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Some Conjectures and Results Concerning the Homology of Nilpotent Lie Algebras

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In this paper we consider the Lie algebra homology of

$$L_k = L \otimes (\mathbb{C}[t]/(t^{k+1}))$$

for L a complex Lie algebra. Our goal is to express the homology of L_k in terms of the homology of L. This problem comes up in previous work by the author on the Macdonald root system conjectures.

We present a number of conjectures related to this problem. The simplest of these conjectures asserts that

$$H(L_k) = H(L)^{\otimes(k+1)} \qquad (*)$$

when L is either semisimple or a nilpotent upper summand of a semisimple Lie algebra. We give a proof of (*) in the case $L = sl_n(\mathbb{C})$. Lastly we present computational evidence in support of our other conjectures.

1. THE STRONG MACDONALD CONJECTURES

1.1. *Lie Algebra Homology*

Let L be a complex Lie algebra and let $U(L)$ denote its universal enveloping algebra. The *Lie algebra homology of* L (with trivial coefficients) is defined to be the graded vector space $H_d(L)$ given by

$$H_d(L) = \mathrm{Tor}_d^{U(L)}(\mathbb{C}, \mathbb{C}).$$

There is an explicit complex called the *Koszul complex* for computing $H_*(L)$. Let $C_d(L)$ be the dth exterior power of L. Define $\partial_d\colon C_d(L) \to C_{d-1}(L)$ by

$$\partial_d(l_1 \wedge \cdots \wedge l_d) = \sum_{1 \leq i < j \leq d} (-1)^{i+j+1} \times [l_i, l_j] \wedge l_1 \wedge \cdots \wedge \hat{l}_i \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_d.$$

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It is easy to check that $\partial_{d-1} \circ \partial_d = 0$. The dth homology group of L is

$$H_d(L) = \ker \partial_d / \mathrm{im}\, \partial_{d+1}.$$

Very little is known in general about the homology of Lie algebras. However, the homology of semisimple lie algebras is known and this computation is important for what follows.

THEOREM 1.1 (Koszul [K]). *Let L be a semisimple Lie algebra of rank n with exponents $m_1, ..., m_n$. Then $H_*(L)$ is an exterior algebra with n generators, one in degree $2m_i + 1$ for each i.*

For more information on the exponents of a semisimple Lie algebra see Kostant [Ko2].

1.2. *The Strong Macdonald Conjectures*

For L a complex Lie algebra and k a non-negative integer let L_k and $\bar{L}_k$ denote the Lie algebras

$$L_k = L \otimes (\mathbb{C}[t]/(t^{k+1}))$$
$$\bar{L}_k = L \otimes (t\mathbb{C}[t]/(t^{k+1})).$$

In both L_k and $\bar{L}_k$ the bracket is given by

$$[x \otimes t^i, y \otimes t^j] = \begin{cases} [x, y] \otimes t^{i+j} & \text{if } \; i+j \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

The strong Macdonald conjectures are statements about the homology of L_k and $\bar{L}_k$ in the case that L is semisimple. Even when L is semisimple L_k and $\bar{L}_k$ are far from semisimple. $\bar{L}_k$ is nilpotent and L_k splits as a semidirect product of $L = L \otimes 1$ and $\bar{L}_k$. It is difficult to see how the homology of L is related to the homologies of L_k and $\bar{L}_k$.

The Lie algebras L_k and $\bar{L}_k$ are $\mathbb{N}$-graded as Lie algebras. In both cases the ith-graded piece is $L \otimes t^i$. This gives a grading on $C_*(L_k)$ (and $C_*(\bar{L}_k)$) as follows: if l_j is in the i_jth-graded piece of L_K (or $\bar{L}_K$) then

$$l_1 \wedge \cdots \wedge l_d \tag{1.2.1}$$

is in the $(i_1 + \cdots + i_d)$th graded piece of $C_d(L_k)$ (or $C_d(\bar{L}_k)$). We call this grading *weight* to distinguish it from homological degree and we say that the vector (1.2.1) has "weight $(i_1 + \cdots + i_d)$."

Note that L is a subalgebra of L_k ($L \cong L \otimes 1 \subseteq L_k$) and that $\bar{L}_k$ is a complementary ideal to L in L_k. So both L_k and $\bar{L}_k$ are L-modules. This L-module structure lifts to the complex $C_*(L_k)$ and commutes with the boundary maps ∂_d. Thus $H_*(L_k)$ and $H_*(\bar{L}_k)$ are (bigraded) L-modules. The strong Macdonald conjectures are statements about the structure of these bigraded L-modules in the case that L is semisimple. If L is semisimple then L is a reductive subalgebra of L_k. Hence the L-module structure of $H_*(L_k)$ is trivial (see Koszul [K]). Thus the strong Macdonald conjectures do not address the L-module structure of $H_*(L_k)$.

The Strong Macdonald Conjectures. Let L be a semisimple complex Lie algebra with exponents $m_1, m_2, ..., m_n$ and let k be a non-negative integer. Then

> $H_{**}(L)$ has the structure of an exterior algebra with $n(k+1)$ generators. For each i there are $k+1$ generators in homological degree $2m_i+1$ and these $k+1$ generators have weigts
>
> $$0, (k+1)\, m_i+1, (k+1)\, m_i+2, ..., (k+1)\, m_i+k. \qquad (1.2.2)$$
>
> As an ungraded L-module, $H(\bar{L}_k)$ is isomorphic to 2^{nk} copies of $V_{k\rho} \otimes V_{k\rho}$ where ρ is half the sum of the positive roots and $V_{k\rho}$ denotes the irreducible L-module with highest weight $k\rho$. (1.2.3)

There is a great deal of evidence in support of Conjecture (1.2.2). It is known to be true for the classical simple Lie algebras in the limit as n tends to infinity (see [H1]). It is also true in a rather trivial way as k tends to infinity. The author has verified Conjecture (1.2.2) by computer in a number of small cases. Also Conjecture (1.2.2) implies Euler characteristic equation which is exactly the Macdonald Root System Conjectures (see [M, H1]). The latter conjectures have been verified for almost all semisimple Lie algebras. This adds further evidence in support of Conjecture (1.2.2). In Section 3 of the paper we will verify Conjecture (1.2.2) for $L = sl_n(\mathbb{C})$. B. L. Feigin has announced a proof of Conjecture (1.2.2) for all semisimple L but that proof has not appeared.

There is much less evidence in support of Conjecture (1.2.2). Although the homology of $\bar{L}_k$, can be computed in the limiting cases $n \to \infty$ and $k \to \infty$ (see [H2, GL]), it is difficult to make sense of $V_{k\rho} \otimes V_{k\rho}$ in those limits. The only firm evidence in support of Conjecture (1.2.3) consists of computations done by the author in small cases.

This paper contains conjectures and results that the author obtained in his efforts to prove Conjectures (1.2.2) and (1.2.3). In Section 2 we consider general Lie algebras L, in Section 3 we look at the special case of $L = sl_n(\mathbb{C})$, and in Section 4 we consider nilpotent upper summands.

The reader will find that there are many more conjectures than results.

There are so few tools for computing homology of nilpotent Lie algebras that most of the conjectures in this paper seem out of reach at the present time.

2. PROPERTY M

Consider $H(L_k)$ as a vector space graded by homological degree only. Conjecture (1.2.2) asserts that if L is semisimple then

$$H_*(L_k) \cong H_*(L)^{\otimes(k+1)} \tag{2.1.1}$$

for any complex Lie algebra L let $L_k(z) = L \otimes (\mathbb{C}[t]/t^{k+1} - z)$. Here z is a complex parameter. The Lie algebra L_k from Section 1 is $L_k(0)$ so $L_k(z)$ is a deformation of the Lie algebra L_k. For $z \neq 0$, we have

$$L_k(z) \cong L \oplus L \oplus \cdots \oplus L,$$

where the direct sum on the right is a Lie algebra direct sum. So for $z \neq 0$ we have

$$H_*(L_k(z)) \cong H_*(L)^{\otimes(k+1)}.$$

At the singular point, $z = 0$, the structure of $L_k(z)$ changes dramatically. Conjecture (1.2.2) implies that the homology remains constant at the singular point when L is a semisimple Lie algebra.

DEFINITION 2.1.2. Let L be a complex Lie algebra. We say L has *property M* if

$$H_*(L_k) \cong H_*(L)^{\otimes(k+1)}.$$

Conjecture (1.2.2) implies that semisimple Lie algebras have property M. There is another broad class of Lie algebras that seem to have property M.

DEFINITION 2.1.3. Let L be a finite dimensional semisimple Lie algebra with root system R and let

$$L = \mathscr{H} \oplus \bigoplus_{\alpha \in R} L_\alpha$$

be the Cartan decomposition of L. Let Δ be a basis of simple roots, let S be a subset of Δ, and let $R_S \subseteq R$ be the root system generated by S. Define N_S to be

$$N_S = \bigoplus_{\alpha \in R^+ \setminus R_S^+} L_\alpha.$$

It is easy to see that N_S is a nilpotent Lie subalgebra of L. We call N_S a *nilpotent upper summand* of L.

Conjecture 2.1.4. Let N be a nilpotent upper summand of a semisimple Lie algebra. Then N has property M.

In Section 4 we will state a stronger version of Conjecture 2.1.4 and discuss the Heisenberg Lie algebras in more detail.

At this point it makes sense to ask if every complex Lie algebras has property M. It is easy to check that the 1-dimensional and 2-dimensional Lie algebras have property M. Among 3-dimensional Lie algebras $sl_2(\mathbb{C})$ and the 3-dimensional Heisenberg $\mathscr{H}_3$ are conjectured to have property M by (1.2.2) and (2.1.4). It is straightforward to check that all others have property M except the following Lie algebra $R(\alpha)$. As a vector space over $\mathbb{C}$, $R(\alpha)$ has basis $\{x, y, h\}$ with bracket

$$[x, h] = x, \qquad [y, h] = \alpha y, \qquad [x, y] = 0.$$

So $R(\alpha)$ is a solvable Lie algebra. One can check that

$$\dim(H_*(R(\alpha))) = \begin{cases} 4, & \alpha = 0, -1 \\ 2, & \text{otherwise.} \end{cases} \tag{2.1.5}$$

PROPOSITION 2.1.6. (A) *If α is* not *a negative rational then $R(\alpha)$ has property M.*

(B) *For certain negative rationals α and certain values of k we have*

$$H_*(R(\alpha)_k) \not\cong H_*(R(\alpha))^{\otimes(k+1)}.$$

In particular, $H_*(R(-1/2)_1) \not\cong H_*(R(-1/2))^{\otimes 2}$.

Proof. First we prove (A). The case where $\alpha = 0$ is easy because in the case $R(0)$ splits as a direct sum of a 2-dimensional and a 1-dimension Lie algebra. Assume α is not a negative rational and that α is not 0. For convenience of notation, let x_i, y_i, h_i denote $x \otimes t^i$, $y \otimes t^i$, and $h \otimes t^i$. Write our complex $C_*(R(\alpha))$ as a (vector space) direct sum

$$C_*(R(\alpha)) = C_*^{(0)}(R(\alpha)) + C_*^{(1)}(R(\alpha)),$$

where $C_*^{(0)}(R(\alpha))$ is the span of all wedges which do not include h_0 and $C_*^{(1)}(R(\alpha))$ is the span of all wedges which do include h_0. Define s on $C_*(R(\alpha))$ by

$$s(C_d^{(i)}(R(\alpha))) = d - i.$$

For each l, let $\Omega^{(l)}$ denote the span of all vectors v with $s(v)=l$. Then

$$\partial_*: \Omega^{(l)} \to \Omega^{(l)} \oplus \Omega^{(l-1)}$$

so s gives a filtration of the complex $C_*(R(\alpha))$. This filtration gives rise to a spectral sequence $(E^{(r)}, \partial^r)$ which abutts to $C_*(R(\alpha))$. The E^0 term is the associated graded module and the ∂^0 differential is that part of the differential which preserves s. It is easy to see that

$$\partial_0(x_{i_1} \wedge \cdots \wedge x_{i_t} \wedge y_{j_1} \wedge \cdots \wedge y_{j_u} \wedge h_{l_1} \wedge \cdots \wedge h_{l_v}) = 0 \quad \text{for} \quad 0 < l_1 < \cdots < l_v$$

whereas

$$\partial_0(x_{i_1} \wedge \cdots \wedge x_{i_t} \wedge y_{j_1} \wedge \cdots \wedge y_{j_u} \wedge h_0 \wedge h_{l_1} \wedge \cdots \wedge h_{l_v}) = (t+\alpha u)\, x_{i_1} \wedge \cdots \wedge x_{i_t} \wedge y_{j_1} \wedge \cdots \wedge y_{j_u} \wedge h_{l_1} \wedge \cdots \wedge h_{l_v}.$$

Since α is not a negative rational, $t+\alpha u$ is nonzero except when $t=u=0$. It follows that

$$E^1 \cong \bigwedge \mathscr{H},$$

where $\mathscr{H} = \langle h_0, h_1, ..., h_k \rangle$. Thus dim $E^1 = 2^{k+1}$. Therefore

$$\dim(H(R(\alpha)_k)) \leqslant 2^{k+1} = \dim(H(R(\alpha))^{\otimes(k+1)}).$$

By deformation theoretic considerations we have an injection

$$H(R(\alpha))^{\otimes(k+1)} \hookrightarrow H(R(\alpha)_k)$$

and so $H(R(\alpha)_k) \cong H(R(\alpha))^{\otimes(k+1)}$.

The proof of (B) is a straightforward computation. ∎

Problem. *Compute $H(R(\alpha)_k)$ for α a negative rational.*

3. The Case $L = sl_n(\mathbb{C})$

3.1. *A Preview of Our Proof*

In this section we give a proof of Conjecture (1.2.2) in the case that $L = sl_n(\mathbb{C})$. Our first step is to note that (1.2.2) is equivalent to a fact about the homology of $\bar{L}_k$. Recall that $H(\bar{L}_k)$ has the structure of an L-module. Let $H^0(\bar{L}_k)$ denote the L-trivial isotopic component of $H(\bar{L}_k)$. A proof of the following Theorem can be found in Guichardet [G].

THEOREM 3.1.1. *Let L be a semisimple Lie algebra. Then*

$$H_{**}(L_k) \cong H_{*,0}(L) \otimes H^0_{**}(\bar{L}_k).$$

Using (3.1.1) we see that Conjecture (1.2.2) is equivalent to:

Conjecture 3.1.2. Let L be a semisimple Lie algebra of rank n with exponents $m_1, ..., m_n$. Then $H^0(\bar{L}_k)$ is an exterior algebra with nk generators. For each i there are k generators of degree $2m_i+1$ and these k generators have weights $(k+1)m_i+s$, $s=1, 2, ..., k$.

Henceforth we asume that $L=sl_n(\mathbb{C})$. The irreducible representations of L are indexed by partitions λ with length less than n. For λ a partition we let $\bar{\lambda}$ denote the dominant weight for L in the irreducible representation indexed by λ.

