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Orthogonal Tableaux and an Insertion Algorithm for $SO(2n+1)$

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A new set of tableaux indexing the weights of the irreducible representations of $SO(2n+1)$ is presented. These tableaux are used to produce an insertion scheme which gives a combinatorial description of the decomposition of the kth tensor power of the natural action of $SO(2n+1)$ into irreducibles. In particular, the multiplicities in this decomposition are described explicitly. © 1990 Academic Press, Inc.

1. INTRODUCTION

We begin by briefly reviewing some well-known facts about the combinatorics of the representation theory of $Gl(n)$, and some recent developments in the analogous theory for the symplectic group $Sp(2n)$.

Recall that for fixed n, the irreducible representations of $Gl(n)$ are indexed by partitions λ with at most n parts, and the weights of the irreducible module V^λ are indexed by column-strict tableaux of shape λ. The corresponding formal character is the Schur function $s_\lambda(x_1, ..., x_n)$, which belongs to the ring $\Lambda_n = \mathbf{Z}[x_1, ..., x_n]^{S_n}$, of all symmetric polynomials in the x's with integer coefficients.

Combinatorics enters in the form of the Knuth–Schensted–Robinson [Kn, S]) insertion algorithm, which gives a bijection establishing the identity

$$(x_1 + \cdots + x_n)^k = \sum_{\substack{\lambda \vdash k \\ l(\lambda) \leqslant n}} f^\lambda s_\lambda(x_1, ..., x_n). \tag{1}$$

This is, of course, simply the character-theoretic version of

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THEOREM 1.1 (Schur) [Sch, Sta, Ha]. *Under the action of $Gl(V) = Gl(n, \mathbf{C})$, $V^{\otimes k}$ decomposes into irreducibles as follows*:

$$V^{\otimes k} = \coprod_{\substack{\lambda \vdash k \\ l(\lambda) \leqslant n}} (f^{\lambda})\, N^{\lambda}, \tag{2}$$

where f^{λ}, the multiplicity of the irreducible $Gl(n, \mathbf{C})$-module N^{λ} in the above decomposition, is the number of standard Young tableaux of shape $\lambda \vdash k$.

For the symplectic group $Sp(2n)$, again the irreducible representations are indexed by partitions λ of length at most n, and the corresponding weights are indexed by symplectic tableaux (first defined in [Kil]) of shape λ, and the formal character is the symplectic Schur function $sp_{\lambda}(x_1, ..., x_n)$, the generating function for the symplectic tableaux of shape λ, suitably weighted.

By establishing the validity of the analogous character identity in the ring $\tilde{\Lambda}_n = \mathbf{Z}[x_1^{\pm 1}, ..., x^{\pm 1}]^{B_n}$, where B_n is the Weyl group of $Sp(2n)$ (i.e., the hyperoctahedral group),

$$(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^k = \sum_{\substack{\mu \\ l(\mu) \leqslant n}} \tilde{f}_k^{\mu}(n)\, sp_{\mu}(_1^{\pm 1}, ..., x_n^{\pm 1}) \tag{3}$$

the ingenious and very natural insertion algorithm due to Berele [Be] gives a combinatorial proof of the following:

THEOREM 1.2. *Let V afford the defining representation for $Sp(2n, \mathbf{C})$. Then the kth tensor power of V decomposes into irreducible $Sp(2n, \mathbf{C})$-modules $\tilde{N}^{\mu}$ as*

$$V^{\otimes k} = \coprod_{\substack{\mu \\ l(\mu) < n}} \tilde{f}_k^{\mu}(n)\, \tilde{N}^{\mu},$$

*where $\tilde{f}_k^{\mu}(n)$ is the number of n-***symplectic up-down tableaux** *of shape μ and length k, that is, sequences of shapes $(\varnothing = \mu^0, \mu^1, ..., \mu^k = \mu)$ such that*

(1) *two consecutive shapes differ by exactly one box, i.e., for all $i = 1, ..., k$, either $\mu^i/\mu^{(i-1)} = (1)$ or $\mu^{(i-1)}/\mu^i = (1)$.*

(2) *$l(\mu^i) \leqslant n$, for all $i = 1, ..., k$.*

For the orthogonal group $SO(2n+1)$, the irreducible representations are also indexed by partitions λ of length at most n. In this paper we present solutions to the following two problems:

(1) Find a set of objects (associated to the partition λ) which index the weights of the irreducible representation corresponding to λ, and

(2) Use these objects to construct an insertion algorithm which describes the decomposition of $V^{\otimes k}$ under the natural action of $SO(V)=SO(2n+1)$ into irreducibles.

