1) \delta(N_R, N'_R) \delta(D, D')$
Source	$S_0 p_s(m) \delta(N_C, N'_C + m) \delta(N_R, N'_R) \delta(D, D')$
Leakage	$r_{CR} N'_C \delta(N_C, N'_C - 1) \delta(N_R, N'_R + 1) \delta(D, D')$
Reflector	
Reflection	$r_{RC} N'_R \delta(N_C, N'_C + 1) \delta(N_R, N'_R - 1) \delta(D, D')$
Capture/loss	$r_{R\infty} N'_R \delta(N_C, N'_C) \delta(N_R, N'_R - 1) \delta(D, D')$
Detection	$r_D N'_R \delta(N_C, N'_C) \delta(N_R, N'_R - 1) \delta(D, D' + 1)$

The stochastic equations for the fluctuations in the state vector quantities from their respective equilibrium values can now immediately be written as:

$$\frac{dn_C(t)}{dt} = -an_C(t) + r_{RC}n_R(t) + s_N(t) \quad (a = r_c + r_{CR} + r_f - \bar{v}r_f), \quad (74)$$

$$\frac{dn_R(t)}{dt} = -bn_R(t) + r_{CR}n_C(t) + s_R(t) \quad (b = r_{RC} + r_{R\infty} + r_D), \quad (75)$$

$$\frac{d\delta D(t)}{dt} = r_D n_R(t) + s_D(t), \quad (76)$$

where

$$\underline{\alpha}(t) = \begin{bmatrix} n_C(t) \\ n_R(t) \\ \delta D(t) \end{bmatrix} = \begin{bmatrix} N_C(t) - N_0 \\ N_R(t) - \bar{N}_R \\ D(t) - \bar{D}(t) \end{bmatrix}, \quad (77)$$

$$\mathbf{S}(t) = \begin{bmatrix} s_N(t) \\ s_R(t) \\ s_D(t) \end{bmatrix} = \text{noise equivalent source} \quad (78)$$

and, from equation (15c), we know $\langle \mathbf{S}(t) \mathbf{S}^T(t') \rangle = \mathbb{D}(\bar{\mathbf{X}}(t)) \delta(t - t')$, where:

$$\mathbb{D}(\bar{\mathbf{X}}(t)) = \begin{bmatrix} 2a\bar{N}_C + r_f\bar{N}_C\overline{v(v-1)} + S_0\overline{m(m-1)} & -(b+r_{RC})\bar{N}_R & 0 \\ -(b+r_{RC})\bar{N}_R & 2b\bar{N}_R & -r_D\bar{N}_R \\ 0 & -r_D\bar{N}_R & r_D\bar{N}_R \end{bmatrix}.$$

By following the same procedure as in Section 5.1.1, Method 2, namely, by applying the Langevin equation method to this reactor system, we can express the PSD of the detector count rate fluctuations as:

$$G_{ii}(\omega) = r_D^2 \langle |n_R(\omega)|^2 \rangle + 2r_D \operatorname{Re} [\langle n_R(\omega) s_D^*(\omega) \rangle] + \langle |s_D(\omega)|^2 \rangle \quad (79)$$

and one finds, through some straightforward manipulations, that the PSD becomes:

$$G_{ii}(\omega) = r_D \bar{N}_R \left[1 + r_{CR} r_D b \frac{r_f \overline{v(v-1)} + \frac{S_0}{\bar{N}_C} \overline{m(m-1)}}{|\Delta(\omega)|^2} \right], \quad (80)$$

where

$$\Delta(\omega) = (a + i\omega)(b + i\omega) - r_{CR}r_{RC}.$$

5.3. One-group, bare, point reactor with three detectors

In this section, we consider two out-of-core detectors and a third in-core detector which exclusively detects the occurrence of a source event. The CPSD of these detectors is calculated in order to determine a ratio of power spectral densities that has been used for experimental determination of core reactivity in subcritical, ^{252}Cf source-driven systems (Mihalcz *et al.*, 1987). This ratio is especially ingenious since it is independent of the out-of-core detector efficiencies. This application is included here to clear up any controversy that may have arisen over the calculation of this CPSD ratio. We consider the simplest reactor model in order to demonstrate how the general formulation we presented above can be implemented to calculate the CPSDs. We do not consider such refinements as neutron importance and the transfer functions of the electronic components for the sake of clarity, because these refinements are not controversial (Difilippo, 1988; Mihalcz *et al.*, 1987).