Let M be any graded Lie algebra with $L \subseteq \mathrm{Aut}(M)$. As usual we refer to the grading on M as *weight* and we carry it over to a grading on $H(M)$ which we also call *weight*. So $H(M)$ has the structure of a bigraded L-module. For any partition λ let $H^\lambda_{d,w}(M)$ denote the subspace of $H_{d,w}(M)$ spanned by the maximal weight vectors of dominant weight $\bar{\lambda}$. Let $P^\lambda(M; z, q)$ denote the Poincaré series for $H^\lambda_{**}(M)$, i.e.,

$$P^\lambda(M; z, q) = \sum_{d,w} \dim(H^\lambda_{d,w}(M))\, z^d q^w.$$

In terms of Poincaré series, Conjecture 3.1.2 is equivalent to

$$\begin{aligned} P^0(\bar{L}_k; z, q) &= \prod_{i=1}^{n-1} \prod_{s=1}^{k} (1 + z^{2m_i+1} q^{(k+1)m_i+s}) \\ &= \prod_{i=1}^{n-1} \prod_{s=1}^{k} (1 + z^{2i+1} q^{(k+1)i+s}) \end{aligned}$$

the latter equality following because the exponents for $sl_n(\mathbb{C})$ are $1, 2, ..., n-1$.

In this section we will compute $P^\lambda(\bar{L}_k; z, q)$ for every hook shape λ (where we consider 0 to be a hook shape). We will use an idea devised by Stembridge [S1, S2] which he used with great success in his papers on the Euler characteristics of the $H^\lambda(\bar{L}_k)$. This idea is to develop a recursion which relates $P^{\beta 1^p}(\bar{L}_k; z, q)$ to $P^{(\beta-1)1^{p+1}}(\bar{L}_k; z, q)$. Using this recursion (and the fact that $\beta 1^{n-1}$ determines the same maximal weight as $(\beta-1)$) one obtains a simple expression relating $P^{kn}(\bar{L}_k; z, q)$ to $P^0(\bar{L}_k; z, q)$. In addition we will determine a simple relationship between $P^{kn}(\bar{L}_k; z, q)$ and $P^0(\bar{M}_k; z, q)$ where $M=sl_{n-1}(\mathbb{C})$. Combining these equations gives a simple relationship between $P^0((sl_n(\mathbb{C}))_k; z, q)$ and $P^0((sl_{n-1}(\mathbb{C}))_k; z, q)$ which is enough to prove (3.1.3) by induction on n.

We end this section by stating the exact recursions we will prove. To do so we need a bit of terminology and notation. We will let $\mathscr{H}$ denote the usual maximal torus in $sl_n(\mathbb{C})$. We will be dealing throughout this section with bigraded L-modules where two parts of the bigrading are called degree and weight. There is a possible confusion between the term "weight" as it is used in the bigrading and the term "weight" as it is usually used in the theory of Lie algebras. For the latter usage we will replace the term weight by *$\mathscr{H}$-weight*. We will refer to "dominant $\mathscr{H}$-weights" and "$\mathscr{H}$-weight vectors," etc.

Note that we can compute $H(\bar{L}_k)$ starting with the Koszul complex $\Lambda(\bar{L}_k)$. It follows that all $\mathscr{H}$-weights which appear in $H(\bar{L}_k)$ occur in the root lattice. So if $H^{\lambda}(\bar{L}_k)$ is nonzero then λ is a partition of some multiple of n. This multiple is called *the layer of* λ and is denoted $l(\lambda)$.

We can now state the recursions that we will prove in this section. They are:

Let $L = sl_n(\mathbb{C})$ and let $M = sl_{n-1}(\mathbb{C})$. Then

$$P^{kn}(\bar{L}_k; z, q) = z^{(n-1)k} q^{(n-1)\binom{k+1}{2}} \left\{ \prod_{s=1}^{k} (1 + zq^s) \right\} P^0(\bar{M}_k; z, q). \tag{3.1.4}$$

Let $\beta 1^p$ $(p < n-1)$ be an lth layer hook shape. Let A denote $z^2 q^{(k+1)}$. Then

$$\begin{aligned}(1 + zq^l A^{p+1})(1 - A^{n-(p+1)})\, P^{\beta 1^p}(\bar{L}_k; z, q) \\ = (A^{n-(p+1)} + zq^l)(1 - A^{p+1})\, P^{(\beta-1)1^{p+1}}(\bar{L}_k; z, q).\end{aligned} \tag{3.1.5}$$

In Section 3.2 we will prove the first of these equations (3.1.4). In Section 3.3 we will prove (3.1.5). Lastly in Section 3.4 we will use these recursions to derive the value of $P^{\lambda}(\bar{L}_k; z, q)$ for all hook shapes λ.

3.2. *Proving* (3.1.4)

In this subsection we let $L = sl_n(\mathbb{C})$, $M = sl_{n-1}(\mathbb{C})$, and $N = gl_{n-1}(\mathbb{C})$. Our proof of (3.1.4) consists of two steps, the first being a straightforward result which compares the homology of $\bar{M}_k$ to the homology of $\bar{N}_k$ as $sl_{n-1}(\mathbb{C})$-modules.

LEMMA 3.2.1. *Let λ be a partition of length less than $n-1$. Then the Poincaré series of $H(\bar{M}_k)$ and $H(\bar{N}_k)$ as bigraded $sl_{n-1}(\mathbb{C})$-modules are related by*

$$P^{\lambda}(\bar{N}_k; z, q) = \left\{ \prod_{s=1}^{k} (1 + zq^s) \right\} P^{\lambda}(\bar{M}_k; z, q).$$

Proof. Write $N = M \oplus \langle I_{n-1} \rangle$ where I_{n-1} is the identity matrix. From this decomposition (and the fact that $\langle I_{n-1} \rangle$ carries a trivial M-module structure) we have

$$H^{\lambda}(\bar{N}_k; z, q) \cong H^0(\overline{(I_{n-1})}_k; z, q) \otimes H^{\lambda}(\bar{M}_k; z, q).$$

Lemma 3.2.1 follows upon observing that

$$P^0(\overline{(I_{n-1})}_k; z, q) = \prod_{s=1}^{k} (1 + zq^s). \quad \blacksquare$$

Next we will need to think of N as sitting inside of L. We describe this inclusion in terms of the decomposition $M \oplus \langle I_{n-1} \rangle = N$ by mapping M to the matrices in $sl_n(\mathbb{C})$ which have nonzero entries in rows and columns 2 through n and by mapping I_{n-1} to $\hat{I}_{n-1} = -(n-1) z_{11} + z_{22} + \cdots + z_{nn}$. Let U denote the subcomplex of $\Lambda(\bar{N}_k)$ spanned by the maximal $\mathscr{H}$-weight vectors of $\mathscr{H}$-weight 0. Let V denote the subcomplex of $\Lambda(\bar{L}_k)$ spanned by maximal $\mathscr{H}$-weight vectors of $\mathscr{H}$-weight $\overline{kn}$. Define $\varphi: U \to V$ in the following way. First let u be a pure wedge in $\Lambda(\bar{N}_k)$ of $\mathscr{H}$-weight 0 having the form

$$u = u_0 \wedge ((I_{n-1} \otimes t^{a_1}) \wedge \cdots \wedge (I_{n-1} \otimes t^{a_r})). \tag{3.2.2}$$

Then

$$\varphi(u) = u_0 \wedge \left\{ \bigwedge_{l=1}^{r} (\hat{I}_{n-1} \otimes t^{a_l}) \right\} \wedge \left\{ \bigwedge_{i=2}^{n} \bigwedge_{j=1}^{k} (z_{1i} \otimes t^j) \right\}. \tag{3.2.3}$$

Here z_{uv} denotes the $n \times n$ matrix with a 1 in position u, v and 0's elsewhere. Extend φ linearly to all vectors in $\Lambda(\bar{N}_k)$ having $\mathscr{H}$-weight 0. Note that φ raises degree by $(n-1)k$ and weight by $(n-1)\binom{k}{2}$. Also φ shifts $\mathscr{H}$-weight by $\overline{kn} = (k(n-1), -k, ..., -k)$. The next fact is not so obvious.

LEMMA 3.2.4. *Let w be in U. Then $\varphi(w)$ is in V.*

Proof. Let Γ be the final exterior factor in (3.2.3),

$$\Gamma = \bigwedge_{i=2}^{n} \bigwedge_{j=1}^{k} (z_{1i} \otimes t^j).$$

Assume w is in U and pick α, β with $1 \leqslant \alpha < \beta \leqslant n$. We need to show that $\mathrm{ad}(z_{\alpha\beta}) \circ \varphi(w) = 0$.

Case 1. $\alpha = 1$. We can write w as a linear combination of pure wedges of the form (3.2.2) so that $\varphi(w)$ is a linear combination of pure wedges of the form (3.2.3). We will show that each of these is annihilated by $\mathrm{ad}(z_{1\beta})$.

It is obvious that the factor of Γ in (3.2.3) is annihilated by $\mathrm{ad}(z_{1\beta})$. Also if $\mathrm{ad}(z_{1\beta})$ acts on any of the other factors $(z_{rs} \otimes t^p)$ which appear in (3.2.3) the result is either 0 or is $(z_{1s} \otimes t^p)$ which is then repeated in Γ. In the latter case the wedge product is 0

Case 2. $2 \leqslant \alpha < \beta \leqslant n$. We can write $\varphi(w)$ in the form $\hat{w} \wedge \Gamma$ where $\hat{w}$ is the same as w except that every occurrence of I_{n-1} is replaced by $\hat{I}_{n-1}$. If $z_{\alpha\beta}$ acts on one of the factors $z_{1i} \otimes t^j$ in Γ then the result is 0 if $i \neq \alpha$ and is $-z_{1\beta} \otimes t^j$ if $i = \alpha$. In the latter case $z_{1\beta} \otimes t^j$ is another of the factors in Γ so the wedge product is 0. Hence $\mathrm{ad}(z_{\alpha\beta}) \cdot \Gamma = 0$.

Also $(\mathrm{ad}\, z_{\alpha\beta} \cdot w) = 0$ since $w \in U$. Thus

$$\mathrm{ad}\, z_{\alpha\beta} \cdot \varphi(w) = (\mathrm{ad}\, z_{\alpha\beta} \cdot \hat{w}) \wedge \Gamma + \hat{w} \wedge (\mathrm{ad}\, z_{\alpha\beta} \cdot \Gamma) = 0.$$

This completes the proof of Lemma 3.2.4. ∎

We now know that φ maps U into V. The next lemma is more straightforward to see

LEMMA 3.2.5. *The map φ is an isomorphism of U onto V.*

Proof. It is clear that φ is $1-1$. To see that φ is onto note that any pure wedge y in $\Lambda(\bar{L}_k)$ with $\mathscr{H}$-weight $\overline{kn} = (k(n-1), -k, ..., -k)$ must include as factors all possible $z_{1i} \otimes t^j$. Hence y must be of the form $\varphi(w) \wedge \Gamma$ where w is a pure wedge in $\Lambda(\bar{N}_k)$. This shows φ is onto. ∎

We now know that U and V are isomorphic as graded vector spaces. We end by showing that they are isomorphic as subcomplexes of the Koszul complexes $(\Lambda\bar{N}_k, \partial_N)$ and $(\Lambda\bar{L}_k, \partial_L)$, respectively.

LEMMA 3.2.6. *Let ∂_N and ∂_L be the boundary maps in the Koszul complexes $\Lambda\bar{N}_k$ and $\Lambda\bar{L}_k$, respectively. Then*

$$\partial_L \varphi(w) = \varphi(\partial_N w).$$

Proof. Let $u = u_0 \wedge \{\bigwedge_{l=1}^r (I_{n-1} \otimes t^{a_l})\}$ be a pure wedge product of $\mathscr{H}$-weight 0 in $\Lambda(\bar{N}_k)$. Then

$$\varphi(u) = u_0 \wedge \left\{\bigwedge_{l=1}^{r} (\hat{I}_{n-1} \otimes t^{a_l})\right\} \wedge \Gamma. \tag{3.2.7}$$

$\partial_L \varphi(u)$ is a signed sum of terms where two factors are removed from the wedge product (3.2.7) and bracketed together in $\bar{L}_k$. All of these terms belong to $\varphi(\partial_N u)$ except those where one or both of the factors removed appear in Γ. But these exceptional terms are all 0 for the following reasons:

(1) If both of the factors belong to Γ then their bracket is 0.

(2) If one (say $z_{1i}\otimes t^j$) appears in Γ and the other (either $z_{rs}\otimes t^p$ or $\hat{I}_{n-1}\otimes t^a$) does not appear in Γ then their bracket is either 0 or it is a different factor $z_{1i'}\otimes t^{j'}$ appearing in Γ. In the latter case that factor now appears twice in the resulting wedge product which forces the wedge product to be 0. ▮

Combining the previous three lemmas we obtain the following theorem:

THEOREM 3.2.8. *The map φ induces an isomorphism $\hat{\varphi}$ between $H^0(\bar{N}_k)$ and $H^{kn}(\bar{L}_k)$. This isomorphism raises degree by $k(n-1)$ and raises weight by $\binom{k}{n}(n-1)$. So, for all k and n,*

$$P^{kn}(\overline{sl_n(\mathbb{C})_k}; z, q)$$
$$= z^{k(n-1)}q^{\binom{k}{2}(n-1)}\left\{\prod_{s=1}^{k}(1+zq^s)\right\}P^0(\overline{sl_{n-1}(\mathbb{C})_k}; z, q).$$

3.3. *Proving* (3.1.5)

We now come to the more difficult part of the proof, namely the verification of the recursion (3.1.5). As above we assume that $L=sl_n(\mathbb{C})$ throughout this subsection. We will be dealing with various Lie algebras which are quotients of subalgebras of $L\otimes\mathbb{C}[t]$. The standard basis for $L\otimes\mathbb{C}[t]$ is

$$\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2,$$

where

$$\mathscr{B}_1=\{z_{uv}\otimes t^w : u\neq v,\ u, v\in\mathbf{n},\ w\geqslant 0\},$$

and

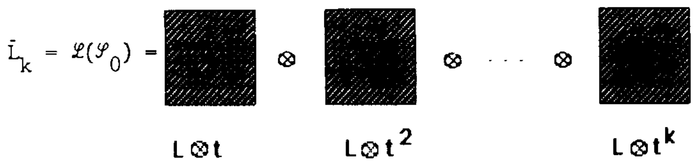
$$\mathscr{B}_2=\{(z_{uu}-z_{(u+1)(u+1)})\otimes t^w : u\in\boldsymbol{n-1},\ w\geqslant 0\}.$$

The Lie algebra we deal with will be indexed by subsets $\mathscr{A}$ of $\mathscr{B}$. Given a subset $\mathscr{A}\subseteq\mathscr{B}$ let $\mathscr{L}(\mathscr{A})$ denote the span of all basis vectors in $\mathscr{A}$. The bracket is given by

$$[x, y]=\prod_{\mathscr{L}(\mathscr{A})}([x, y]_{L\otimes\mathbb{C}[t]}). \tag{3.3.1}$$

Here $[\ ,\ ]_{L\otimes\mathbb{C}[t]}$ denotes the bracket in $L\otimes\mathbb{C}[t]$ and $\prod_{\mathscr{L}(\mathscr{A})}$ is the orthogonal projection onto $\mathscr{L}(\mathscr{A})$ (orthogonal with respect to the form which makes $\mathscr{B}$ an orthonormal basis). An example of this is if we take $\mathscr{A}_0$ to be the subset of all basis elements of the form $u \otimes t^w$ where $1 \leqslant w \leqslant k$. In this case $\mathscr{L}(\mathscr{A}_0) = \bar{L}_k$. Given an arbitrary subset $\mathscr{A} \subseteq \mathscr{B}$ the vector space $\mathscr{L}(\mathscr{A})$, with the bracket (3.3.1), may or may not be a Lie algebra. The reader can easily check that $\mathscr{L}(\mathscr{A})$ is a Lie algebra for all sets $\mathscr{A}$ that we use in this paper.