We remark that prior to our work, at least two kinds of tableaux which index weights for $SO(2n+1)$ were known; in [Ki1], King worked out such descriptions for all the classical Lie groups. More recently, Proctor [Pr1] found Gelfand pattern formulations for these groups, which, when translated into tableaux, at least for the $SO(2n+1)$ case, turn out to be identical to the independently discovered tableaux of [KT].

2. New Tableaux for $SO(2n+1)$

In this section we describe the new orthogonal tableaux and show that they do indeed correctly index the weights for $SO(2n+1)$. Our description has the advantage of being considerably simpler than those mentioned at the end of the preceding section.

We fix an alphabet of $(2n+1)$ symbols $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$.

Definition 2.1. An **so-tableau** T^0_λ of shape λ, $l(\lambda)\leqslant n$, is a filling of the Ferrers diagram of λ with the letters of the alphabet $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$, such that:

(1) the entries are weakly increasing along rows and, when restricted to $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}$, strictly increasing down the columns;

(2) all entries in row i are larger than, or equal to, i;

(3) the entries equal to ∞ form a skew-shape λ/μ which is a **vertical strip**; that is (cf. [Macd]), no two symbols ∞ appear in the same row.

Observe that (1) simply requires that the tableau be column-strict in the first $2n$ letters of the alphabet. We refer to (2) as the **symplectic condition**; the first two conditions define the symplectic tableaux of [Ki1], which index the weights for $Sp(2n)$.

Example 2.2. Let $n=3$, $\lambda=(4,3,3)\vdash 10$. Then

$$T^0_{(4,3,3)}=\begin{matrix} 1 & \bar{1} & 3 & \infty \\ \bar{2} & \bar{2} & \infty & \\ \bar{3} & \bar{3} & \infty & \end{matrix}$$

is an so-tableau.

The next step is to specify a suitable weighting scheme for these tableaux. We do this simply by assigning a weight to each entry of the tableau according to the rule

$$i \to x_i, \qquad \bar{i} \to x_i^{-1}, \qquad \infty \to 1.$$

Then the weight of an so-tableau T^0 is

$$wt(T^0) = \prod_{i=1}^{n} x_i^{\text{number of } i\text{'s in } T^0} (x_i^{-1})^{\text{number of } \bar{i}\text{'s in } T^0},$$

a monomial in $\tilde{\Lambda}_n$.

In the example above, $wt(T^0_{(4,3,3)}) = x_2^{-2} x_3^{-1}$. We now make a

DEFINITION 2.3. *For all partitions λ of length at most n, define*

$$so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1) = \sum_{\substack{T^0_\lambda \\ \text{so-tableau} \\ \text{of shape } \lambda}} wt(T^0_\lambda). \tag{4}$$

3. A CONNECTION BETWEEN ORTHOGONAL AND SYMPLECTIC CHARACTERS

The following observation is immediate from the definitions:

PROPOSITION 3.1.

$$so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu \text{ vertical strip}}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}). \tag{5}$$

Remark. Since this is effectively a statement about the characters of the groups $SO(2n+1)$ and $Sp(2n)$, it should be pointed out that to the best of this author's knowledge, it does not seem possible to attach any representation-theoretic significance to the above identity. For an overview of other such relations between the characters of the classical Lie groups, see [Pr2].

Let $E(n)$ be the matrix whose rows and columns are indexed by partitions of length at most n, ordered lexicographically, say (any total order will do), such that the (λ, μ)-entry of $E(n)$ is

$$\begin{cases} 1 & \text{if } \lambda/\mu \text{ is a vertical strip;} \\ 0 & \text{else.} \end{cases}$$

The following result will enable us to write down the inverse $E(n)^{-1}$ of $E(n)$:

LEMMA 3.2. *Let $F(t)$ be in $\Lambda(t)$, the ring of symmetric formal power series (over $\mathbf{Z}$) in the n variables $t_1, ..., t_n$, such that $F(0)=1$ (so $F^{-1}(t)$ is also in $\Lambda(t)$). Then the matrices $(\langle s_{\lambda/\mu}, F(t)\rangle)_{\lambda,\mu}$ and $(\langle s_{\lambda/\mu}, F^{-1}(t)\rangle)_{\lambda,\mu}$ (rows and columns indexed by* $\mathrm{Par}_n = \{\nu \in \mathrm{Par}: l(\nu) \leqslant n\}$ *with some total ordering) are inverses of each other.*

Proof. We compute

$$\sum_{\nu} \langle s_{\lambda/\nu}, F(t)\rangle \langle s_{\nu/\mu}, F^{-1}(t)\rangle.$$

Recall that in the ring $\Lambda(t)\otimes\Lambda(s)$ of formal power series (over $\mathbf{Z}$) in two sets of variables $\{t\}$ which are separately symmetric in $\{t\}$ and $\{s\}$, the following identity holds [Macd, Chap. 1, p. 41, (5.10)]:

$$s_{\lambda/\mu}(t, s) = \sum_{\lambda \supseteq \nu \supseteq \mu} s_{\lambda/\nu}(s)\, s_{\nu/\mu}(t). \tag{6}$$