For this reactor system, the state vector is written as:

$$\mathbf{X}(t) = \begin{bmatrix} N(t) \\ D_1(t) \\ D_2(t) \\ D_3(t) \end{bmatrix}, \quad (81)$$

where $N(t)$ and $D(t)$ are defined as before, but now accumulated counts are defined for detectors 1 (in-core source detector), 2 and 3 (out-of-core detectors). A source event can now be described as one in which the number of neutrons in the core increases by m and the number of accumulated counts in detector 1 increases by ε_1 , where ε_1 assumes values of 1 and 0, depending on whether or not the source event is detected. As before, we define $p(\varepsilon_1 = 1) = p_1 =$ source detection probability. The transition probability rate of the source event can then be written as:

$$W_{\text{source}}(N, D_1, D_2, D_3 | N', D'_1, D'_2, D'_3) = S_0 p_s(m) p(\varepsilon_1) \delta(N, N' + m) \delta(D_1, D'_1 + \varepsilon_1) \delta(D_2, D'_2) \delta(D_3, D'_3). \quad (82)$$

A leakage event is defined as one in which a neutron leaves the core and is either counted in detector 2, counted in detector 3 or not counted at all. The transition probability rate for this event can be expressed as:

$$W_{\text{leakage}}(N, D_1, D_2, D_3 | N', D'_1, D'_2, D'_3) \\ = r_L N' p(\varepsilon_2, \varepsilon_3) \delta(N, N' - 1) \delta(D_1, D'_1) \delta(D_2, D'_2 + \varepsilon_2) \delta(D_3, D'_3 + \varepsilon_3). \quad (83)$$

The calculation of the necessary power spectral densities is performed by following the previously outlined Langevin method. We can write down the fluctuation vector as $\alpha(t) = \text{col}[\delta N(t), \delta D_1(t), \delta D_2(t), \delta D_3(t)]$ and the noise equivalent source as $\mathbf{S}(t) = \text{col}[s_N(t), s_{D1}(t), s_{D2}(t), s_{D3}(t)]$. The appropriate stochastic equations in this case are:

$$\frac{d\delta N(t)}{dt} = -a \delta N(t) + s_N(t), \quad (84a)$$

$$\frac{d\delta D_1(t)}{dt} = s_{D1}(t), \quad (84b)$$

$$\frac{d\delta D_2(t)}{dt} = p_2 r_L \delta N(t) + s_{D2}(t), \quad (84c)$$

$$\frac{d\delta D_3(t)}{dt} = p_3 r_L \delta N(t) + s_{D3}(t). \quad (84d)$$

Here p_i ($i = 2, 3$) is defined as the probability of neutron detection for detector i . The diffusion matrix is then found to be:

$$\mathbb{D}(\bar{\mathbf{X}}) = \mathbb{G}_{SS} = \begin{bmatrix} a\bar{N} + \overline{v(v-1)r_f\bar{N}} + S_0\overline{m^2} & p_1\bar{m}S_0 & -p_2r_L\bar{N} & -p_3r_L\bar{N} \\ p_1\bar{m}S_0 & p_1S_0 & 0 & 0 \\ -p_2r_L\bar{N} & 0 & p_2r_L\bar{N} & 0 \\ -p_3r_L\bar{N} & 0 & 0 & p_3r_L\bar{N} \end{bmatrix}. \quad (85)$$

The ratio of power spectral densities to be calculated is defined by:

$$S_R = \frac{G_{12}^*(\omega)G_{13}(\omega)}{G_{11}(\omega)G_{23}(\omega)}, \quad (86)$$

where $G_{12}(\omega)$ and $G_{13}(\omega)$ are the CPSDs for the in-core and each of the out-of-core detectors, $G_{23}(\omega)$ is the CPSD between the out-of-core detectors, and $G_{11}(\omega)$ is the auto power spectral density (APSD) of the in-core source detector. The transition probability rates uniquely define these PSDs to be:

$$G_{11}(\omega) = p_1S_0, \quad (87a)$$

$$G_{12}(\omega) = \frac{p_1p_2r_L\bar{m}S_0(a-i\omega)}{a^2 + \omega^2}, \quad (87b)$$

$$G_{13}(\omega) = \frac{p_1p_3r_L\bar{m}S_0(a-i\omega)}{a^2 + \omega^2}, \quad (87c)$$

$$G_{23}(\omega) = \frac{p_2p_3r_L^2[\overline{v(v-1)r_f\bar{N}} + S_0\overline{m(m-1)}]}{a^2 + \omega^2}. \quad (87d)$$