To help the reader get a better feel for the Lie algebras $\mathscr{L}(\mathscr{A})$ we will sometimes visualize them in the following way. We will represent $sl_n(\mathbb{C})$ by an $n \times n$ grid and $sl_n(\mathbb{C}) \otimes \mathbb{C}[t]$ as a sequence of grids (the wth grid representing $sl_n(\mathbb{C}) \otimes t^w$). We then "draw" $\mathscr{L}(\mathscr{S})$ by shading the u, v square in the wth grid for all $z_{uv} \otimes t^w \in \mathscr{S}$. For example, with $\mathscr{S}_0$ as above we have



DEFINITION 3.3.2. Define $\mathscr{Y}^+$, $\mathscr{Y}^-$, $\mathscr{Z}^+$, $\mathscr{Z}^- \subseteq \mathscr{B}$ as

$$\mathscr{Y}^+ = \{z_{ij} \otimes 1 : n \geqslant i > j \geqslant 1\}$$

$$\mathscr{Y}^- = \{z_{ij} \otimes 1 : n-1 \geqslant i > j \geqslant 1\} \cup \{z_{in} \otimes 1 : 1 \leqslant i \leqslant n-1\}.$$

$$\mathscr{Z}^+ = \{z_{in} \otimes t^{k+1} : 1 \leqslant i \leqslant n\}$$

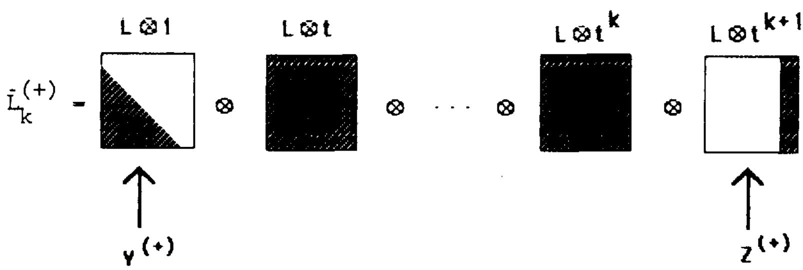
$$\mathscr{Z}^- = \{z_{ni} \otimes t^{k+1} : 1 \leqslant i \leqslant n\}.$$

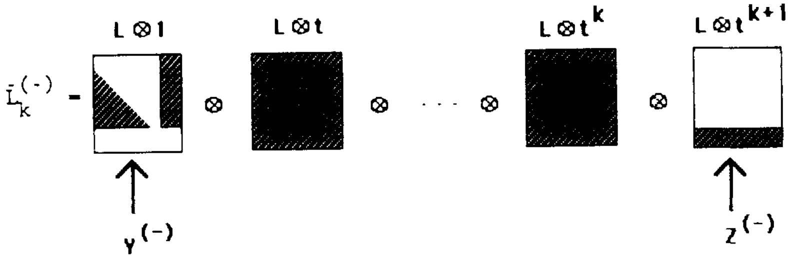
Let $Y^{(+)} = \mathscr{L}(\mathscr{Y}^+)$, $Y^{(-)} = \mathscr{L}(\mathscr{Y}^-)$, $Z^{(+)} = \mathscr{L}(\mathscr{Z}^+)$, and $Z^{(-)} = \mathscr{L}(\mathscr{Z}^-)$. Lastly define $\bar{L}_k^{(+)}$ and $\bar{L}_k^{(-)}$ by

$$\bar{L}_k^{(+)} = \mathscr{L}(\mathscr{Y}^+ \cup \mathscr{S}_0 \cup \mathscr{Z}^+)$$

$$\bar{L}_k^{(-)} = \mathscr{L}(\mathscr{Y}^- \cup \mathscr{S}_0 \cup \mathscr{Z}^-).$$

In terms of our visual representations of these Lie algebras we have





Define the shift operator $\Omega: \bar{L}_k^{(+)} \to \bar{L}_k^{(-)}$ to be the linear map satisfying

$$\Omega(z_{ij} \otimes t^u) = \begin{cases} z_{in} \otimes t^{u-1} & \text{if } j = n, \quad i \neq n \\ z_{nj} \otimes t^{u+1} & \text{if } i = n, \quad j \neq n \\ z_{nn} \otimes t^u & \text{if } i = j = n \\ z_{ij} \otimes t^u & \text{if } i < n \quad \text{and} \quad j < n. \end{cases}$$

The next lemma is straightforward to check.

LEMMA 3.3.3. *Ω is a Lie algebra isomorphism between $\bar{L}_k^{(+)}$ and $\bar{L}_k^{(-)}$.*

We are now ready to begin the proof of (3.1.5). Before diving into technical details, it is worth giving an overview of the proof. Roughly speaking, we will compute $P^{[\beta, 1^p]}(\bar{L}_k^{(+)}; z, q)$ and $P^{[\beta, 1^p]}(\bar{L}_k^{(-)}; z, q)$ and set them equal by virtue of Lemma 3.3.3. There are two subtleties. First we will compute $H^{[\beta, 1^p]}(\bar{L}_k^{(+)})$ and $H^{[\beta, 1^p]}(\bar{L}_k^{(-)})$ by means of a spectral sequence which has the effect of isolating the contributions made to them by the $H^\lambda(\bar{L}_k)$. This spectral sequence will allow us to express the Poincaré series $P^{[\beta, 1^p]}(\bar{L}_k^{(+)}; z, q)$ and $P^{[\beta, 1^p]}(\bar{L}_k^{(-)}; z, q)$ in terms of the $P^\lambda(\bar{L}_k; z, q)$. As we will see, the only $P^\lambda(\bar{L}_k; z, q)$ which arise in these expressions are $P^{[\beta, 1^p]}(\bar{L}_k; z, q)$ and $P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; z, q)$. Therefore when we set $P^{[\beta, 1^p]}(\bar{L}_k^{(+)}; z, q)$ equal to $P^{[\beta-1, 1^{p+1}]}(\bar{L}_k^{(-)}; z, q)$ we obtain an equation which relates $P^{[\beta, 1^p]}(\bar{L}_k; z, q)$ to $P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; z, q)$. This equation will turn out to be (3.1.5).

There is a second subtlety. The isomorphism Ω does not preserve weight. In addition we will introduce a third grading (which we call skew-degree) which is also shifted by Ω. It will be necessary to determine how Ω effects weight and skew-degree.

The computations needed to determine $H^{[\beta, 1^p]}(\bar{L}_k^{(+)})$ and $H^{[\beta, 1^p]}(\bar{L}_k^{(-)})$ are very similar. Therefore we will do the full computation of $H^{[\beta, 1^p]}(\bar{L}_k^{(+)})$ and indicate where the computation for $\bar{L}_k^{(-)}$ is different. The computation of $H^{[\beta, 1^p]}(\bar{L}_k^{(+)})$ will be done in steps.

Step 1. There is a spectral sequence (E^r, e_r) which abutts to $H(\bar{L}_k^{(+)})$ and which has E^1 term

$$E^1 = \Lambda(Y^{(+)}) \otimes H(\bar{L}_k) \otimes \Lambda(Z^{(+)}).$$

Proof. Let ∂ be the differential in the Koszul complex $\Lambda(\bar{L}_k^{(+)})$. Since $\bar{L}_k^{(+)} = Y^{(+)} \oplus \bar{L}_k \oplus Z^{(+)}$ we have

$$\Lambda(\bar{L}_k^{(+)}) \cong \Lambda(Y^{(+)}) \otimes \Lambda(\bar{L}_k) \otimes \Lambda(Z^{(+)}). \tag{3.3.4}$$

It is easy to check that ∂ has the following behavior with respect to the tensor product decomposition given by (3.3.4):

$$\begin{aligned} &\partial(\Lambda^a(Y^{(+)}) \otimes \Lambda^b(\bar{L}_k) \otimes \Lambda^c(Z^{(+)})) \\ &\quad \subseteq \{\Lambda^a(Y^{(+)}) \otimes \Lambda^{b-1}(\bar{L}_k) \otimes \Lambda^c(Z^{(+)})\} \\ &\qquad \oplus \{\Lambda^{a-1}(Y^{(+)}) \otimes \Lambda^b(\bar{L}_k) \otimes \Lambda^c(Z^{(+)})\} \\ &\qquad \oplus \{\Lambda^a(Y^{(+)}) \otimes \Lambda^{b-2}(\bar{L}_k) \otimes \Lambda^{c+1}(Z^{(+)})\}. \end{aligned} \tag{3.3.5}$$

Define a grading β on $\Lambda(\bar{L}_k^{(+)})$ by

$$\beta(\Lambda^a(Y^{(+)}) \otimes \Lambda^b(\bar{L}_k) \otimes \Lambda^c(Z^{(+)})) = a - c.$$

It follows from (3.3.5) that β gives a filtering on the complex $(\Lambda(\bar{L}_k^{(+)}), \partial)$. Let (E^r, e_r) be the associated spectral sequence. It is straightforward to check, using (3.3.5), that for $y \in \Lambda^a(Y^{(+)})$, $x \in \Lambda^b(\bar{L}_k)$, $z \in \Lambda^c(Z^{(+)})$ we have

$$e_0(y \wedge x \wedge z) = y \wedge (\bar{\partial} x) \wedge z,$$

where $\bar{\partial}$ is the differential in the Koszul complex $(\Lambda(\bar{L}_k), \bar{\partial})$. It follows immediately that

$$E^1 \cong \Lambda(Y^{(+)}) \otimes H(\bar{L}_k) \otimes \Lambda(Z^{(+)}). \tag{3.3.6}$$

This completes Step 1.

Note that we have accomplished what we set out to do—we have isolated in E^1 the contribution made by $H(\bar{L}_k)$. Our next step is to obtain some information about the E^2 term. We will not need to do a full computation of E^2. Before proceeding we introduce some new terminology and notation.

DEFINITION. Define a new grading s on E^1 using the isomorphism (3.3.6) by

$$s(\Lambda^a(Y^{(+)}) \otimes H_b(\bar{L}_k) \otimes \Lambda^c(Z^{(+)})) = b + 2c.$$

We call the grading s *skew-degree*.

Since $b + 2c = (a + b + c) - (a - c)$ it follows that the boundary map e_1 preserves skew degree. For the next step in the computation we will need the notion of Lie algebra homology with non-trivial coefficients.

DEFINITION. Let N be a Lie algebra and let M be an N-module. For each r let $C_r(N; M)$ denote

$$C_r(N; M) = (\Lambda^r N) \otimes M.$$

Define $\delta_r: C_r(N; M) \to C_{r-1}(N; M)$ by

$$\delta_r((n_1 \wedge \cdots \wedge n_r) \otimes m) = (\partial_r(n_1 \wedge \cdots \wedge n_r)) \otimes m + \sum_{i=1}^{r} (-1)^{r-i} (n_1 \wedge \cdots \wedge \hat{n}_i \wedge \cdots \wedge n_r) \otimes (n_i \circ m).$$

It is easy to check that $\delta_{r-1} \circ \delta_r = 0$. Let $H_r(N; M)$ be defined by

$$H_r(N; M) = \ker \delta_r / \operatorname{im} \delta_{r+1}.$$

We call $H_*(N; M)$ the *homology of N with coefficients in M.*

We now are ready for the second step in our computation.

Step 2. Now we will consider the computation of the E^2 term in the spectral sequence (E^r, e_r). The differential e_1 is induced by that part of the original boundary ∂ which maps

$$\Lambda^a(Y^{(+)}) \otimes (\Lambda^b(\bar{L}_k)) \otimes \Lambda^c(Z^{(+)})$$

to

$$(\Lambda^{a-1}(Y^{(+)}) \otimes (\Lambda^b(\bar{L}_k)) \otimes \Lambda^c(Z^{(+)})) \oplus (\Lambda^a(Y^{(+)}) \otimes (\Lambda^{b-2}(\bar{L}_k)) \otimes \Lambda^{c+1}(Z^{(+)})).$$

Filter this complex (E^1, e_1) by γ where

$$\gamma(\Lambda^a(Y^{(+)}) \otimes (\Lambda^b(\bar{L}_k)) \otimes \Lambda^c(Z^{(+)})) = c.$$

This is an upward filtration which gives rise to a second spectral sequence (F^r, f_r) that abutts to E^2. The f_0 boundary is induced by that part of the original boundary ∂ which maps

$$\Lambda^a(Y^{(+)}) \otimes (\Lambda^b(\bar{L}_k)) \otimes \Lambda^c(Z^{(+)})$$

to

$$\Lambda^{a-1}(Y^{(+)}) \otimes (\Lambda^b(\bar{L}_k)) \otimes \Lambda^c(Z^{(+)}).$$

It is easy to see that f_0 is exactly the boundary for computing the homology of $Y^{(+)}$ with coefficients in $H(\bar{L}_k) \otimes \Lambda(Z^{(+)})$. Thus

$$F^1 = H(Y^{(+)}; H(\bar{L}_k) \otimes \Lambda(Z^{(+)})). \tag{3.3.7}$$

Equation (3.3.7) gives us a means to compute F^1. Unfortunately it is not refined enough for our purposes. We need to break up the boundary f_0 into a sum of two finer mappings.

Consider the f_0 boundary. For $u \in H(\bar{L}_k)$, $v \in \Lambda(Z^{(+)})$, and $y_1, ..., y_a \in Y^{(+)}$ we have

$$f_0(y_1 \wedge \cdots \wedge y_a \otimes u \otimes v) = (f_0^{(1)} + f_0^{(2)} + f_0^{(3)})(y_1 \wedge \cdots \wedge y_a \otimes u \otimes v),$$

where

$$f_0^{(1)}(y_1 \wedge \cdots \wedge y_a \otimes u \otimes v) = \sum_{1 \leqslant i < j \leqslant a} (-1)^{i+j} [y_i, y_j] \wedge y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_a \otimes u \otimes v$$

$$f_0^{(2)}(y_1 \wedge \cdots \wedge y_a \otimes u \otimes v) = \sum_{i=1}^{a} (-1)^{a-i} y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_a \otimes (y_i \cdot u) \otimes v$$

$$f_0^{(3)}(y_1 \wedge \cdots \wedge y_a \otimes u \otimes v) = \sum_{i=1}^{a} (-1)^{a-i} y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_a \otimes u \otimes (y_i \cdot v).$$

Let v be a vector which is homogeneous of $\mathscr{H}$-weight λ. Write $\lambda = \sum a_i \alpha_i$ where $\alpha_1, ..., \alpha_{n-1}$ is the usual basis for the root system A_n. Recall that the *height of* λ is $\sum a_i \in \mathbb{Z}$. If $y \in Y^{(+)}$ is homogeneous of $\mathscr{H}$-weight μ then μ is negative so the height of the $\mathscr{H}$-weight of $y \cdot v$ is less than the height of the $\mathscr{H}$-weight of v. Thus $f_0^{(1)} + f_0^{(2)}$ preserves the height of the $\mathscr{H}$-weight of v and $f_0^{(3)}$ lowers it.

Filter F^0 by δ where $\delta(y \otimes u \otimes v)$ is the height of the $\mathscr{H}$-weight of v. This filtration gives rise to a third spectral sequence (K^r, k_r) which abutts to F^1 and which has k_0 differential $f_0^{(1)} + f_0^{(2)}$.

