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A NEW DERIVATION OF THE GENERATING FUNCTION FOR THE MAJOR INDEX

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We present a new proof of the well-known combinatorial result $\left[{n \atop k}\right]_q = \sum_w q^{\mathrm{Maj}(w)}$ where w is a permutation of 0^k1^{n-k}, by showing a bijection between the set of partitions of an integer m that fit in a $k \times n-k$ rectangle and the set consisting of all permutations w of 0^k1^{n-k} having $\mathrm{Maj}(w) = m$.

1. Introduction

Consider the sequence $w = w_1w_2 \cdots w_n$ where $w_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$. The *Descent Set* of w, $D(w)$ is $\{i \mid 1 \leqslant i < n,\ w_i > w_{i+1}\}$, and the *major index*, $\mathrm{Maj}(w)$ is the sum of all the elements (possibly zero) of $D(w)$. MacMahon [4, 5] showed that the major indices of the set of all permutations of w has the same generating function as the *inversion numbers*, $\mathrm{Inv}(w)$ of these permutations. A combinatorial proof of this correspondence between $\mathrm{Maj}(w)$ and $\mathrm{Inv}(w)$ was obtained by Foata [1]. Rawlings [7] uses a statistic called r-major index to describe a bijection that takes the major index to the inversion number.

This paper considers the case where the w's are permutations of 0^k1^{n-k}. A new combinatorial proof for the generating function of the major indices of these sequences is derived by showing a bijection between the set of partitions of a positive integer m that fit inside a $k \times n-k$ rectangle and the set of permutations w of 0^k1^{n-k} which have $\mathrm{Maj}(w) = m$. The bijections in [1, 7] can be shown, with some effort, to reduce to the bijection described in this paper in this case.

2. Notation and definitions

A *partition* of a positive integer m into k parts is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = m$. The *size* of the partition, denoted by $|\lambda|$, is m, and the *length*, $L(\lambda)$, is k.

The *diagram* of the partition is the set of lattice points $\{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant L(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}$ and is denoted by D_λ. The diagram D_λ and the partition λ may be used interchangeably in this paper. The *rank* of a partition λ is the length of the largest square subdiagram of D_λ. Suppose the rank of λ is r. Then λ can be

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expressed as $(\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)$ [Frobenius notation, see Macdonald [3] page 3], where α_i is the number of nodes in the ith row of D_λ to the right of (i, i), and β_i is the number of nodes in the ith column of D_λ below (i, i). A list of standard notation used to describe partitions and q-combinations is presented below.

- $p(m)$ = Number of partitions of m.
- $p_k(m)$ = Number of partitions, λ, of m, where $L(\lambda) \leqslant k$.
- $p_{n,k}(m)$ = Number of partitions of m that fit inside a $n \times k$ rectangle.
- $[k] = 1 + q + \cdots + q^{k-1}$, $k \geqslant 1$; $[0] = 1$.
- $[n]! = [1][2] \cdots [n]$.
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!}{[k]!\,[n-k]!}$.

3. Theorem and proof

Theorem. *There is a bijection between the set of partitions of m (>0) that fit inside a $k \times n - k$ rectangle and the set consisting of permutations w of $0^k 1^{n-k}$ having* $\mathrm{Maj}(w) = m$.

This theorem and the well-known result $[{n \atop k}]_q = \sum_{m \geqslant 0} p_{k,n-k}(m) q^m$ (Stanley [8], page 29) directly imply that the generating function for the major index of permutations w of $0^k 1^{n-k}$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_w q^{\mathrm{Maj}(w)}.$$

Proof. Let $w = w_1 w_2 \cdots w_n$ be a permutation of $0^k 1^{n-k}$ such that $\mathrm{Maj}(w) = m$. Suppose $D(w) = \{i_1, \ldots, i_j, \ldots, i_r\}$ where $i_1 < i_2 < \cdots < i_r$. For each j, $1 \leqslant j \leqslant r$, let $\mathrm{Zero}(j)$ denote the number of zeros, and $\mathrm{One}(j)$ denote the number of ones, respectively, found in the substring $w_1 w_2 \cdots w_{i_j - 1}$. Let partition λ_w be defined as $(\mathrm{One}(r), \mathrm{One}(r-1), \ldots, \mathrm{One}(1) \mid \mathrm{Zero}(r), \mathrm{Zero}(r-1), \ldots, \mathrm{Zero}(1))$. The height of λ_w is $(\mathrm{Zero}(r) + 1)$, which is at most k. Likewise, the width of λ_w is $(\mathrm{One}(r) + 1)$, which is at most $n - k$. Also, $|\lambda_w| = \sum_{j=1}^r (\mathrm{One}(j) + \mathrm{Zero}(j) + 1) = \sum_{j=1}^r i_j = m$. Thus, $w \to \lambda_w$ defines a mapping from the set of permutations of $0^k 1^{n-k}$ having $\mathrm{Maj}(w) = m$ to the set of partitions $p_{k,n-k}(m)$.