Consequently, in $\Lambda(t)\otimes\Lambda(s)$,

$$\begin{aligned}
\langle s_\lambda(t, s), s_\mu(s)\, s_\nu(t)\rangle &= \sum_{\substack{\tau \\ \lambda \supseteq \tau}} \langle s_{\lambda/\tau}(s)\, s_\tau(t), s_\mu(s)\, s_\nu(t)\rangle \\
&= \sum_{\substack{\tau \\ \lambda \supseteq \tau}} \langle s_{\lambda/\tau}(s), s_\mu(s)\rangle \langle s_\tau(t), s_\nu(t)\rangle \\
&= \langle s_{\lambda/\nu}(s), s_\mu(s)\rangle \\
&\quad (\{s_\lambda(t)\} \text{ being an orthonormal basis for } \Lambda(t)) \\
&= \langle s_{\lambda/\nu}, s_\mu\rangle \\
&= \langle s_\lambda, s_\mu s_\nu\rangle \\
&= c^\lambda_{\mu,\nu}.
\end{aligned}$$

Since the functions $\{s_\mu(t)\, s_\nu(s)\}_{\mu,\nu}$ clearly form a basis for the ring $\Lambda(t)\otimes\Lambda(s)$, this means

$$\langle h(t, s), f(t)\, g(s)\rangle = \langle h, fg\rangle \tag{7}$$

for all symmetric functions f, g and all $h(t, s)$ in $\Lambda(t, s)$ (jointly symmetric in $\{t\}$, $\{s\}$).

Consider $\langle s_{\lambda/\mu}(t, s), F(t)\, F^{-1}(s)\rangle$. By (7), this equals

$$\langle s_{\lambda/\mu}, FF^{-1}\rangle = \langle s_{\lambda/\mu}, s_\varnothing\rangle = \delta_{\lambda\mu}.$$

On the other hand, by (6) we have

$$\langle s_{\lambda/\mu}(t, s), F(t)\, F^{-1}(s)\rangle = \sum_{\substack{\tau \\ \lambda \supseteq \tau \supseteq \mu}} \langle s_{\lambda/\tau}(t)\, s_{\tau/\mu}(s), F(t)\, F^{-1}(s)\rangle$$

$$= \sum_{\substack{\tau \\ \lambda \supseteq \tau \supseteq \mu}} \langle s_{\lambda/\tau}(t), F(t)\rangle \langle s_{\tau/\mu}(s), F^{-1}(s)\rangle. \quad \blacksquare$$

We can no write down the expansion of $sp_\lambda(x)$ in terms of $so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)$, for partitions λ of length at most n:

THEOREM 3.3.

$$sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu \text{ horizontal strip}}} (-1)^{|\lambda/\mu|}\, so_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1). \tag{8}$$

Proof. We need only recall an easy symmetric function identity,

$$\prod_{i=1}^{n} (1-t_i)^{-1} = \sum_{r \geqslant 0} h_r(t_1, ..., t_n),$$

where h_r is the complete homogeneous symmetric function (see [Macd]).

Now observe that for the matrix $E(n) = (d_\mu^\lambda)$, $(l(\lambda), l(\mu) \leqslant n)$, taking

$$F(t) = \prod_{1 \leqslant i \leqslant n} (1 + t_i)$$

gives, from elementary facts about symmetric functions,

$$d_\mu^\lambda = \langle s_{\lambda/\mu}, F(t)\rangle,$$

so that, applying Lemma (3.2) to this choice of F, we get

$$\begin{aligned}(E(n)^{-1})_{\lambda,\mu} &= \langle s_{\lambda/\mu}, F^{-1}(t)\rangle \\ &= \sum_{r \geqslant 0} \langle s_{\lambda/\mu}, (-1)^r h_r\rangle \qquad \text{by the above remark} \\ &= \begin{cases} (-1)^r & \text{if } \lambda/\mu \text{ is a horizontal strip of size } r; \\ 0 & \text{else,} \end{cases}\end{aligned}$$

where the final equality is a consequence of the Littlewood–Richardson rule for multiplying Schur functions [Macd].

Treating (5) as a matrix equation and inverting the matrix on the right-hand side gives the desired result. $\blacksquare$

COROLLARY 3.4. *The functions* $\{so_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}, 1): l(\lambda) \leqslant n\}$ *form an integral basis for the ring* $\tilde{\Lambda}_n$ *of Laurent polynomials with integer coefficients in* $\{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$ *which are invariant under the action of the hyperoctahedral group* B_n.