Note that there is an action of $\mathscr{H}$ on the Lie algebra $\bar{L}_k^{(+)}$ which induces an action of $\mathscr{H}$ on F^0. This action commutes with the differential k_r so we can split the spectral sequence (K^r, k_r) into a direct sum of spectral sequences

$$(K^r, k_r) \cong \bigoplus_{\lambda} (K^r(\lambda), k_r),$$

where $K^r(\lambda)$ is the span of all $\mathscr{H}$-weight vectors in K^r of weight λ. We will compute the terms in the spectral sequence $(K^r([\beta, 1^p], k_r)$ explicitly (the sequence collapses at $K^n([\beta, 1^p])$.

We will need the following elegant theorem due to Kostant.

THEOREM 3.3.8 (Kostant [Ko1]). *Let* $Y^{(+)}$ *be as above and let* V^λ *denote the irreducible L-module with lowest weight* λ. *Then* $H(Y^{(+)}; V^\lambda)$ *has dimension n! and the homology classes have a natural indexing by elements of the Weyl group* S_n. *Under the indexing* C_π, *the class corresponding to the permutation* π *satisfies*:

(1) C_π *has homological degree* $l(\pi)$, *the length of* π *as an element of the Coxeter group* S_n.

(2) C_π *has* $\mathscr{H}$-*weight* $\pi(\lambda+\rho)-\rho$ *where* ρ *is half the sum of the positive roots of* A_{n-1}.

(3) C_π *has homology representative* $y_\pi \otimes v_\pi$ *where* v_π *is the unique* $\mathscr{H}$-*weight vector in* V^λ *of weight* $\pi\lambda$ *and where* y_π *is the wedge of all* z_{ij} *in* $\mathscr{Y}^{(+)}$ *such that* $\pi(z_{ij})$ *is not in* $\mathscr{Y}^{(+)}$ (*i.e., such that* $\pi j < \pi i$).

Moreover if v *is any* $\mathscr{H}$-*weight vector in* V^λ, *if* y *is any pure wedge product in* $\Lambda(Y^{(+)})$, *if* $y \otimes v$ *is in the kernel of the Koszul boundary, and if* $y \otimes v$ *is* not *of the form* $y_\pi \otimes v_\pi$ *then* $y \otimes v$ *is in the image of the Lie algebra boundary map on* $(\Lambda Y^{(+)}) \otimes V^\lambda$.

Observe that the complex (K^0, k_0) is isomorphic to the tensor product of $\Lambda(Z^{(+)})$ with the Koszul complex for computing the homology of $Y^{(+)}$ with coefficients in $H(\bar{L}_k)$. So we can apply Kostant's theorem to determine all the $\mathscr{H}$-weight vectors of weight $[\beta, 1^p]$ in $K^1 = H(Y^{(+)}; H(\bar{L}_k)) \otimes \Lambda(Z^{(+)})$.

For any $S \subseteq (\boldsymbol{n-1})$ let z_S be the vector in $\Lambda(Z^{(+)})$ given by

$$z_S = \bigwedge_{i \in S} z_{in}.$$

Note that z_S has $\mathscr{H}$-weight e_S where $e_S \in \mathbb{Z}^n$ is given by

$$(e_S)_i = \begin{cases} 1 & \text{if } i<n \quad \text{and} \quad i \in S \\ 0 & \text{if } i<n \quad \text{and} \quad i \notin S \\ -|S| & \text{if } i=n. \end{cases}$$

As observed above we have that

$$K^1 = \bigoplus_{S \subseteq (\boldsymbol{n-1})} (\{H(Y^{(+)}; H(\bar{L}_k)) \otimes \langle z_S \rangle\}$$
$$\oplus \{H(Y^{(+)}; H(\bar{L}_k)) \otimes \langle z_S \wedge z_{nn} \rangle\}). \tag{3.3.9}$$

We will determine which of the summands in (3.3.9) contribute to $K^1([\beta, 1^p])$. By Kostant's Theorem (3.3.8), the summands in (3.3.9) which contribute to $K^1([\beta, 1^p])$ correspond to sets S which satisfy

$$[\beta, 1^p] - l(1, 1, ..., 1) = e_S + \pi(\lambda+\rho) - \rho \tag{3.3.10}$$

for some $\pi \in S_n$ and some highest weight λ. Given such a triple S, π, and λ the corresponding contribution to $K^1([\beta, 1^p])$ is then

$$(y^{(+)} \otimes H^\lambda(\bar{L}_k) \otimes z_S) \otimes (y_\pi \otimes H^\lambda(\bar{L}_k) \otimes (z_S \wedge z_{nn})).$$

It follows from (3.3.10) that

$$[\beta, 1^p] - l(1, 1, ..., 1) + \rho - e_S$$

has distinct entries or equivalently that

$$[\beta, 1^p] + (n-1, n-2, ..., 1, 0) + e_S \tag{3.3.11}$$

has distinct entries. So if $i \in S$ then $(i+1) \in S$ for $(p+1) < i < (n-1)$ and for $1 < i < (p+1)$. This implies that S must be of the form

$$S = \{i, i+1, ..., p+1\} \cup \{j, j+1, ..., n-1\}$$

or

$$S = \{1\} \cup \{i, i+1, ..., p+1\} \cup \{j, j+1, ..., n-1\}. \tag{3.3.12}$$

In fact, not all the sets in (3.3.12) are possible because we have not yet taken into account that the nth coordinate of (3.3.11) must be distinct from the first $(n-1)$. On can easily show that the only sets S for which all entries of (3.3.11) are distinct are given in the following chart:

S	π	λ
ϕ	id	$[\beta, 1^p]$
$\{j, j+1, ..., n-1\}$ for $p+2 \leqslant j \leqslant n-1$	$(j, j+1, ..., n)$	$[\beta, 1^p]$
$\{1, i, i+1, ..., n-1\}$ for $2 \leqslant i \leqslant p+2$	$(i, i+1, ..., n)$	$[\beta-1, 1^{p+1}]$

For $\pi = (j, j+1, ..., n)$ we have

$$y_\pi = z_{nj} \wedge z_{n(j+1)} \wedge \cdots \wedge z_{n(n-1)}.$$

Let $\mathscr{A}_j$, $\mathscr{B}_j$, $\mathscr{C}_j$, and $\mathscr{D}_j$ be defined by

$$\mathscr{A}_j = z_{nj} \wedge z_{n(j+1)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta, 1^p]}(\bar{L}_k) \otimes z_{jn} \wedge \cdots \wedge z_{(n-1)n}$$

$$\mathscr{B}_j = z_{nj} \wedge z_{n(j+1)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta, 1^p]}(\bar{L}_k)$$
$$\otimes z_{jn} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}$$

$$\mathscr{C}_j = z_{nj} \wedge z_{n(j+1)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k)$$
$$\otimes z_{1n} \wedge z_{jn} \wedge \cdots \wedge z_{(n-1)n}$$

$$\mathscr{D}_j = z_{nj} \wedge z_{n(j+1)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k)$$
$$\otimes z_{1n} \wedge z_{jn} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}.$$

In terms of this notation we have

$$K^1([\beta, 1^p]) = \left\{ \bigoplus_{j=p+1}^{n} (\mathscr{A}_j \oplus \mathscr{B}_j) \right\} \oplus \left\{ \bigoplus_{j=2}^{p+1} (\mathscr{C}_j \oplus \mathscr{D}_j) \right\}. \qquad (3.3.13)$$

The form of $K^1([\beta, 1^p])$ is simple enough that we can compute the higher differentials k_r in the spectral sequence $(K^r([\beta, 1^p]), k_r)$. We will not actually need this computation but we state it here with a brief proof because we will return later to comment on the form of $K^\infty([\beta, 1^p]) = F^1([\beta, 1^p])$.

LEMMA 3.3.14. (i) *For* $1 \leqslant r \leqslant n-p$ *we have*

$$K^r([\beta, 1^p]) = \{\mathscr{A}_n \oplus \mathscr{B}_{n-r+1}\} \oplus \bigoplus_{j=p+1}^{n-r} \{\mathscr{A}_j \oplus \mathscr{B}_j\} \oplus \bigoplus_{j=2}^{p+1} \{\mathscr{C}_j \oplus \mathscr{D}_j\}.$$

(ii) *The* k_r *differential is an isomorphism from* $\mathscr{A}_{n-r}$ *onto* $\mathscr{B}_{n-r+1}$ *and is* 0 *elsewhere (for* $1 \leqslant r \leqslant n-p$).

(iii) *For* $n-p+1 \leqslant r$ *we have*

$$K^r([\beta, 1^p]) = \{\mathscr{A}_n \otimes \mathscr{B}_{p+1}\} \oplus \{\mathscr{C}_{p+1} \oplus \mathscr{D}_{n-r+1}\} \oplus \bigoplus_{j=2}^{n-r} \{\mathscr{C}_j \oplus \mathscr{D}_j\}.$$

(iv) *The* k_r *differential is an isomorphism from* $\mathscr{C}_{n-r}$ *onto* $\mathscr{D}_{n-r+1}$ *and is zero everywhere else.*

Proof. The proof goes by induction on r. Statements (i) and (iii) which give an expressions for the $K^r([\beta, 1^p])$ follow immediately from statements (ii) and (iv) which explain how to compute the differentials k_r. Also we consider only the case $r < n-p+1$ as the proofs for $r \geqslant n-p+1$ are similar. So we fix a value of $r < n-p+1$ and we assume $K^r([\beta, 1^p])$ has the form given in (i). We will show that the differential k_r satisfies (ii).

The map k_r is induced by that part of the boundary map $f_0^{(3)}$ consisting of all terms where the height of the $\mathscr{H}$-weight of the third tensor position drops by exactly r. Note that the $\mathscr{H}$-weight of z_{nj} is $e_n - e_j$ which has height $j-n$. Referring to the form of $K^1([\beta, 1^p])$ given in (3.3.13) it is clear that the only terms where the height drops by r are those terms where a $z_{n(n-r)}$ from the first tensor position brackets with a $z_{(n-r)n}$ in the third tensor position leaving a z_{nn} in the third tensor position. So immediately we have

$$k_r(\beta_j) = k_r(\mathscr{D}_j) = 0.$$

Now let v be an element of $H^{[\beta, 1^p]}(\bar{L}_k)$. By Kostant's Theorem we have

$$k_0(z_{nj} \wedge \cdots \wedge z_{n(n-1)} \otimes v \otimes z_{jn} \wedge \cdots \wedge z_{(n-1)n}) = 0$$

so

$$z_{ns} \cdot v = 0 \qquad \text{for} \quad j \leqslant s \leqslant (n-1). \tag{3.3.15}$$

Note that

$$\begin{aligned} &k_r(z_{nj} \wedge \cdots \wedge z_{n(n-1)} \otimes v \otimes z_{jn} \wedge \cdots \wedge z_{(n-1)n}) \\ &\quad = 0 \qquad \text{if} \quad j > (n-r) \qquad (3.3.16a) \\ &\quad = z_{nj} \wedge \cdots \wedge \hat{z}_{n(n-r)} \wedge \cdots \wedge z_{n(n-1)} \otimes v \\ &\qquad \otimes z_{jn} \wedge \cdots \wedge z_{(n-r)n} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}. \qquad (3.3.16b) \end{aligned}$$

Consider the vector A in (3.3.16b). For $j \neq n-r$ this vector A is not of the form $y_\pi \otimes v_\pi \otimes X$ for $\pi \in S_n$ and for $X \in \Lambda Z^{(+)}$. So by the second half of Kostant's theorem A is an eigenvector for the Laplacian $L = k_0 k_0^t + k_0^t k_0$ with nonzero eigenvalue λ. Also by (3.3.15) we have $k_0 A = 0$. Thus $A = k_0((k_0^t/\lambda)A)$, so A is in the image of k_0. Hence for $j < (n-r)$ the vector given in (3.3.16) is zero in $K^r([\beta, 1^p])$.

For $j = n-r$ we have

$$\begin{aligned} &k_r(z_{n(n-r)} \wedge \cdots \wedge z_{n(n-1)} \otimes v \otimes z_{(n-r)n} \wedge \cdots \wedge z_{(n-1)n}) \\ &\quad = z_{n(n-r+1)} \wedge \cdots \wedge z_{n(n-1)} \otimes v \\ &\qquad \otimes z_{(n-r+1)n} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}. \end{aligned} \tag{3.3.17}$$

Note that (3.3.17) shows that k_r is an isomorphism of $\mathscr{A}_{n-r}$ onto $\mathscr{B}_{n-r+1}$ which completes the induction step. ∎

COROLLARY 3.3.18. *The spectral sequence* $(K^r([\beta, 1^p]), k_r)$ *collapses at* $K^{n-1}([\beta, 1^p])$. *We have*

$$K^{n-1}([\beta, 1^p]) = K^\infty([\beta, 1^p]) = \mathscr{A}_n \oplus \mathscr{B}_{p+1} \oplus \mathscr{C}_{p+1} \oplus \mathscr{D}_2.$$

We have just shown that $K^\infty([\beta, 1^p]) = F^1([\beta, 1^p])$ is the direct sum of four spaces which have representations from the original E^1 complex given by

$$\begin{aligned} F^1([\beta, 1^p]) = &\{1 \times H^{[\beta, 1^p]}(\bar{L}_k) \otimes 1\} \\ &\oplus \{z_{n(p+2)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta, 1^p]}(\bar{L}_k) \otimes z_{(p+2)n} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}\} \\ &\oplus \{z_{n(p+2)} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k) \otimes z_{1n} \wedge z_{(p+2)n} \wedge \cdots \wedge z_{(n-1)n}\} \\ &\oplus \{z_{n2} \wedge \cdots \wedge z_{n(n-1)} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k) \otimes z_{1n} \wedge z_{2n} \wedge \cdots \wedge z_{(n-1)n} \wedge z_{nn}\}. \end{aligned}$$

Observe that the skew degree s of each of these four spaces is very close to the actual homological degree d. In fact the difference $d-s$ is $0, -1, -1, -2$ on the four spaces above (respectively).

At this point it becomes difficult to do further computation with the spectral sequences (F^r, δ_r) and (E^r, ∂_r). Fortunately we have no need for further computations as we will pass to an Euler characteristic. For each d, w, s let $F^1_{d,w,s}([\beta, 1^p])$ denote the subspace of $F^1_{d,w}([\beta, 1^p])$ spanned by vectors with skew degree s.

DEFINITION 3.3.19. Define the *Euler characteristic* $\chi(F^1([\beta, 1^p]))$ by

$$\chi(F^1([\beta, 1^p])) = \sum_{d,w,s} \dim(F^1_{d,w,s}([\beta, 1^p]))(-1)^d q^w z^s.$$

From (3.3.13) it follows easily that

$$\begin{aligned}\chi(F^1([\beta, 1^p]))& \\ &= P^{[\beta, 1^p]}(\bar{L}_k; -z, q)(1-(q^{k+1}z^2)^{n-p+1}) \\ &\quad + P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; -z, q)(-(q^{k+1}z^2)^{n-p+1} + (q^{k+1}z^2)^n). \qquad (3.3.20)\end{aligned}$$

Our goal is to derive a recursion which expresses $P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; z, q)$ in terms of $P^{[\beta, 1^p]}(\bar{L}_k; z, q)$. Note that the Euler characteristic $\chi(F^1([\beta, 1^p]))$ depends exactly on these two quantities.