The ratio S_R now becomes:

$$S_R = \frac{p_1\bar{m}^2S_0}{\overline{v(v-1)r_f\bar{N}} + S_0\overline{m(m-1)}}. \quad (88)$$

This method for measuring subcriticalities of an assembly of fissile material has been studied and performed at ORNL (Mihalcz *et al.*, 1987). We find that the ratio of CPSDs, S_R , in a paper by Mihalcz *et al.* (1987) differs from the calculation of our ratio in equation (88). In equation (A.5) of this reference, the denominator does not vanish when $v \equiv 1$ and $m \equiv 1$.

5.4. Bare source with three detectors

In this section, we calculate the CPSD ratio, S_R , for a bare source accounting now for the angular correlation of the emitted source neutrons. We define a source event as a sequence of random points on the time axis. At each random point t_k , m_k neutrons are produced, m_{2k} of which are intercepted by detector 2 and ε_{2k} of m_{2k} are detected instantaneously. Also, m_{3k} of the m_k neutrons are intercepted by detector 3 where ε_{3k} of m_{3k} are detected simultaneously. Each source event is either recorded in detector 1 ($\varepsilon_{1k} = 1$) or not recorded ($\varepsilon_{1k} = 0$). The appropriate state vector in this problem is:

$$\mathbf{X}(t) = \begin{bmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \end{bmatrix}. \quad (89)$$

The associated transition probability per unit time for this single, extended source event can be written as:

$$W(D_1, D_2, D_3 | D'_1, D'_2, D'_3) = S_0 p(\varepsilon_1, \varepsilon_2, \varepsilon_3, m_2, m_3, m, \theta) \delta(D_1, D'_1 + \varepsilon_1) \delta(D_2, D'_2 + \varepsilon_2) \delta(D_3, D'_3 + \varepsilon_3). \quad (90)$$

The joint probability distribution function in equation (90) can be broken up as:

$$p(\varepsilon_1, \varepsilon_2, \varepsilon_3, m_2, m_3, m, \theta) = p(\varepsilon_1)B(\varepsilon_2 | m_2)B(\varepsilon_3 | m_3)p(m_2, m_3, \theta | m)p(m), \quad (91)$$

where B represents the binomial distribution. Here we assume that the detectors are perfect so that neutrons impinging on them can be counted independently with the same probability (no dead time, etc.). For this

system, the drift vector simply reduces to $\eta(\mathbf{X}) = \eta_0 = S_0 \text{col}[\langle \varepsilon_1 \rangle, \langle \varepsilon_2 \rangle, \langle \varepsilon_3 \rangle]$, so that the relaxation matrix becomes $\Lambda = \mathbf{0}$. The elements of the diffusion matrix can be written as:

$$D_{ij} = \langle \varepsilon_i \varepsilon_j \rangle S_0, \quad (92)$$

so that the solution of this problem reduces to the evaluation of $\langle \varepsilon_i \varepsilon_j \rangle$ for $i, j = 1, 2, 3$. One could now write the stochastic equations for the fluctuations in the state vector and continue with the Langevin equation method as discussed above. One finds that the PSD matrix \mathbb{G} of the detector rates is simply equal to the diffusion matrix. We, however, return to the problem formulation that was originally introduced to calculate the CPSD ratio, S_R .

Since the detection rate in all three detectors can be represented as a sequence of random points on the time axis, we can define the following detector count rates as:

$$I_{D_i}(t) = \sum_k \varepsilon_{ik} \delta(t - t_k), \quad (93)$$

where $i (= 1, 2, 3)$ represents the particular detector. As we did in the previous two-detector problem, we now apply equation (93) and cross-correlate fluctuations in the detector count rates to determine the desired PSD matrix. We find that the APSD for detector 1 becomes:

$$G_{11} = \langle \varepsilon_1^2 \rangle S_0 = p_1 S_0. \quad (94)$$

In addition, the CPSDs of the in-core and out-of-core detectors are found to be:

$$G_{12} = \langle \varepsilon_1 \varepsilon_2 \rangle S_0, \quad (95)$$

$$G_{13} = \langle \varepsilon_1 \varepsilon_3 \rangle S_0. \quad (96)$$

We can determine $\langle \varepsilon_1 \varepsilon_2 \rangle$ and $\langle \varepsilon_1 \varepsilon_3 \rangle$ using the joint probability distribution in equation (91) to obtain the following results:

$$G_{12} = p_1 p_2 \Omega_2 \bar{m} S_0, \quad (97)$$

$$G_{13} = p_1 p_3 \Omega_3 \bar{m} S_0, \quad (98)$$

where Ω_i is the probability that an original source neutron will be intercepted by detector i .