Proof. Equation (8) shows the $\{so_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}, 1): l(\lambda) \leqslant n\}$ span $\tilde{\Lambda}_n$, since the symplectic Schur functions do so (see [Su1] for an elementary proof); Equation (5) shows they are linearly independent, by the independence of the $sp_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1})$. ∎

We now proceed to show that the functions so_λ defined in the preceding section are in fact the formal characters for the group $SO(2n+1)$. We shall need the following two classical identities, known to Weyl [We] and Littlewood [Li].

THEOREM 3.5 (The Cauchy identity for $Sp(2n)$).

$$\prod_{1 \leqslant i < j \leqslant n} (1 - t_i t_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1}$$
$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu({}_1^{\pm 1}, \dots, x_n^{\pm 1})\, s_\mu(t_1, \dots, t_n). \tag{9}$$

THEOREM 3.6 (The Cauchy identity for $SO(2n+1)$). *Denote by* $\mathrm{char}_\mu(x_1^{\pm 1}, \dots, x_n^{\pm 1}, 1)$ *the character of* $SO(2n+1)$ *corresponding to the irreducible representation indexed by the partition* μ, *where* $l(\mu) \leqslant n$. *Then*

$$\prod_{1 \leqslant i \leqslant j \leqslant n} (1 - t_i T_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1} \prod_{i=1}^{n} (1 - t_i)^{-1}$$
$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} \mathrm{char}_\mu(x_1^{\pm 1}, \dots, x_n^{\pm 1}, 1)\, s_\mu(t_1, \dots, t_n). \tag{10}$$

LEMMA 3.7. *The functions* so_λ *satisfy the following identity in the ring* $\mathbf{Z}[t_1, \dots, t_n]^{S_n} \otimes \mathbf{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{B_n}$:

$$\prod_{i=1}^{n} (1 + t_i) \prod_{1 \leqslant i < j \leqslant n} (1 - t_i t_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1}$$
$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} so_\mu(x_1^{\pm 1}, \dots, x_n^{\pm 1}, 1)\, s_\mu(t_1, \dots, t_n). \tag{11}$$

Proof. We start with the right-hand side of this equation. Using Proposition (3.1) we get

$$\sum_{\substack{\lambda \\ l(\lambda)\leqslant n}} so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)\, s_\lambda(t_1, ..., t_n)$$

$$= \sum_{\substack{\lambda \\ l(\lambda)\leqslant n}} \sum_{\substack{\mu\subseteq\lambda \\ \lambda/\mu \text{ vertical strip}}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\lambda(t_1, ..., t_n)$$

$$= \sum_{\substack{\mu \\ l(\mu)\leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}) \sum_{\substack{\lambda \\ \lambda/\mu \text{ vertical strip}}} s_\lambda(t_1, ..., t_n).$$

But it is well known from the theory of symmetric functions (e.g., [Macd]) that

$$\sum_{\substack{\lambda \\ \lambda/\mu \text{ vertical strip}}} s_\lambda(t_1, ..., t_n) = s_\mu(t_1, ..., t_n) \prod_{i=1}^{n} (1+t_i).$$

Using this and the Cauchy identity for $Sp(2n)$ above immediately yields the desired identity. ∎

Hence, by comparing the coefficient of the Schur function $s_\mu(t_1, ..., t_n)$ in (8) and in the Cauchy identify for $SO(2n+1)$, we get

THEOREM 3.8. *For $l(\lambda)\leqslant n$ the function $so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)$ is the character of $SO(2n+1)$ corresponding to the irreducible representation indexed by λ.*

4. AN INSERTION ALGORITHM

We begin by defining the objects which will be shown to play the rôle of standard Young tableaux in the $Gl(n)$ theory:

DEFINITION 4.1. For a partition μ of length at most n, let $So_\mu^k(n)$ be the set of all k-sequences of shapes $(\varnothing=\mu^0, \mu^1, ..., \mu^k=\mu)$ such that

(1) $l(\mu^i)\leqslant n$ for all $i=1, ..., k$,

(2) either $\mu^i/\mu^{(i-1)}=(1)$ or $\mu^{(i-1)}/\mu^i=(1)$, or

(3) $l(\mu^{(i-1)})=n$ and $\mu^{(i-1)}=\mu^i$.

The first two conditions define the n-symplectic up–down tableaux of shape μ, length k studied in [Be] and [Su1).