Step 3. At this point we must repeat the computations that appear in Steps 1 and 2 for the Lie algebra $\bar{L}_k^{(-)}$. In exactly the same way we will define spectral sequences $(E^r_{(-)}, e_r)$, $(F^r_{(-)}, f_r)$, and $(K^r_{(-)}, k_r)$. As in Step 2 above our goal is to compute the component of $F^1_{(-)}$ having $\mathscr{H}$-weight $[\beta, 1^p]$. The computations of the $K^r_{(-)}$ and $F^1_{(-)}$ are exactly the same. To see that the same methods apply one needs to note that $Y^{(-)}$ is the nilpotent lower summand of $L = sl_n(\mathbb{C})$ with respect to the basis of the root system A_n given by

$$B_{(-)} = \{e_1 - e_2, e_2 - e_3, ..., e_{n-2} - e_{n-1}, e_n - e_1\}.$$

The reader must take care when applying Kostant's Theorem to compute $K^1_{(-)}$. The subtle point is that Kostant's Theorem describes the decomposition of $K^1_{(-)}$ as an $\mathscr{H}_{(-)}$ is the torus with respect to the basis $B_{(-)}$. We want the $[\beta, 1^p]$ component with respect to the action of $\mathscr{H}$. It is not difficult to account for this during the computation since $\mathscr{H}_{(-)}$ and $\mathscr{H}$ are S_n-conjugate. From these computations one obtains

$$\begin{array}{ll|l} & & d-s \\ F^1_{(-)} = & \{z_{1n} \wedge \cdots \wedge z_{(n-1)n} \otimes H^{[\beta, 1^p]}(\bar{L}_k) \otimes z_{n1} \wedge \cdots \wedge z_{n(n-1)} \wedge z_{nn}\} & -1 \\ & \oplus \{z_{1n} \wedge \cdots \wedge z_{(p+1)n} \otimes H^{[\beta, 1^p]}(\bar{L}_k) \otimes z_{n1} \wedge \cdots \wedge z_{n(p+1)}\} & 0 \\ & \oplus \{z_{1n} \wedge \cdots \wedge z_{(p+1)n} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k) \otimes z_{n2} \wedge \cdots \wedge z_{n(p+1)} \wedge z_{nn}\} & 0 \\ & \oplus \{z_{1n} \otimes H^{[\beta-1, 1^p, 0^s, 1]}(\bar{L}_k) \otimes 1\} & 1. \end{array} \tag{3.3.21}$$

Note that once again the skew degree is very close to the actual degree. Taking Euler characteristics we obtain

$$\chi(F^1_{(-)}([\beta, 1^p])) = P^{[\beta, 1^p]}(\bar{L}_k; -z, q)((q^{k+1}z^2)^{p+1} - (q^{k+1}z^2)^n) + P^{[\beta-1, 1^{p+1}]}(\bar{L}_k: -z, q)(-1 + (q^{k+1}z^2)^{p+1}).$$

At this point we are going to compare $\chi(F^1([\beta, 1^p]))$ and $\chi(F^1_{(-)}([\beta, 1^p]))$.

Step 4. Pushing one step further. We would like to prove that the Euler characteristics $\chi(F^1([\beta, 1^p]))$ and $\chi(F^1_{(-)}([\beta, 1^p]))$ are identical (up to a shift in the parameters w and s) using the map Ω. However, it is not clear that the map Ω induces an isomorphism between F^1 and $F^1_{(-)}$. The problem is this. Both F^1 and $F^1_{(-)}$ are the result of taking the homology of the original complexes $\Lambda(L_k^{(+)})$ and $\Lambda(L_k^{(-)})$ with respect to some portion of the original boundary. In order to derive an isomorphism between F^1 and $F^1_{(-)}$ we must have that the map Ω matches those portions of the boundaries used to compute F^1 and $F^1_{(-)}$. Unfortunately this is not the case. In the process of computing F^1 we allowed that part of the boundary map on $\Lambda(L_k^{(+)})$ induced by bracketting together elements of the form $z_{ni} \otimes 1$ and $z_{in} \otimes t^{k+1}$ to get a $z_{nn} \otimes t^{k+1}$. The corresponding bracket in $L_k^{(-)}$ under the map Ω is the bracketting of elements $z_{ni} \otimes t$ and $z_{in} \otimes t^k$ to get a $z_{nn} \otimes t^{k+1}$. This was not allowed. Going the other way we allowed $[z_{in} \otimes 1, z_{ni} \otimes t^{k+1}] = z_{nn} \otimes t^{k+1}$ when we computed $F^1_{(-)}$ but we did not allow the Ω-counterpart $[z_{in} \otimes t, z_{ni} \otimes t^k] = z_{nn} \otimes t^{k+1}$ when we computed F^1. The reader will note that otherwise we used the Ω-counterpart boundaries to compute F^1 and $F^1_{(-)}$. The point of this next step in our computation is to make up for these deficiencies without altering the Euler characteristics.

DEFINITION 3.3.22. Define $\lambda: \Lambda^r \bar{L}_k \to \Lambda^{r-2} \bar{L}_k$ by

$$\lambda((z_{i_1 j_1} \otimes t^{l_1}) \wedge \cdots \wedge (z_{i_r j_r} \otimes t^{l_r})) = \sum_{u<v} (-1)^{u+v+1} \varepsilon(u, v)(z_{i_1 j_1} \otimes t^{l_1}) \wedge \cdots \wedge \widehat{(z_{i_u j_u} \otimes t^{l_u})} \wedge \cdots \wedge \widehat{(z_{i_v j_v} \otimes t^{l_v})} \wedge \cdots \wedge (z_{i_r j_r}) \otimes t^{l_r}),$$

where the sum is over all $u < v$ and where

$$\varepsilon(u, v) = \begin{cases} 1 & \text{if } i_u = j_v = n,\ j_u = i_v < n,\ l_u + l_v = k+1 \\ -1 & \text{if } i_v = j_u = n,\ i_u = j_v < n,\ l_u + l_v = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

The idea here is that λ represents that portion of the original boundary of $\Lambda(L_k^{(+)})$ and $\Lambda(L_k^{(-)})$ which allows two elements of $\bar{L}_k$ to bracket together to give $z_{nn} \otimes t^{k+1}$.

LEMMA 3.3.23. *Let ∂ be the Lie algebra homology boundary on $\Lambda \bar{L}_k$. Then $\partial \circ \lambda = \lambda \circ \partial$.*

Proof. Both of the maps $\partial \circ \lambda$ and $\lambda \circ \partial$ applied to a pure wedge $w_1 \wedge \cdots \wedge w_r$ produce a sum of terms where three or four of the w_i have been removed, two have been bracketted together to form the contribution of the ∂, and two have been replaced by ± 1 or 0 to form the contribution of λ. It is easy to check that the corresponding terms in the sum are equal if they involved four of the w_i. Also any term which occurs in $\partial \circ \lambda(w_1 \wedge \cdots \wedge w_r)$ involves four of the w_i. So we only check that those terms in $\lambda \circ \partial(w_1 \wedge \cdots \wedge w_r)$ which involve exactly three of the w_i's cancel out.

To see this note that those terms in $\lambda \circ \partial$ which involve only three of the w_i's come about when there is a triple w_s, w_u, w_v with

$$w_s = z_{ij} \otimes t^a$$
$$w_u = z_{jn} \otimes t^b$$
$$w_v = z_{ni} \otimes t^{(k+1)-(a+b)}.$$

There are two terms involving w_s, w_u, w_v which occur in $\lambda \circ \partial(w_1 \wedge \cdots \wedge w_r)$ but not in $\partial \circ \lambda(w_1 \wedge \cdots \wedge w_r)$. The first comes about when w_s and w_u are bracketted together by ∂ to give $z_{in} \otimes t^{a+b}$ and this $z_{in} \otimes t^{a+b}$ is removed along with $z_{ni} \otimes t^{(k+1)-(a+b)}$ by λ. The resulting term is

$$(-1)^{(s+u-1)+(1+(v-1)+1)}(-1)\, w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge \hat{w}_u \wedge \cdots \wedge \hat{w}_v \wedge \cdots \wedge w_r.$$

The second comes about when w_s and w_v are bracketted together by ∂ to give a $z_{nj} \otimes t^{(k+1)-b}$ and then this $z_{nj} \otimes t^{(k+1)-b}$ is removed along with $z_{jn} \otimes t^b$ by λ. The resulting term is

$$(-1)^{(s+v+1)+(1+u)+1}(-1)\, w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge \hat{w}_u \wedge \cdots \wedge \hat{w}_v \wedge \cdots \wedge w_r.$$

It is clear that these terms cancel out which means that $(\lambda \circ \partial)(w_1 \wedge \cdots \wedge w_r)$ and $(\partial \circ \lambda)(w_1 \wedge \cdots \wedge w_r)$ are the same. ∎

By Lemma 3.3.23 we have that λ induces a map on $H_{**}(\bar{L}_k)$ which reduces degree by 2, preserves $\mathscr{H}$-weight, and reduces weight by $k+1$. We write λ to denote this induced map on $H(\bar{L}_k)$. Now define the maps $\tilde{\lambda}_{(+)}$ and $\tilde{\lambda}_{(-)}$ on $F^1([\beta, 1^p])$ and $F^1_{(-)}([\beta, 1^p])$ by

$$\tilde{\lambda}_{(+)}(y \otimes v \otimes z) = y \otimes (\lambda v) \otimes (z \wedge z_{nn}) \quad \text{for} \quad y \in \Lambda Y^{(+)} \quad \text{and} \quad z \in \Lambda Z^{(+)}$$

and

$$\tilde{\lambda}_{(-)}(y \otimes v \otimes z) = y \otimes (\lambda v) \otimes (z \wedge z_{nn}) \quad \text{for} \quad y \in \Lambda Y^{(-)} \quad \text{and} \quad z \in \Lambda Z^{(-)}.$$

It is easily seen that $\tilde{\lambda}^2_{(+)} = \tilde{\lambda}^2_{(-)} = 0$ because both involve wedging by z_{nn} in the third tensor position. Let $\tilde{F}^1([\beta, 1^p])$ and $\tilde{F}^1_{(-)}([\beta, 1^p])$ denote the homologies of $F^1([\beta, 1^p])$ and $F^1_{(-)}([\beta, 1^p])$ with respect to the maps $\tilde{\lambda}^2_{(+)}$ and $\tilde{\lambda}^2_{(-)}$. Note that both $\tilde{\lambda}_{(+)}$ and $\tilde{\lambda}_{(-)}$ preserve the value of s. This together with our previous observation about λ show that

$$\chi(\tilde{F}^1([\beta, 1^p])) = \chi(F^1([\beta, 1^p]))$$

and

$$\chi(\tilde{F}^1_{(-)}([\beta, 1^p])) = \chi(F^1_{(-)}([\beta, 1^p])). \tag{3.3.24}$$

Moreover a careful examination of all the differentials used to compute $\tilde{F}^1$ and $\tilde{F}^1_{(-)}$ yields the following theorem.

THEOREM 3.3.25. *The map Ω induces an isomorphism between $\tilde{F}^1$ and $\tilde{F}^1_{(-)}$.*

Step 5. Comparing $\chi(F^1([\beta, 1^p]))$ and $\chi(F^1_{(-)}([\beta, 1^p]))$. Combining (3.3.24) with Theorem 3.3.25 we see that $\chi(F^1([\beta, 1^p]))$ and $\chi(F^1_{(-)}([\beta, 1^p]))$ are Euler characteristics of isomorphic complexes. However, we cannot equate the Euler characteristics because the isomorphism Ω between the complexes does not preserve weight and skew degree. So we need to examine how Ω shifts weight and skew degree. To determine the shift in skew degree we must introduce a new induction hypothesis.

Recall the standard bases for $\bar{L}_k$

$$\mathscr{B}^{(k)} = \mathscr{B}^{(k)}_1 \cup \mathscr{B}^{(k)}_2,$$

where

$$\mathscr{B}_1^{(k)} = \{z_{uv} \otimes t^w : u \neq v,\ u,\ v \in \mathbf{n},\ 1 \leqslant w \leqslant k\}$$
$$\mathscr{B}_2^{(k)} = \{(z_{uu} - z_{(u+1)(u+1)}) \otimes t^w : 1 \leqslant u \leqslant n-1,\ 1 \leqslant w \leqslant k\}.$$

For each $j = 1, 2, ..., n$ define disjoint subsets $\mathscr{A}_j^{(+)}$, $\mathscr{C}_j^{(+)}$, $\mathscr{R}_j^{(+)}$ of $\mathscr{B}^{(k)}$ by

$$\mathscr{A}_j^{(+)} = \{z_{ji} \otimes t : i \in \mathbf{n},\ i \neq j\}$$
$$\mathscr{C}_j^{(+)} = \{z_{ij} \otimes t^k : i \in \mathbf{n},\ i \neq j\}$$
$$\mathscr{R}_j^{(+)} = \mathscr{B}^{(k)} \backslash (\mathscr{A}_j^{(+)} \cup \mathscr{C}_j^{(+)}).$$

Define $A_j^{(+)}$, $C_j^{(+)}$, and $R_j^{(+)}$ to be the spans of the sets $\mathscr{A}_j^{(+)}$, $\mathscr{C}_j^{(+)}$, and $\mathscr{R}_j^{(+)}$. Note that

$$\Lambda(\bar{L}_k) \cong \Lambda(A_j^{(+)}) \otimes \Lambda(R_j^{(+)}) \otimes \Lambda(C_j^{(+)}).$$

Define $\alpha_j^{(+)}$ on $\Lambda(\bar{L}_k)$ by

$$\alpha_j^{(+)}(\Lambda^a(A_j^{(+)}) \otimes \Lambda^r(R_j^{(+)}) \otimes \Lambda^c(C_j^{(+)})) = a - c.$$

Note that $\alpha_j^{(+)}$ takes on values between $-(n-1)$ and $(n-1)$. For each l let $M_{j,l}^{(+)}$ denote the span of all vectors m in $\Lambda(\bar{L}_k)$ with $\alpha_j^{(+)}(m) = l$. Let ∂ denote the boundary map in the Koszul complex for $\bar{L}_k$. Note that

$$\partial(M_{j,l}^{(+)}) \subseteq M_{j,l}^{(+)} \oplus M_{j,l-1}^{(+)}.$$

So $\alpha_j^{(+)}$ gives us a filtering of the Koszul complex $(\Lambda(\bar{L}_k), \partial)$. Let

$$H_{**}(\bar{L}_k) = \bigoplus_l H_{**l}(\bar{L}_k; j, +)$$

be the associated grading of the homology of $\bar{L}_k$. The induction hypothesis (H) that will help us compute the shift in skew degree is:

(i) For all j we have

$$H^{\phi}(\bar{L}_k) = H_{**}(\bar{L}_k; j, +) \tag{H}$$

(ii) Let $[\beta, 1^p]$ be a hook partition. Then

(a) for $2 \leqslant j \leqslant p+1$ we have

$$H^{[\beta, 1^p]}(\bar{L}_k) = H_{**}(\bar{L}_k; j, +)$$

(b) for $p+2 \leqslant j \leqslant n$ we have

$$H^{[\beta, 1^p]}(\bar{L}_k) = H_{**-1}(\bar{L}_k; j, +).$$

This hypothesis asserts that, with respect to the grading given by $\alpha_j^{(+)}$, the $[\beta, 1^p]$ component is concentrated in the 0th graded piece for $2 \leq j \leq p+1$ and in the (-1)st graded piece for $p+2 \leq j \leq n$.

As mentioned, we will prove (H) along with our main theorem simultaneously by induction. This will be a double induction—first by induction on k and then by reverse induction on β. To start the induction for (H) we will prove the case $k=1$ directly.