Example. Let $n = 10$, $k = 6$, and, $w = 0010010110$. Then $D(w) = \{3, 6, 9\}$, $\mathrm{Zero}(1) = 2$, $\mathrm{One}(1) = 0$, $\mathrm{Zero}(2) = 4$, $\mathrm{One}(2) = 1$, $\mathrm{Zero}(3) = 5$, $\mathrm{One}(3) = 3$. *Fig. 1* shows the partition diagram for the example.

The inverse function can be obtained by using the following method. Let λ be a partition of m that fits inside a $k \times n - k$ rectangle. Suppose $\lambda =$

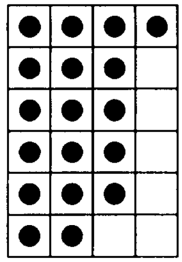

Fig. 1. D_λ for $w = 0010010110$.

$(\alpha_r, \alpha_{r-1}, \ldots, \alpha_1 \mid \beta_r, \beta_{r-1}, \ldots, \beta_1)$. The string $w_\lambda = w_1 w_2 \cdots w_n$ can be constructed as follows. Place β_1 zeros and α_1 ones before the first **10** descent pair. All these zeros must precede all the ones since they cannot give rise to any descents. Thus, the partial string w_λ up to and including the first descent pair is

$$\underbrace{00\cdots 0}_{\beta_1}\overbrace{11\cdots 1}^{\alpha_1}\mathbf{10}.$$

Place $\beta_2 - \beta_1 - 1$ zeros and $\alpha_2 - \alpha_1 - 1$ ones between the first and second descent pairs, such that the zeros precede the ones. The partial string w_λ up to the second descent pair is therefore

$$\underbrace{0\cdots 0}_{\beta_1}\overbrace{1\cdots 1}^{\alpha_1}\mathbf{10}\underbrace{0\cdots 0}_{\beta_2-\beta_1-1}\overbrace{1\cdots 1}^{\alpha_2-\alpha_1-1}\mathbf{10}.$$

Continuing similarly, the string w_λ can be compiled up to the last descent pair. Placing $k - \beta_r - 1$ zeros and $(n-k) - \alpha_r - 1$ ones in sequence after the last descent pair completes string w_λ. Note that, $\mathrm{Maj}(w_\lambda) = \sum_{i_j \in D(w)} i_j = \sum_{j=1}^{r} (\alpha_j + \beta_j + 1) = |\lambda| = m$. Thus, the bijection has been established. □

Consider permutations of $0^k 1^{n-k}$ which have exactly r descents. The following Corollary establishes the generating function for the major indices of these permutations. MacMahon [6] (pp. 169–170) describes a different bijection for the case $q = 1$. Goulden [2] provides a bijective proof of Stanley's shuffling theorem, which in a special case, provides an alternate proof, similar to MacMahon's, of this corollary.

Corollary.

$$q^{r^2}\begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q = \sum_w q^{\mathrm{Maj}(w)},$$

summed over permutations w of $0^k 1^{n-k}$ having r descents.

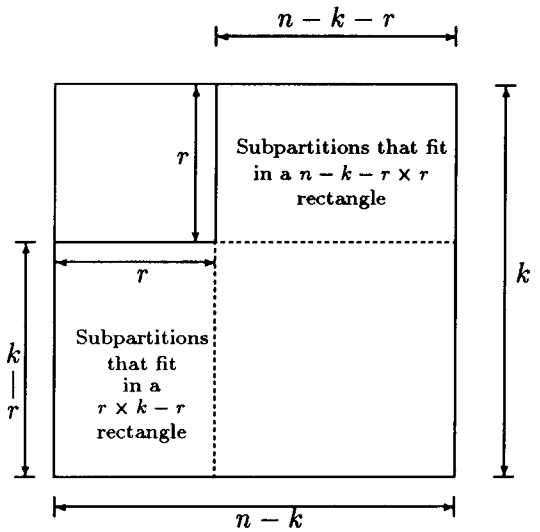

Fig. 2. Partitions that fit inside a $k \times n - k$ rectangle with rank r.

Proof. Use the bijection technique of the theorem. Each permutation, w, of $0^k 1^{n-k}$ having r descents is mapped to a partition with rank r. The generating function for partitions of rank r that fit inside a $k \times n - k$ rectangle is

$$q^{r^2} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q,$$

as is evident from *Fig. 2.* □

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