Set $\tilde{F}_\mu^k(n)=|So_\mu^k(n)|$. We are now ready to state our main result:

THEOREM 4.2. *There is an insertion scheme $\mathscr{S}$ which proves the identity, for k a non-negative integer,*

$$(x_1+x_1^{-1}+\cdots+x_n+x_n^{-1}+1)^k=\sum_{\substack{\mu\\ l(\mu)\leqslant n}} \tilde{F}_\mu^k(n)\, so_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1). \tag{12}$$

Proof. The algorithm $\mathscr{S}$ is the obvious extension of Berele insertion suggested by the very definition of so-tableaux and by Proposition (3.1). We shall assume familiarity with ordinary Knuth–Schensted–Robinson (KSR) row insertion (see [Kn, S]) and the "bumping process," and while we describe the modification which defines Berele's algorithm, we refer the reader to [Be] for the proof of its correctness.

We define $\mathscr{S}$ to be a mapping from the set of all words of length k in the alphabet of $(2n+1)$ symbols $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$, to the set of all pairs $(T_\mu^0, (\varnothing=\mu^0, \mu^1, ..., \mu^k=\mu))$, where the first component is an so-tableau of shape μ, and the second is a sequence in the set $So_\mu^k(n)$. The map $\mathscr{S}$ works essentially like KSR insertion, by successively inserting the letters of the k-word w to produce the appropriate tableau of shape μ, and a k-sequence of shapes ending in the shape μ. To ensure that the resulting tableau T_μ^0 satisfies the symplectic condition ((2) in Definition (2.1)), we need to incorporate the additional elements of Berele's algorithm; to ensure that the condition on this ∞'s holds, we simply require the following exception to the ordinary bumping rules:

(A) ∞ always bumps ∞.

It suffices to describe one step of the insertion process, that of inserting a single letter x from the alphabet into a legal so-tableau T_μ^0 with entries in this alphabet. We indicate such a step by writing $(T_\mu^0 \leftarrow x)$. Either ordinary KSR (row) insertion of x into T_μ^0, modulo rule (A), produces an so-tableau of shape one cell larger than μ, or else we have the following possibilities:

Case 1. A symplectic violation occurs; that is, at some point in the ordinary bumping path, an i is about to bump an $\bar{i}$ from row i into row $(i+1)$. Instead of letting the i displace the $\bar{i}$, Berele's algorithm now proceeds by replacing the last $\bar{i}$ in the ith row by a blank square, which is then slid out to the southeast boundary of the shape μ by a sequence of *jeu de taquin* [Schü] moves, again respecting the condition that the ∞'s form a vertical strip. Clearly the resulting tableau has shape one cell less than μ, and it is not hard to see that this process preserves weights; that is, the weight of the resulting tableau is weight$(x)\cdot$weight(T_μ^0).

Case 2. If $l(\mu)=n$, the bumping path of $(T_\mu^0 \leftarrow x)$ may terminate by bumping an ∞ from the nth row into the $(n+1)$th row. That is, the output of the ordinary KSR insertion is a tableau of shape $\bar{\mu} = \mu$ plus a box in row $(n+1)$, where the first n rows form an so-tableau T^0_μ, and the box in row $(n+1)$ contains an ∞. In this case we simply erase the offending box in the $(n+1)$th row, and declare the result of the insertion to be T^0_μ. The shape has remained unchanged, and clearly the weights have been preserved (trivially). Figure 1 describes the insertion process for $n=3$.

To check that the mapping is invertible, it suffices to show that the invertibility of Berele insertion is not destroyed by the exceptional bumping rule. Thus consider the situation where we are given a pair consisting of an so-tableau $T^0_{\mu^i}$ of shape μ^i, and the preceding shape μ^{i-1}; we need only describe how to retrieve the preceding so-tableau of shape μ^{i-1} and the letter x, such that

$$(T^0_{\mu^{i-1}} \leftarrow x) = T^0_{\mu^i} \qquad (\text{via } \mathscr{S}\text{-insertion}),$$

in the case when $\mu^{i-1} = \mu^i$, i.e., when the successive shapes are equal. From Case 2 above this forces $l(\mu^i) = n$, and it is clear that the required information is recovered by adding an ∞ to row $(n+1)$ of $T^0_{\mu^i}$ and reversing the ordinary bumping process from this cell of the tableau. ∎

In [Su1], the author used Berele's algorithm to construct a bijection establishing the Cauchy identity for $Sp(2n)$. We now show that the above insertion scheme easily extends to a bijection for the analogous identity for $SO(2n+1)$ ((8) of Lemma (3.4)). We refer the reader to [Su2], where the essential details of the proof have been verified.