In the case $k=1$, the value of $\alpha_j^{(+)}$ on an $\mathscr{H}$-weight vector is just the jth component of the corresponding weight. Hence the value of $\alpha_j^{(+)}$ on $H^{[\beta, 1^p]}(\bar{L}_1)$ equals the jth component of the maximal weight corresponding to $[\beta, 1^p]$. The maximal weight is

$$(\beta - l, \underbrace{1-l, 1-l, ..., 1-l}_{p}, \underbrace{-l, -l, -l, ..., -l}_{n-1-p}),$$

where l is the layer of $[\beta, 1^p]$. So the value of $\alpha_j^{(+)}$ on $H^{[\beta, 1^p]}(\bar{L}_1)$ is

$$\begin{cases} 1-l & \text{if } 2 \leq j \leq p+1 \\ -l & \text{if } p+2 \leq j \leq n. \end{cases} \tag{3.3.26}$$

It is straightforward to check that the only hook partitions which have non-zero components in $\Lambda(\bar{L}_1)$ (hence in $H(\bar{L}_1)$) are the first layer hooks. This observation together with (3.3.26) proves (H) in the case $k=1$.

At this point we can also check that the hypothesis (H) holds as we take the induction step from the case where we consider the 0 $\mathscr{H}$-weight component of $H(\bar{L}_{k-1})$ to where we consider the $[kn, 0^{n-1}]$ component of $H(\bar{L}_k)$. We want to verify that

$$H_{**0}^{[kn]}(\bar{L}_k; j, +) = H^{[kn]}(\bar{L}_k) \qquad \text{for} \quad 2 \leq j \leq n$$

given that $H_{**0}^{\phi}(\bar{L}_{k-1}; j, +) = H^{\phi}(\bar{L}_{k-1})$ for $1 \leq j \leq n$. This follows easily by examining the isomorphism φ used in the proof of Theorem 3.2.8.

We will therefore assume that (H) holds for $H^{[\beta, 1^p]}(\bar{L}_k)$. Recall that $F^1([\beta, 1^p])$ is a sum of four spaces

$$F^1([\beta, 1^p]) = \mathscr{A}_n \oplus \mathscr{B}_{p+1} \oplus \mathscr{C}_{p+1} \oplus \mathscr{D}_2$$

(see Corollary 3.3.18). The map $\bar{\lambda}_{(+)}$ maps $\mathscr{A}_n$ into $\mathscr{B}_{p+1}$ and $\mathscr{C}_{p+1}$ into $\mathscr{D}_2$. We write $\tilde{F}^1([\beta, 1^p])$ as

$$\tilde{F}^1([\beta, 1^p]) = \tilde{\mathscr{A}} \oplus \tilde{\mathscr{B}} \oplus \tilde{\mathscr{C}} \oplus \tilde{\mathscr{D}},$$

where

$$\tilde{\mathscr{A}} = (\ker \tilde{\lambda}_{(+)}) \cap \mathscr{A}_n$$
$$\tilde{\mathscr{B}} = \mathscr{B}_{p+1}/(\mathscr{B}_{p+1} \cap \operatorname{im} \tilde{\lambda}_{(+)})$$
$$\tilde{\mathscr{C}} = (\ker \tilde{\lambda}_{(+)}) \cap \mathscr{C}_{p+1}$$

and

$$\tilde{\mathscr{D}} = \mathscr{D}_2/(\mathscr{D}_2 \cap \operatorname{im} \tilde{\lambda}_{(+)}).$$

Note that the map λ on $H(\bar{L}_k)$ preserves $\alpha_n^{(+)}$ so $\alpha_n^{(+)}$ is well-defined on $\tilde{F}^1([\beta, 1^p])$.

From (3.3.21) we have that $F^1_{(-)}([\beta, 1^p])$ is also a sum of four spaces $F^1_{(-)}([\beta, 1^p]) = \mathscr{W}_{(-)} \oplus \mathscr{X}_{(-)} \oplus \mathscr{Y}_{(-)} \oplus \mathscr{Z}_{(-)}$ (where these are the four summands in (3.3.21) taken from top to bottom). So we can write $F^1_{(-)}([\beta, 1^p])$ as

$$\tilde{F}^1_{(-)}([\beta, 1^p]) = \tilde{\mathscr{W}} \oplus \tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}} \oplus \tilde{\mathscr{Z}}$$

as above. We now want to determine how Ω maps $\tilde{F}^1([\beta, 1^p])$ to $\tilde{F}^1_{(-)}([\beta, 1^p])$. Since $\Omega(z_{nn} \otimes t^{k+1}) = z_{nn} \otimes t^{k+1}$ we have

$$\Omega(\tilde{\mathscr{A}} \oplus \tilde{\mathscr{C}}) \subseteq \tilde{\mathscr{X}} \oplus \tilde{\mathscr{Z}}$$

and

$$\Omega(\tilde{\mathscr{B}} \oplus \tilde{\mathscr{D}}) \subseteq \tilde{\mathscr{W}} \oplus \tilde{\mathscr{Y}}.$$

But now by hypothesis (H) we have

$$\Omega(\tilde{\mathscr{A}}) = \tilde{\mathscr{Z}} \qquad \Omega(\tilde{\mathscr{C}}) = \tilde{\mathscr{X}}$$
$$\Omega(\tilde{\mathscr{B}}) = \tilde{\mathscr{Y}} \qquad \Omega(\tilde{\mathscr{D}}) = \tilde{\mathscr{W}}.$$

It follows that Ω decreases $d-s$ by 1. Since Ω preserves degree we have

The map Ω from $\tilde{F}^1([\beta, 1^p])$ to $\tilde{F}^1_{(-)}([\beta, 1^p])$ increases s by 1. (3.3.27)

It remains to determine how Ω effects weight. Recall that l is the layer of $[\beta, 1^p]$ so the dominant weight associated with $[\beta, 1^p]$ is

$$\lambda = (\beta - l, 1 - l, 1 - l, ..., 1 - l, -l, -l, ..., -l).$$

Let $v = (z_{i_1 j_1} \otimes t^{l_1}) \wedge \cdots \wedge (z_{i_d j_d} \otimes t^{l_2})$ be a pure wedge in $\Lambda(L_k^{(+)})$ with $\mathscr{H}$-weight λ. Let p and r be the number of $z_{i_s j_s} \otimes t^{l_s}$ with $i_s = n$, $j_s \neq n$ and

$i_s \neq n$, $j_s = n$, respectively. Then $p - r = -l$ (the nth component of λ). Under Ω each factor counted by p increases in weight by 1 and each factor counted by r decreases in weight by 1. So the total shift in weight from v to Ωv is $r - p = l$. Hence

The map Ω from $\tilde{F}^1([\beta, 1^p])$ to $\tilde{F}^1_{(-)}([\beta, 1^p])$ increases weight by $l =$ (layer of $[\beta, 1^p]$). (3.3.28)

Combining (3.3.27) with (3.3.28) we have the main result in this step of the computation.

LEMMA 3.3.29. *Let l be the layer of $[\beta, 1^p]$. Then*

$$\chi(\tilde{F}^1_{(-)}([\beta, 1^p]) = zq^l \chi(\tilde{F}^1_{(-)}([\beta, 1^p]).$$

To finish this step in the computation we note that the induction hypothesis (H) (which we knew to hold for $[\beta, 1^p]$) also holds for $[\beta - 1, 1^{p+1}]$ because Ω maps $\tilde{\mathscr{A}}$ to $\tilde{\mathscr{X}}$, $\tilde{\mathscr{B}}$ to $\tilde{\mathscr{Y}}$, $\tilde{\mathscr{C}}$ to $\tilde{\mathscr{X}}$, and $\tilde{\mathscr{D}}$ to $\tilde{\mathscr{W}}$.

Step 6. The Endgame. Combining Lemma 3.3.29 with (3.3.24) we have

$$\chi(F^1([\beta, 1^p])) = zq^l \chi(F^1_{(-)}([\beta, 1^p])). \tag{3.3.30}$$

Now combining (3.3.30) with (3.3.21) and (3.3.20), and setting $z = -z$ yields

$$\begin{aligned} &P^{[\beta, 1^p]}(\bar{L}_k; z, q)(1 - (q^{k+1}z^2)^{n-p-1}) \\ &\quad + P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; z, q)(-(q^{k+1}z^2)^{n-p-1} + (q^{k+1}z^2)^n) \\ &= -zq^l\{P^{[\beta, 1^p]}(\bar{L}_k; z, q)((q^{k+1}z^2)^{p+1} - (q^{k+1}z^2)^n) \\ &\quad + P^{[\beta-1, 1^{p+1}]}(\bar{L}_k; z, q)(-1 + (q^{k+1}z^2)^{p+1})\}, \end{aligned}$$

which is equivalent to the desired recursion (3.1.5).

We pause now for an example of how the recursion (3.1.5) can be used to compute the $P^{[\beta, 1^p]}(\bar{L}_k; z, q)$. Consider the case where $n = 3$, $k = 2$, $\beta = 6$, and $p = 0$. Assume that we have already computed that

$$P^{[00]}(\bar{M}_1; z, q) = (1 + z^3q^4)(1 + z^3q^5), \qquad \text{where} \quad M = sl_2(\mathbb{C}).$$

Then by (3.1.4) we have

$$P^{[6,0,0]}(\bar{L}_2; z, q) = z^4q^6(1 + zq)(1 + zq^2)(1 + z^3q^4)(1 + z^3q^5). \tag{3.3.31}$$

The reader can check in the appropriate table in Section 5 that (3.3.31) gives the correct value of $P^{[6,0,0]}(\bar{L}_2; z, q)$. Now by (3.1.5) we have

$$(1 + zq^2(z^2q^3))(1 - (z^2q^3)^2)\, P^{[6,0,0]}(\bar{L}_2; z, q) = ((z^2q^3)^2 + zq^2)(1 - (z^2q^3))\, P^{[5,1,0]}(\bar{L}_2; z, q).$$

Simplifying this gives

$$(zq^2)(1 + z^3q^4)\, P^{[5,1,0]}(\bar{L}_2; z, q) = (1 + z^3q^5)(1 + z^2q^3)\, P^{[6,0,0]}(\bar{L}_2; z, q). \tag{3.3.32}$$

Combining (3.3.31) with (3.3.32) yields

$$P^{[5,1,0]}(\bar{L}_2; z, q) = z^3q^4(1 + z^2q^3)(1 + zq)(1 + zq^2)(1 + z^3q^5)^2. \tag{3.3.33}$$

Again the reader can check using the appropriate table that this is the correct value for $P^{[5,1,0]}(\bar{L}_k; z, q)$.

At first it might seem surprising that the proper divisibility conditions hold to make (3.1.5) possible. In fact, a closer examination of the recursion (3.1.5) reveals that the polynomials $P^{[\beta, 1^p]}(\bar{L}_k; z, q)$ have a very simple form. For each hook shape $[\beta, 1^p]$ define the *multiset* of pairs, $S[\beta, 1^p]$, recursively as

$$S([kn, 0^{n-1}]) = \{(2i+1, (k+1)\, i + j) : 0 \leqslant i \leqslant n-2, 1 \leqslant j \leqslant k\} \tag{R1}$$

$S([\beta - 1, 1^{p+1}])$ is obtained from $S([\beta, 1^p])$ by removing the pair $(2(n-p-1)-1, (k+1)(n-p-1)-l)$ and adding the pair $(2(p+1)+1, (k+1)(p+1)+l)$. (R2)

The following formula is a straightforward consequence f the recursion (3.1.5):

$$P^{[\beta, 1^p]}(\bar{L}_k; z, q) = z^{(n-1)l - p} q^{(n-1)\binom{l+1}{2} - pl} \times \begin{bmatrix} n-1 \\ p \end{bmatrix}_{z^2q^{k+1}} \prod_{(u, v) \in S([\beta, 1^p])} (1 + z^u q^v).$$

3.4. *The First-Layer Case*

There is one further conjecture which probably can be solved using methods similar to those used in Section 3.3. This conjecture gives an explicit formula for $P^\lambda(\bar{L}_k; z, q)$ in the case that λ is a first-layer partition.

Conjecture 3.4.1. Suppose λ is a first-layer partition. Then

$$P^\lambda(\bar{L}_k; z, q) = A^\lambda(z, q)\, B^\lambda(z, q)\, C^\lambda(z, q),$$

where

$$A^\lambda(z,q)=\left[\frac{n!}{\prod h_{ij}}\right]_{z^2q^{k+1}}$$

$$B^\lambda(z,q)=\prod_{(i,j)\in\lambda}\{(z^2q^{k+1})^{j-1}(1+z^{2i-1}q^{(k+1)i-k})\},$$

and

$$C^\lambda(z,q)=\prod_{u=1}^{n}\prod_{v=1}^{k-1}(1+z^{2u-1}q^{(k+1)u-v}).$$

Stembridge [S1] proves that the equality in 3.4.1 holds when $z=-1$. In this case Conjecture 3.4.1 reduces to the First Layer Conjectures of Gupta and Hanlon (see "Combinatorics and Algebra" (C. Greene, Ed.), Contemporary Mathematics, Vol. 34, pp. 305–307).

4. Conjectures about Laplacians

4.1. *Laplacians*

In this section we will present some conjectures concerning the eigenvalues of Laplacians connected with the Koszul complex. The presentation will be brief because this topic will be discussed in more detail in a separate paper (see [H3]).

Let M be any Lie algebra and let $\langle\ ,\ \rangle$ be a positive definite Hermitian form on M. We can extend $\langle\ ,\ \rangle$ to a positive definite Hermitian form on ΛM (also denoted $\langle\ ,\ \rangle$) by

$$\langle x_1\wedge\cdots\wedge x_r,\ y_1\wedge\cdots\wedge y_r\rangle=\det(\langle x_i,y_j\rangle)$$

(here it is understood that $\langle\Lambda^r M,\Lambda^s M\rangle=0$ if $r\neq s$).

Let ∂ denote the boundary in the Koszul complex for M. Define δ to be the transpose of $-\partial$ with respect to $\langle\ ,\ \rangle$. In other words, δ is defined on ΛM by the condition

$$\langle\varphi u,v\rangle=-\langle u,\partial v\rangle.$$

Note that $\delta\colon\Lambda^r M\to\Lambda^{r+1}M$. It is easy to check that $\delta^2=0$. We can define the homology of M as the kernel of δ modulo its image.

Definition 4.1.1. Define the *Laplacian* of M, denote Λ, by

$$\Lambda=\partial\delta+\delta\partial.$$

Note that Λ is a degree preserving linear map on ΛM which depends both on M and on the original form $\langle\ ,\ \rangle$. In this section we will be concerned with the *spectral resolution of* Λ, i.e., the eigenvalues and eigenspaces of Λ acting on the Koszul complex of M, for $M = L_k$ and $\bar{L}_k$. It is well known that the vectors in the nullspace of Λ are simultaneously a complete set of homology and cohomology representatives. So the problem of computing the spectral resolution of Λ for $M = L_k$ and $\bar{L}_k$ is more general than problems considered in Sections 1–3 of this paper.

4.2. *Conjectures for* $L = sl_2(\mathbb{C})$

Throughout this section we will assume that $L = sl_2(\mathbb{C})$. Let K denote the killing form on L, i.e.,

$$K(\alpha, \beta) = \mathrm{tr}(\mathrm{ad}(\alpha)\,\mathrm{ad}(\beta)).$$

Define $\langle\ ,\ \rangle$ on L_k by

$$\langle \alpha \otimes t^i, \beta \otimes t^j \rangle = \begin{cases} K(\alpha, \beta) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let Λ be the Laplacian of L_k with respect to the above form.