Finally, the CPSD between detectors 2 and 3 was found to be:

$$\begin{aligned} G_{23} &= \langle \varepsilon_2 \varepsilon_3 \rangle S_0 \\ &= p_2 p_3 \Omega_2 \Omega_3 \overline{m(m-1)} G(\theta) S_0, \end{aligned} \quad (99)$$

where $G(\theta)$ is a function which depends on the angle between detectors 2 and 3 as discussed by Difilippo (1988). The form of equation (99) can be obtained if: (1) one assumes a binomial distribution for $p(m_2, m_3, \theta | m)$; or (2) by the argument that $\langle \varepsilon_2 \varepsilon_3 \rangle$ depends, in general, on the number of source pairs produced ($m(m-1)$), the probability of source interception by each of the out-of-core detectors, and some function of the angle between the two detectors.

The CPSD ratio immediately reduces to:

$$S_R = \frac{p_1 \bar{m}^2}{m(m-1)G(\theta)}. \quad (100)$$

Although the constant spectrum accounting for detector noise was absent in equation (12) by Difilippo (1988) for the APSD, equation (100) is identical to Difilippo's (1988) results in equation (47), since the detector noise term vanishes when cross correlations are used. However, this result is at variance with equation (A.5) by Mihalczko (1987).

6. DISCUSSION

The main idea behind the Langevin equation approach to the calculation of power spectral densities of fluctuations of dynamical variables in linear systems is to regard these fluctuations as the response of the systems to a set of stochastic inputs, referred to in nuclear engineering as the noise equivalent source (NES).

The PSD matrix of the dynamical variables can then be expressed in terms of the transfer matrix of the time-invariant linear system and the PSD matrix of the vector NES. We have emphasized in this paper that the Langevin equation method and the Master equation approach are equivalent to each other, provided the fluctuations in the dynamical variables can be considered to be a vector Markov process. Then the PSD matrix of the NES in the stochastic Langevin equations is unambiguously determined in terms of the transition probability rates of the underlying vector Markov process. The relationship between the PSD of the NES and the transition probability rates has been customarily referred to in nuclear engineering as the Schottky formula. Its 1-D form was used by Cohn (1960) to calculate the PSD of the fluctuations in the number of neutrons $N(t)$ in a point reactor.† We think that the ambiguity of having to introduce two different interpretations of the Schottky formula depending on the mode of detection or the location of the detector might have arisen from trying to express the PSD of the detection rates in terms of the PSD of a fictitious NES in the neutron balance equation, rather than using the multi-component ($N(t)$ and $D(t)$) Langevin description. The examples we have considered have hopefully demonstrated that the Schottky formula (if one wishes to give a name to it) in vector form can be used for all modes of detection and detector locations. Some of the examples we considered were chosen to point out the subtleties involved in calculating the PSD of the detection rate when the detector is outside of the core. The distinction between the in-core and out-of-core detector cases arises only in the diffusion approximation. If one uses the stochastic transport equation to describe the fluctuations in the space- and velocity-dependent neutron density, this distinction becomes moot, because one can always define the outer boundary of the reactor to include the detectors. The PSD of the noise equivalent source in the stochastic Boltzmann equation with and without delayed neutrons is available in the early literature on reactor noise analysis (Matthes, 1962; Natelson *et al.*, 1966; Sheff and Albrecht, 1966; Akcasu and Osborn, 1966; Saito, 1967). However, the calculation of the PSD of the detection rate at the transport level is complicated, and therefore simple reactor models are often preferable and in fact often sufficient to interpret the noise experiments if high numerical accuracy is not an issue.

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† A historical note: it seems that the concept of "noise equivalent source" as a recipe for calculating pile noise approximately was first introduced by H. Hurwitz in 1950 in a letter to S. Hanauer (private communication).