THEOREM 4.3. *There is a bijection establishing*

$$\prod_{i=1}^{n} (1+t_i) \prod_{1 \leqslant i<j \leqslant n} (1-t_it_j) \prod_{i,j=1}^{n} (1-t_ix_j)^{-1}(1-t_ix_j^{-1})^{-1}$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} so_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)\, s_\mu(t_1, ..., t_n).$$

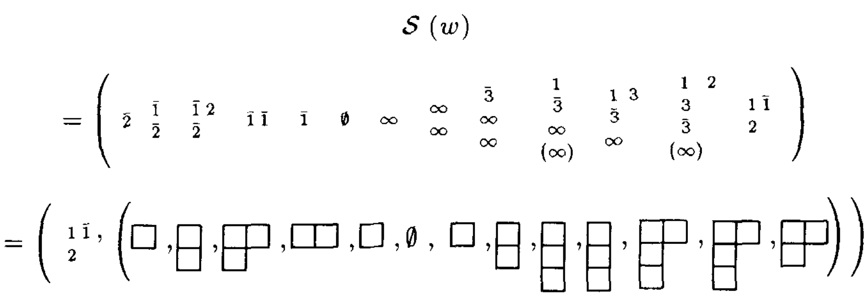

FIG. 1. $\mathscr{S}$-insertion of the word $w = \bar{2}\bar{1}2\bar{1}11\infty\infty\bar{3}132\bar{1}$ for $n=3$, and $k=13$.

Proof. We may enumerate the left-hand side by means of Knuth two-line arrays [Kn]

$$\mathscr{T} = \begin{pmatrix} t_{i_1} \cdots y_{i_k} \\ y_{i_1} \cdots y_{i_k} \end{pmatrix},$$

where the y_{i_j}'s are in the set $\{x_1^{\pm 1}, ..., x_n^{\pm 1}, \infty\}$; such a two-line array would correspond to the term $(t_{i_1} \cdots t_{i_k} y_{i_1} \cdots y_{i_k})$ in the expansion of the left side as a formal power series, the ∞'s now being interpreted as one's coming from the product $\prod_{i=1}^{n}(1+t_i)$. We impose the usual [Kn] lexicographic ordering on the arrays, viz., (1) $t_{i_1} \leqslant \cdots \leqslant t_{i_k}$ and (2) $t_{i_j} = t_{i_{j+1}}$ implies $y_{i_j} \leqslant y_{i_{j+1}}$.

The right-hand side clearly counts the set of all triples $(P_\mu^0(x), P_\mu(t), P_\beta(t))$, where $P_\mu^0(x)$ is an so-tableau of shape μ with entries in the alphabet

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty,$$

corresponding to the variables $\{x_1^{\pm 1}, ..., x_n^{\pm 1}\}$, $P_\mu(t)$ is an ordinary tableau with entries in $[n]$ of the same shape μ as P_μ, and $P_\beta(t)$ is an ordinary tableau of shape β with entries in $[n]$, where β has even columns.

Following the steps in the symplectic case [Su2], we insert the bottom row of the two-line array using the algorithm $\mathscr{S}$, to produce the appropriate so-tableau of shape μ; we keep track of the resulting sequence of shapes $So_\mu^k(n)$ by using the labels in the top row [Su2]. (Here k is the length of either row of the two-line array.) The following observations are made:

(a) the two-line array satisfies the additional condition

$$y_{i_j} = \infty = y_{i_{j+1}} \text{ implies } t_{i_j} \text{ is strictly less than } t_{i_{j+1}}.$$

(b) the sequence of shapes $So_\mu^k(n)$ is in fact an ordinary up–down tableau of length k and shape μ, i.e., there is no repetition of shapes in the sequence, and

(c) $\mathscr{S}$ has the property that insertion of an ∞ symbol into an so-tableau never causes a symplectic cancellation.

To see why (b) is true, observe that otherwise at step $(j+1)$ of the insertion, the previous shape μ^j of length n will be repeated. This can only happen if the so-tableau P_j^0 at step j contained an ∞ in row n, and $\mathscr{S}$-insertion of the next letter $y_{i_{j+1}}$ resulted in bumping this ∞ out of row n. Hence the length of the shape that would have resulted from ordinary Schensted insertion of the word $w = y_{i_1} \cdots y_{i_{j+1}}$ is at least $n+1$. By a well-known property of Schensted insertion [S], this means that the word w contains a decreasing subsequence of length at least $n+1$. But from the ordering imposed on the two-line array, this is impossible since there are at most n distinct t's in the top row.

The sequence of shapes So^k_μ can thus be encoded by means of the labels in the top row, exactly as in [Su2], into a pair of ordinary tableaux (P_μ, P_β), where β is a partition with even columns, and $|\mu| + |\beta| = k$. The observation (a) above guarantees that P_μ will be column-strict, in spite of the modified bumping rule which makes ∞ bump itself.

Observation (c) ensures that the labels in the top row corresponding to those insertions which caused a cancellation, satisfy the same requirements which allowed us to argue in [Su2] that the up–down tableau can be recovered uniquely from the pair (P_μ, P_β).