Conjecture 4.2.1. The eigenvalues of Λ are non-negative integers.

We wish to describe the eigenvalues of Λ more explicitly. Note that the form $\langle\ ,\ \rangle$ on L_k is L-invariant so the same is true of its extension to ΛL_k. Hence Λ commutes with the adjoint action of L on ΛL_k. We will state a conjecture which describes the eigenvalues of Λ in terms of the L-module structure of ΛL_k.

Recall that ΛL has the L-module decomposition

$$\Lambda L = (\Lambda^0 L) \oplus (\Lambda^1 L) \oplus (\Lambda^2 L) \oplus (\Lambda^3 L),$$

where $(\Lambda^0 L)$ and $(\Lambda^3 L)$ are isomorphic to the trivial module $V_{(0,0)}$ and (Λ^1) and $(\Lambda^2 L)$ are isomorphic to the adjoint representation $V_{(1,-1)}$. As an L-module we have

$$\Lambda L_k \cong \underbrace{(\Lambda L) \otimes (\Lambda L) \otimes \cdots \otimes (\Lambda L)}_{(k+1) \text{ times.}}$$

Hence

$$\Lambda^{r,w} L_k \cong \bigoplus_{(i_0, i_1, \ldots, i_k)} (\Lambda^{i_0} L) \otimes (\Lambda^{i_1} L) \otimes \cdots \otimes (\Lambda^{i_k} L), \tag{4.2.2}$$

where the sum on the right is over all sequences $(i_0, i_1, ..., i_k)$ such that $\sum i_j = r$ and $\sum j i_j = w$. Of course the summands on the right are not irreducible as L-modules. We introduce some further notation for the irreducible which appear in their decomposition.

DEFINITION 4.2.3. For each sequence $(i_0, i_1, ..., i_k) = \mathbf{i}$ and each non-negative integer l let $E_{\mathbf{i},l}$ denote the $V_{(l,-l)}$ isotypic component of the summand

$$(\Lambda^{i_0} L) \otimes (\Lambda^{i_1} L) \otimes \cdots \otimes (\Lambda^{i_k} L).$$

For each irreducible L-module $V_{(l,-l)}$ let $C(V_{(l,-l)})$ denote the value of the Casimir operator on the module $V_{(l,-l)}$. Explicitly we have

$$C(V_{(l,-l)}) = l^2 + l.$$

Conjecture 4.2.4. The Koszul complex ΛL_k can be written as a direct sum of subspaces $U_{\mathbf{i},l}$ indexed by sequences $(i_0, i_1, ..., i_k)$ and non-negative integers l. These subspaces satisfy:

(a) $U_{\mathbf{i},l} \subseteq \Lambda^{r,w} L_k$ where $r = \sum_{j=0}^{k} i_j$ and $w = \sum_{j=0}^{k} j i_j$.

(b) Each subspace $U_{\mathbf{i},l}$ is L-invariant. As an L-module $U_{\mathbf{i},l}$ is isomorphic to $E_{\mathbf{i},l}$ (so in particular $U_{\mathbf{i},l}$ is a direct sum of copies of the irreducible $V_{(l,-l)}$).

(c) Each subspace $U_{\mathbf{i},l}$ is an eigenspace for the Laplacian Δ. The corresponding eigenvalue is

$$\left\{ \sum_{j=0}^{k} j C(\Lambda^{i_j} L_k) \right\} + C(V_{l,-l}).$$

Conjecture 4.2.4 is based on overwhelming computational evidence. The author does not know if there are corresponding conjectures for other simple Lie algebras L.

4.3. *Conjectures for the Three-Dimensional Heisenberg*

Throughout this subsection we will assume that L is $\mathscr{H}_3$, the three-dimensional Heisenberg Lie algebra. Recall that $\mathscr{H}_3$ has basis $\{e, f, x\}$ with brackets

$$[e, f] = -[f, e] = x$$
$$[e, x] = -[x, e] = [f, x] = -[f, x] = 0.$$

It is easy to check that $H_*(L)$ has dimension 6 with homology representatives 1, e, f, $e\Lambda x$, $f\Lambda x$, $e\Lambda f\Lambda x$. Let $\langle\ ,\ \rangle$ be the form whose matrix with respect to the basis $\{e, f, x\}$ is the identity form. The Laplacian Δ on ΛL contains 1, e, f, $e\Lambda x$, $f\Lambda x$, $e\Lambda f\Lambda x$ in its nullspace. It is easy to check that x and $e\Lambda f$ are both eigenvectors of Δ associated to the eigenvalue 1.

DEFINITION 4.3.1. Let M be a complex Lie algebra equipped with a positive definite Hermitian form. Let Δ_M denote the Laplacian of M with respect to that form. Assume all the eigenvalues of Δ_M are non-negative integers. Define the E-polynomial of M, $E_M(y, z)$ by

$$E_M(y, z) = \sum_{n=0}^{\infty} \sum_{r=0}^{\dim(M)} m(n, r)\, y^r z^n,$$

where $m(n, r)$ is the multiplicity of n as an eigenvalue of Δ_M restricted to $\Lambda^r M$.

According to this notion, for $M = L = \mathscr{H}_3$ we have

$$E_L(y, z) = (1 + 2y + 2y^2 + y^3) + z(y + y^2).$$

Now we turn to L_k. The following conjecture is again based on overwhelming computational evidence.

Conjecture 4.3.2. Let $L = \mathscr{H}_3$. Let $\langle\ ,\ \rangle$ be the form on L_k whose matrix with respect to the basis $\{e \otimes t^i, f \otimes t^i, x \otimes t^i\}$ is the identity. Let Δ be the Laplacian of L_k with respect to that form. Then

(a) The eigenvalues of Δ are non-negative integers.

(b) $E_{L_k}(y, z) = \prod_{i=1}^{k+1} E_L(y, z^i)$.

Conjecture 4.3.2 can be sharpened and certain partial results can be proven. This work will appear in a separate paper [H3] which deals specifically with the Laplacian of $(\mathscr{H}_3)_k$.

Note that L is a nilpotent upper summand of $sl_3(\mathbb{C})$. Let N be an arbitrary nilpotent upper summand in a semisimple Lie algebra. Kostant [Ko1] has shown that there exists a positive definite Hermitian form $\{\ ,\ \}$ on N so that the Laplacian Δ of N with respect to $\{\ ,\ \}$ has non-negative integer eigenvalues (in what follows we call $\{\ ,\ \}$ the *Kostant form* on N). In view of Conjectures 2.1.4 and 4.3.2 the following statement is plausible. We label it a conjecture for lack of a better name but we warn the reader that there is no computational evidence in support of this conjecture except in the case $L = \mathscr{H}_3$.

Conjecture 4.3.3. Let N be a nilpotent upper summand of a semisimple Lie algebra. Let $\{\ ,\ \}$ be the Kostant form on N and let $\langle\ ,\ \rangle$ be the form on N_k defined by $\langle u\otimes t^i, v\otimes t^j\rangle = \{u, v\}\,\delta_{ij}$. Let Δ be the Laplacian of N_k with respect to $\langle\ ,\ \rangle$. Then

(a) The eigenvalues of Δ are non-negative integers.

(b) $E_{N_k}(y, z) = \prod_{i=1}^{k+1} E_N(y, z^i)$.

5. Computational Evidence

The conjectures in this paper are based in large part on computational evidence. This section contains a brief discussion of our computational methods as well as tables with results of some of our computations. The algorithms were designed and implemented by the author using SUN workstations and a CRAY-2. Substantial support for the computer work came from the National Science Foundation, SUN Microsystems, and the CRAY Research Foundation.

5.1. *Algorithms*

The homology computations were done using Koszul complexes. Of course the exact computation differes depending on what Lie algebra is considered. However, the difficulties one encounters are roughly the same so we will limit our discussion to one specific computation—the computation of $H(\bar{L}_k)$ as bigraded L-module.

In theory this computation is straightforward. For each $r, w \in \mathbb{N}$ and each dominant $\mathscr{H}$-weight λ let $\Lambda^{r,w,\lambda}\bar{L}_k$ denote the subspace of $\Lambda\bar{L}_k$ spanned by all vectors of degree r, weight w, and $\mathscr{H}$-weight λ. A straightforward combinatorial algorithm will compute a basis for each of these spaces $\Lambda^{r,w,\lambda}\bar{L}_k$. It is equally straightforward to compute the matrix $D_{r,w,\lambda}$ of the boundary map $\partial: \Lambda^{r,w,\lambda}\bar{L}_k \to \Lambda^{r-1,w,\lambda}\bar{L}_k$ with respect to these bases. The ranks of these matrices $D_{r,w,\lambda}$ then give the homology $H_{*,*,\lambda}(\bar{L}_k)$ for each dominant $\mathscr{H}$-weight λ. This finishes the computation of $H(\bar{L}_k)$ as a bigraded $\mathscr{H}$-module and a simple inclusion–exclusion yields the decomposition of $H(\bar{L}_k)$ as a bigraded L-module.

The practical constraints to this computation are the large sizes of the matrices $D_{r,w,\lambda}$. For example, when $k=3$ and $L = sl_3(\mathbb{C})$ one runs into matrices $D_{r,w,\lambda}$ with approximately 33,000 rows and columns. Some steps must be taken to reduce the size of the matrices $D_{rw\lambda}$. We have tried two different methods for achieving this reduction which we will briefly describe.

Method 1. Let $\mathscr{B}_{r,w,\lambda}$ denote the basis for $\Lambda^{rw\lambda}\bar{L}_k$ computed earlier. The idea in this method is to compute from $\mathscr{B}_{rw\lambda}$ a basis $\mathscr{M}_{rw\lambda}$ for the maximal weight vectors of weight λ. Each vector in $\mathscr{M}_{rw\lambda}$ is a linear combination of vectors in $\mathscr{B}_{rw\lambda}$. So $\mathscr{M}_{rw\lambda}$ can be specified by a matrix $M_{rw\lambda}$ with $m_{rw\lambda}$ rows and $b_{rw\lambda}$ columns where

$$m_{rw\lambda} = |\mathscr{M}_{rw\lambda}|$$

and

$$b_{rw\lambda} = |\mathscr{B}_{rw\lambda}|.$$

The differential ∂ maps $\mathscr{M}_{rw\lambda}$ to $\mathscr{M}_{(r-1)w\lambda}$ so once the matrices $M_{rw\lambda}$ are known, this restriction of ∂ can be computed. This computation is complicated to describe and implement but it involves significantly smaller matrices (the $M_{rw\lambda}$) and is much more efficient in general. To actually compute $\mathscr{M}_{rw\lambda}$ we need to find the intersection of the nullspaces of the maps

$$\mathrm{ad}(z_\alpha)\colon \mathscr{B}_{rw\lambda} \to \mathscr{B}_{rw(\lambda+\alpha)}$$

for α in a basis of the root system R of L. On the face of it, this idea seems impractical—it replaces the task of computing the rank of one matrix $D_{r,w,\lambda}$ with the task of computing the intersection of the nullspaces of several matrices (the $\mathrm{ad}(z_\alpha)$) which have approximately the same size. However, you don't need to work with the entire space $\mathscr{B}_{rw\lambda}$ when you compute the kernels of the $\mathrm{ad}(z_\alpha)$. This lends to a significant reduction in the sizes of the matrices involved. To explain this in more detail, let $\mathrm{Par}(r, k, w)$ denote the set of partitions of w which fit inside an r by k rectangle. For each $\mu = \mu_1 \mu_2 \cdots \mu_r$ in $\mathrm{Par}(r, k, w)$ let $\mathscr{B}_{\mu,\lambda}$ denote the span of all weight vectors in $\mathscr{B}_{r,w,\lambda}$ of the form

$$(z_1 \otimes t^{\mu_1}) \wedge (z_2 \otimes t^{\mu_2}) \wedge \cdots \wedge (z_r \otimes t^{\mu_r}),$$

where $z_1, z_2, \ldots, z_r \in L$. The key observation is that $\mathrm{ad}(z_\alpha)$ maps $\mathscr{B}_{\mu,\lambda}$ to $\mathscr{B}_{\mu,\lambda+\alpha}$. So we can split $\mathscr{B}_{r,w,\lambda}$ into the direct sum of the $\mathscr{B}_{\mu,\lambda}$ and we can compute a basis for each space $\mathscr{M}_{\mu\lambda}$ separately. The spaces $\mathscr{B}_{\mu\lambda}$ are significantly smaller than the original space $\mathscr{B}_{rw\lambda}$ when the size of $P(r, k, w)$ is large. Fortunately this occurs exactly when this reduction is needed the most, i.e., when the dimension of $\mathscr{B}_{rw\lambda}$ is largest.

Method 2. Return now to the original matrices $D_{rw\lambda}$ and the problem of computing their rank. If one applies the boundary map ∂ to a basis vector in $\mathscr{B}_{rw\lambda}$ of the form $(z_1 \otimes t^{a_1}) \wedge \cdots \wedge (z_r \otimes t^{a_r})$ then the result is a linear combination of no more than $\binom{r}{2}$ terms. So every column of $D_{rw\lambda}$ has a maximum of $\binom{r}{2}$ nonzero entries. As the size of $D_{rw\lambda}$ grows, it becomes sparse. For example, in the case mentioned earlier where $D_{rw\lambda}$ had 33,000 rows and columns we have $r=12$. So $\binom{r}{2}=66$ is an upper bound to the number of nonzero entries in each column.

It is straightforward to reorder your bases for $\mathscr{B}_{rw\lambda}$ and $\mathscr{B}_{(r-1)w\lambda}$ so that the matrices $D_{rw\lambda}$ are banded. After the reordering, the form of the matrix $D_{rw\lambda}$ will be

$$D_{rw\lambda}=\begin{bmatrix} D_{11} & & & & & & & \\ D_{21} & D_{22} & & & 0 & & & \\ & D_{32} & D_{33} & & & & & \\ & & D_{43} & D_{44} & & & & \\ 0 & & & D_{54} & D_{55} & & & \\ & & & & & \ddots & & \\ & & & & & & D_{l(l-1)} & D_{ll} \end{bmatrix}, \quad (*)$$

where the blocks D_{ii} are small with respect to $D_{rw\lambda}$. The actual reordering of the bases needed to achieve this form depends on the combinatorial structure of the root system of L.

Once the matrix $D_{rw\lambda}$ is put in the banded block-bidiagonal form $(*)$ then the computation of its rank can be done by the usual methods.

General speaking Method 1 works best when the dimension of $\mathscr{M}_{rw\lambda}$ is small with respect to the dimension of $\mathscr{B}_{rw\lambda}$. This occurs when λ is close to 0 in the dominance order. Our experience has been that Method 2 should be used to compute the λ-isotopic component of $H(\bar{L}_k)$ when λ is large in the dominance order. As you move down in the dominance order Method 1 becomes more practical at some point.

One obvious idea is to mix the two methods, i.e., try to band the matrix for ∂ mapping $\mathscr{M}_{rw\lambda}$ to $\mathscr{M}_{(r-1)w\lambda}$. This appears to be difficult because the vectors in a basis for $\mathscr{M}_{rw\lambda}$ are complicated linear combinations of the basis elements $(z_1 \otimes t^{a_1}) \wedge \cdots \wedge (z_r \otimes t^{a_r})$ and so the argument which gave us sparsity no longer applies. However, it is possible to use banded matrix techniques to speed up the computation of the $\mathrm{ad}(z_\alpha)$.

5.2. *Computational Results*

In this subsection we include a sampling of some of our data.

Part I

In this part we list the polynomials $P^\lambda(\bar{L}_k; z, q)$ for $L=sl_2(\mathbb{C})$, $1 \leqslant k \leqslant 3$, and $L=sl_3(\mathbb{C})$, $1 \leqslant k \leqslant 3$.

(A) $L = sl_2(\mathbb{C})$:

k	λ	$P^\lambda(sl_2(\mathbb{C})_k, z, q)$
1	(0, 0)	$1+z^3q^3$
	(1, −1)	$zq(1+zq)$
2	(0, 0)	$(1+z^3q^5)(1+z^3q^4)$
	(1, −1)	$zq(1+z^3q^5)(1+zq^2)$
	(2, −2)	$z^2q^3(1+zq)(1+zq^2)$
3	(0, 0)	$(1+z^3q^5)(1+z^3q^6)(1+z^3q^5)$
	(1, −1)	$zq(1+z^3q^7)(1+z^3q^6)(1+zq^3)$
	(2, −2)	$z^2q^3(1+z^3q^7)(1+zq^2)(1+zq^3)$
	(3, −3)	$z^3q^6(1+zq)(1+zq^2)(1+zq^3)$

(B) $L = sl_3(\mathbb{C})$:

k	λ	$P^\lambda(sl_3(\mathbb{C})_k; z, q)$
1	(0, 0, 0)	$(1+z^3q^3)(1+z^5q^5)$
	(1, 0, −1)	$zq(1+zq)(1+z^3q^3)(1+z^2q^2)$
	(1, 1, −2) (1, −1, −1)	$z^2q^2(1+zq)(1+z^3q^3)$
	(2, 0, −2)	$z^3q^3(1+zq)^2$
2	(0, 0, 0)	$(1+z^3q^4)(1+z^3q^5)(1+z^5q^7)(1+z^5q^8)$
	(1, 0, −1)	$zq(1+z^3q^4)(1+z^3q^5)(1+zq^2)(1+z^5q^8)(1+z^2q^3)$
	(1, 1, −2) (1, −1, −1)	$z^2q^2(1+z^3q^5)(1+z^3q^5)(1+zq^2)(1+z^5q^8)$
	(2, 0, −2)	$z^3q^4(1+zq)(1+z^3q^5)(1+zq^2)(1+z^5q^8)$ $+z^4q^6(1+z^2q^3)(1+z^3q^5)(1+zq^2)(1+zq)^2$
	(2, 1, −3) (3, −1, −2)	$z^3q^4(1+z^2q^3)(1+zq)(1+zq^2)(1+z^3q^5)^2$
	(3, 0, −3)	$z^4q^5(1+z^2q^3)(1+z^3q^5)(1+zq^2)^3$
	(3, 1, −4) (4, −1, −3)	$z^5q^7(1+zq)(1+z^3q^5)(1+zq^2)^2$
	(2, 2, −4) (4, −2, −2)	$z^4q^6(1+zq)(1+zq)^2(1+z^3q^4)(1+z^3q^5)$
	(4, 0, −4)	$z^6q^9(1+zq)^2(1+zq^2)^2$

For $k=3$ we include only a selection of those λ for which $P^\lambda(sl_3(\mathbb{C}); z, q)$ is nonzero.

k	λ	$P^\lambda(sl_3(\mathbb{C})_k; z, q)$
3	(0, 0, 0)	$(1+z^3q^5)(1+z^3q^6)(1+z^3q^7)(1+z^5q^9)(1+z^5q^{10})(1+z^5q^{11})$
	(1, 0, −1)	$zq(1+z^3q^5)(1+z^3q^6)(1+z^3q^7)(1+zq^3)$ $\cdot(1+z^5q^{10})(1+z^5q^{11})(1+z^2q^4)$

k	λ	$P^{\lambda}(sl_3(\mathbb{C})_k; z, q)$
and	$(2, -1, -1)$ $(1, 1, -2)$	$z^2q^2(1+z^3q^6)(1+z^3q^7)^2(1+zq^3)(1+z^5q^{10})(1+z^5q^{11})$
and	$(3, -1, -2)$ $(2, 1, -3)$	$z^3q^4(1+z^3q^6)(1+z^3q^7)^2(1+zq^3)(1+zq^2)(1+z^5q^{11})$
and	$(3, -1, -2)$ $(2, 1, -3)$	$z^3q^4(1+z^3q^6)(1+z^3q^7)^2(1+zq^3)(1+zq^2)(1+z^5q^{11})$
and	$(4, -2, -2)$ $(2, 2, -4)$	$z^4q^6(1+z^3q^6)(1+z^3q^7)^2(1+zq^3)(1+zq^2)(1+z^5q^{11})$
and	$(5, -2, -3)$ $(3, 2, -5)$	$z^5q^9(1+z^3q^6)(1+z^3q^7)^2(1+zq^3)(1+zq^2)(1+z^5q^{11})$
and	$(6, -2, -4)$ $(4, 2, -6)$	$z^7q^{13}(1+zq)(1+zq^2)(1+zq^3)^2(1+z^3q^6)(1+z^3q^7)$
	$(5, 0, -5)$	$z^7q^{12}(1+zq^3)^3(1+zq^2)^2(1+z^3q^7)(1+z^2q^4)$
and	$(5, 1, -6)$ $(6, -1, -5)$	$z^8q^{15}(1+zq)(1+zq^2)^2(1+zq^3)^2(1+zq^7)$
	$(6, 0, -6)$	$z^9q^{18}(1+zq)^2(1+zq^2)^2(1+zq^3)^2$

Part II. *Eigenvalues of Laplacians*

(A) $L = sl_2(\mathbb{C})$. Below we see eigenvalues of the Laplacian Δ of $sl_2(\mathbb{C}) \otimes (\mathbb{C}[t]/t^{k+1})$. The Laplacian is constructed with respect to the positive definite Hermitian form described in Section 4.2. The list of eigenvalues is split up according to degree, weight, and $sl_2(\mathbb{C})$-irreducible. In other words, the eigenvalues that appear on the following lists under degree d, weight w, and dominant $\mathscr{H}$-weight λ are the eigenvalues of Δ restricted to the V^{λ}-isotypic component of $\Lambda^{d,w}L_k$.

$k = 1$:

		Dominant $\mathscr{H}$-weight λ	
Weight w	Degree d	$(0, 0)$	$(1, -1)$
0	0	0	
0	1		2
0	2		2
0	3	0	

$k = 2$:

Weight w	Degree d	Dominant $\mathscr{H}$-weight λ (0, 0)	(1, −1)	(2, −2)
0	0	0		
	1		2	
	2		2	
	3	0		
1	1		4	
	2	2	4	8
	3	2	4	8
	4		4	
2	2		4	
	3	2	4	8
	4	2	4	8
	5		4	
3	3	0		
	4		2	
	5		2	
	6	0		

$k = 3$:

Weight w	Degree d	Dominant $\mathscr{H}$-weight λ (0, 0)	(1, −1)	(2, −2)	(3, −3)
0	0	0			
	1		2		
	2		2		
	3	0			
1	1		4		
	2	2	4	8	
	3	2	4	8	
	4		4		
2	1		6		
	2	4	6, 6	10	
	3	4, 4	6, 6	10, 10	
	4	4	6, 6	10	
	5		6		
3	2	6	4	12	
	3	2, 6	4, 8, 8	8, 12	18
	4	2	4, 8, 8, 8	8, 12	18
	5	6	4, 8	12	
	6	6			

k = 3 (*continued*)

Weight w	Degree d	Dominant $\mathscr{H}$-weight λ $(0, 0)$	$(1, -1)$	$(2, -2)$	$(3, -3)$
4	2		8		
	3	0, 6	8, 8	12, 12	
	4	6, 6	2, 8, 8, 8	12, 12, 12	18
	5	6	2, 8, 8, 8	12, 12	18
	6	0	8	12	
5	3	0	8	12	
	4	6	2, 8, 8, 8	12, 12	18
	5	6, 6	2, 8, 8, 8	12, 12, 12	18
	6	0, 6	8, 8	12, 12	
	7		8		
6	3	6			
	4	6	4, 8	12	
	5	2	4, 8, 8, 8	8, 12	18
	6	2, 6	4, 8, 8	8, 12	18
	7	6	4	12	
7	4		6		
	5	4	6, 6	10	
	6	4, 4	6, 6	10, 10	
	7	4	6, 6	10	
	8	6			
8	5		4		
	6	2	4	8	
	7	2	4	8	
	8		4		
9	6	0			
	7		2		
	8		2		
	9	0			

(B) $L = \mathscr{H}_3$, the 3-dimensional Heisenberg. Recall from Section 4.3 the standard basis $\mathscr{B}$ for L_k, $\mathscr{B} = \{e \otimes t^i, f \otimes t^i, x \otimes t^i : 0 \leqslant i \leqslant k\}$. Let E_k, F_k, and X_k denote the span of the basis elements $\{e \otimes t^i\}$, $\{f \otimes t^i\}$, and $\{x \otimes t^i\}$, respectively. Then

$$\Lambda^d L_k = \bigoplus_{a+b+c=d} \Lambda^a(E_k) \otimes \Lambda^b(F_k) \otimes \Lambda^c(X_k).$$

It is easy to check that the Laplacian Δ preserves the subspace $\Lambda^a(E_k) \otimes \Lambda^b(F_k) \otimes \Lambda^c(X_k)$. For each a, b, c, w, and n let $m_n(a, b, c; w)$ denote the multiplicity of n as an eigenvalue of the restriction of Δ to the component of $\Lambda^a(E_k) \otimes \Lambda^b(F_k) \otimes \Lambda^c(X_k)$ having weight w.

For example, in the case $k=0$ we have the following basis of eigenvectors of Λ:

Eigenvector	Eigenvalue	a	b	c	w
1	0	0	0	0	0
e_0	0	1	0	0	0
f_0	0	0	1	0	0
x_0	1	0	0	1	0
$e_0 \wedge f_0$	1	1	1	0	0
$e_0 \wedge x_0$	0	1	0	1	0
$f_0 \wedge x_0$	0	0	1	1	0
$e_0 \wedge f_0 \wedge x_0$	0	1	1	1	0

So we have $m_0(a, b, c; w)=1$ for 6 values of $(a, b, c; w)$ and $m_1(a, b, c; w)=1$ for 2 values of $(a, b, c; w)$.

It can be shown that

$$m_n(a, b, c; w)=m_n\left(k+1-a, k+1-b, k+1-c, 3\binom{k+1}{2}-w\right) \qquad (5.2.1)$$

for all (a, b, c, w). So in fact the $m_n(a, b, c; w)$ are determined for all values of $(a, b, c; w)$ once they are known for all $(a, b, c; w)$ with $a+b+c \leqslant 3(k+1)/2$. Using this fact it is easy to compile a table of multiplicities $m_n(a, b, c; w)$ for $k=1$. Below we see a table for $k=2$. In view of (5.2.1) we include only those parameter sets $(a, b, c; w)$ with $a+b+c \leqslant 4$. This table has rows indexed by the parameter sets $(a, b, c; w)$. In the $(a, b, c; w)$ row each positive integer n is repeated $m_n(a, b, c; w)$ times.

d	w	a	b	c	Eigenvalues
0	0	0	0	0	0
1	0	1	0	0	0
		0	1	0	0
		0	0	1	1
	1	1	0	0	0
		0	1	0	0
		0	0	1	2
	2	1	0	0	0
		0	1	0	0
		0	0	1	3
2	0	1	1	0	1
		1	0	1	0
		0	1	1	0

d	w	a	b	c	Eigenvalues
	1	2	0	0	0
		0	2	0	0
		0	0	2	3
		1	1	0	0, 2
		1	0	1	0, 2
		0	1	1	0, 2
	2	2	0	0	0
		0	2	0	0
		0	0	2	4
		1	1	0	0, 0, 3
		1	0	1	0, 1, 3
		0	1	1	0, 1, 3
	3	2	0	0	0
		0	2	0	0
		0	0	2	5
		1	1	0	0, 0
		1	0	1	1, 3
		0	1	1	1, 3
2	4	1	1	0	0
		1	0	1	2
		0	1	1	2
3	0	1	1	1	0
	1	2	1	0	2
		1	2	0	2
		2	0	1	1
		1	0	2	1
		0	2	1	0
		0	1	2	1
		1	1	1	0, 0, 3
	2	2	1	0	1, 3
		1	2	0	1, 3
		2	0	1	0, 0
		1	0	2	1, 3
		0	2	1	0, 0
		0	1	2	1, 3
		1	1	1	0, 0, 0, 0, 4, 4
	3	3	0	0	0
		0	3	0	0
		0	0	3	6
		2	1	0	0, 1, 3
		1	2	0	0, 1, 3
		2	0	1	0, 0, 3
		1	0	2	5, 2, 2
		0	2	1	0, 0, 3
		0	1	2	5, 2, 2
		1	1	1	0, 0, 0, 0, 3, 3, 5

d	*w*	*a*	*b*	*c*	Eigenvalues
3	4	2	1	0	0, 2
		1	2	0	0, 2
		2	0	1	0, 2
		1	0	2	3, 3
		0	2	1	0, 2
		0	1	2	3, 3
		1	1	1	0, 0, 0, 2, 2, 5
	5	2	1	0	0
		1	2	0	0
		2	0	1	1
		1	0	2	4
		0	2	1	1
		0	1	2	4
		1	1	1	0, 1, 3
	6	1	1	1	1
4	1	2	1	1	1
		1	2	1	1
		1	1	2	0
	2	2	2	0	4
		2	0	2	0
		0	0	2	0
		2	1	1	0, 1, 3
		1	2	1	0, 1, 3
		1	1	2	0, 0, 3
4	3	3	1	0	3
		1	3	0	3
		3	0	1	0
		1	0	3	3
		0	3	1	0
		0	1	3	3
		2	2	0	3, 3
		2	0	2	0, 2
		0	2	2	0, 2
		2	1	1	0, 0, 0, 2, 2, 5
		1	2	1	0, 0, 0, 2, 2, 5
		1	1	2	0, 0, 2, 2, 2, 6
	4	3	1	0	2
		1	3	0	2
		3	0	1	0
		1	0	3	4
		0	3	1	0
		0	1	3	4
		2	2	0	2, 2, 5
		2	0	2	0, 1, 3
		0	2	2	0, 1, 3
		2	1	1	0, 0, 0, 0, 3, 3, 5
		1	2	1	0, 0, 0, 0, 3, 3, 5
		1	1	2	0, 1, 1, 1, 3, 3, 6

d	w	a	b	c	Eigenvalues
	5	3	1	0	1
		1	3	0	1
		3	0	1	0
		1	0	3	5
		0	3	1	0
		0	1	3	5
		2	2	0	1, 3
		2	0	2	1, 3
		0	2	2	1, 3
		2	1	1	0, 0, 0, 0, 4, 4
		1	2	1	0, 0, 0, 0, 4, 4
		1	1	2	1, 1, 1, 3, 3, 6
4	6	2	2	0	1
		2	0	2	2
		0	0	2	2
		2	1	1	0, 0, 3
		1	2	1	0, 0, 3
		1	1	2	2, 2, 2
4	7	2	1	1	0
		1	2	1	0
		1	1	2	3

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