The word

$$(t_1 x_1^{-1})^2 (t_2 x_1)^2 t_2 x_1^{-1} t_3 x_1 t_3 x_2^{-1} (t_4 x_2)\, t_5 x_1^{-1} (t_5)\, t_6 x_1 (t_6)$$

corresponds to the two-line array

$$\mathscr{T} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 6 \\ \bar{1} & \bar{1} & 1 & 1 & \bar{1} & 1 & \bar{2} & 2 & \bar{1} & \infty & 1 & \infty \end{pmatrix}.$$

We proceed with $\mathscr{S}$-insertion of the word in the bottom row of $\mathscr{T}$, from left to right as usual. In the schematic that follows, the computation is arranged so that:

- the first row contains the successive so-tableaux, ending in the final tableau P^0_μ,
- the second row encodes the up–down tableau resulting from the insertion, ending in the final tableau P_μ,

$$\mathcal{T} \to \begin{pmatrix} \bar{1} & \bar{1}\bar{1} & \bar{1} & \emptyset & \bar{1} & \emptyset & \bar{2} & \begin{smallmatrix} 2 \\ \bar{2} \end{smallmatrix} & \bar{1} & \bar{1}\infty & \infty & \begin{smallmatrix} \infty \\ \infty \end{smallmatrix} & P^o{}_\mu \\ 1 & 11 & 1 & \emptyset & 2 & \emptyset & 3 & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} & 3 & 35 & 5 & \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} & P_\mu \\ & & \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\} & \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\} & & \left\{ \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right\} & & & \left\{ \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \right\} & & \left\{ \begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right\} & & \mathcal{L} \end{pmatrix}$$

$$\to \left(\begin{smallmatrix} \infty \\ \infty \end{smallmatrix},\ \begin{smallmatrix} 5 \\ 6 \end{smallmatrix},\ \left\{ \begin{smallmatrix} 2\,2\,3\,5\,6 \\ 1\,1\,2\,4\,3 \end{smallmatrix} \right\} \right)$$

$$\to \left(\begin{matrix} \infty \\ \infty \end{matrix},\ \begin{matrix} 5 \\ 6 \end{matrix},\ \begin{matrix} 1\,1\,2\,3 \\ 2\,2\,3\,4 \\ 5 \\ 6 \end{matrix} \right) = (P^o{}_{(1,1)}, P_{(1,1)}, P_{(4,4,1,1)}).$$

FIG. 2. One direction of the Cauchy identity bijection.

- the third row encodes the removals in the form of pairs which, at the end of the process, may be put together into a column-strict tableau P_β whose shape has even columns (via Burge's correspondence, cf. [Su1, Su2, Bu]).

To illustrate the inverse of this mapping, by virtue of Theorem 4.2 it suffices to show that the pair $(P_{(1,1)}, P_{(4,4,1,1)})$ uniquely determines the k-sequence of shapes; by remark (c) above, this follows exactly as in [Su2]. ∎

5. Pieri Rules

Recall that for the ordinary Schur functions, the Pieri rules describe how to write the products $s_\lambda \cdot h_r$ and $s_\lambda \cdot e_r$ as a non-negative integer combination of Schur functions; here h_r, e_r are respectively the homogeneous and elementary symmetric functions of degree r (cf. [Macd]). The rules reduce to the following elementary properties about KSR row-insertion:

Lemma 5.1. *Suppose P is an ordinary column-strict tableau of shape μ, and KSR row-insertion of some word $w = w_1 \cdots w_n$ into P produces a column-strict tableau of shape λ, and suppose insertion of w_i adds a cell in row p_i.*

(1) *If $\omega = \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_n$ is an increasing word, then $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and hence λ/μ is a horizontal strip (cf. [Macd].)*

(2) If *$\omega = \omega_1 > \omega_2 > \cdots > \omega_n$ is a strictly decreasing word, then $p_1 < p_2 < \cdots < p_n$ and hence λ/μ is a vertical strip.*

We shall need a few facts about the mapping $\mathscr{S}$: In what follows we fix n, the rank of the orthogonal group.

Lemma 5.2. *Let y_1, y_2 be letters in $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty$, and P_λ^0 an so-tableau of shape λ. Consider the sequence of $\mathscr{S}$-insertions $((P_\lambda^0 \leftarrow y_1) \leftarrow y_2)$.*

(1) *If $y_1 \leqslant y_2 < \infty$, or $y_1 < y_2 = \infty$, then the bumping path of y_2 lies strictly to the right of the path of y_1, hence ends weakly above it.*

(2) *If $y_1 = y_2 = \infty$, then the bumping paths coincide and that of y_2 ends strictly below that of y_1.*

(3) *If $\infty \geqslant y_1 > y_2$ then the path of y_2 lies weakly to the left of that of y_1.*

(4) *If $y_1 > y_2$ and y_1 causes a symplectic violation, then so does y_2 (in particular, y_2 cannot cause a repeat of the previous shape).*

(5) *If $y_1 > y_2$ and (inserting) y_1 repeats the previous shape λ, then y_2 causes either a repeat or a cancellation.*

Proof. (1) If y_1 bumps an ∞ out of row 1 the result is clear, since y_1 now occupies the right-most position in the first row. If not, let r be the last (highest) row index where the bumping path of y_1 bumps out an entry $c_r < \infty$; so that the rest of the path consists of bumping ∞'s down successive rows until there a row with no symbol ∞ is reached. In particular, note that c_r now occupies the right-most square in row $(r+1)$. Let s be the analogous row index for the path of y_2. (If $y_2 = \infty$ then $s \leqslant 1$). If $s \leqslant r$, it is clear that the path of y_2 must end in row $(r+1)$ to the right of the corner square containing c_r. If $s > r$ this is still true because c_r occupies a corner square.

(2) Obvious.

(3) If r, c_r are defined for the path of y_1 as in (1), then clearly the path of y_2 lies weakly to the left of that of y_1 up to row $(r+1)$, after which the two may coincide (but the y_2-path never moves to the right of the y_1-path).

(4) The hypothesis that y_1 causes a cancellation implies that its ordinary bumping path hit column 1 in some row i (and bumped an $\bar{i}$ into row $(i+1)$). By (3), the path of y_2 must hit the first column at some higher row, hence knocks the entry z in row $(i-1)$, column 1, into row i. But clearly $(i-1) \leqslant z < i$, so this is a symplectic violation.

(5) This time we know that the bumping path of y_1 ends in row $(n+1)$ (and column 1). Again by (3) the path of y_2 must end below this, hence the insertion resulted in either a repeat of shapes or a cancellation. ∎

We now have

THEOREM 5.3.

$$so_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)\, so_{(k)}(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)$$
$$= \sum_{l(\nu)' \leqslant n} so_\nu \begin{pmatrix} \sum_{r=0}^{k} |\{\mu \vdash (|\lambda| - r);\ \lambda/\mu,\ \nu/\mu \text{ horiz. strips};\ |\nu/\mu| = (k-r)\}| \\ + \sum_{r=0}^{k-1} |\mu \vdash (|\lambda| - r),\ l(\mu) = n\text{: } \lambda/\mu,\ \nu/\mu \text{ horiz. strips};\ |\nu/\mu| = (k-r-1)\}| \end{pmatrix}.$$

Proof. The left-hand side enumerates pairs (P_λ^0, ω), where P_λ^0 is an so-tableau of shape λ and ω is an increasing k-word in $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty$ with at most one occurrence of ∞, i.e., $\omega_1 \leqslant \cdots \leqslant \omega_k$, unless $\omega_k = \infty$, in which case $\omega_{k-1} < \omega_k$,

So let us consider the effect of $\mathscr{S}$-inserting ω into the so-tableau P_λ^0. Observe that

(1) if there are no repetitions of shape, but symplectic cancellations occur during the insertion, we can appeal to the case of Berele insertion

[Sul, Su2] to conclude that the coefficient of so_ν in the product on the left-hand side is the number of sequences $\{\lambda \supseteq \mu \subseteq \nu\}$ such that λ/μ is a horizontal strip of size r, while ν/μ is a horizontal strip of size $(k-r)$, for some $0 \leqslant r \leqslant k$. Conversely each such sequence may be uniquely expanded to a k-sequence of shapes beginning in λ and ending in ν, and inverting the Berele algorithm will produce an increasing word of length k.

(2) On the other hand, if there are repetitions of shape, statements (1) and (2) of Lemma (5.2) tell us that there must be exactly one and that it must occur after a string of symplectic cancellations and before the string of additions. Hence we get a contribution to the second term in the coefficient on the right-hand side of (10). Conversely, given any sequence $\{\lambda \supseteq \mu \subseteq \nu\}$ such that $l(\mu)=n$, λ/μ is a horizontal strip of size r, while ν/μ is a horizontal strip of size $(k-r-1)$, for some $0 \leqslant r < k$, this expands uniquely to a k-sequence of shapes $\lambda=\mu^0 \supset \mu_1 \supset \cdots \mu^r = \mu = \mu^{r+1} \subset \nu^1 \subset \cdots \subset \nu^{k-r-1} = \nu$ which produces an increasing k-word upon inverting $\mathscr{S}$-insertion: Lemma 5.2, (3) shows that the recovered letters are weakly increasing in going from a cancellation to a repeat, and statement (4) shows this in going from a repeat to an addition. The fact that the skew-shapes are horizontal strips guarantees that two consecutive ∞'s will not occur. ∎

Denote by $\bar{e}_k$ the generating function of all tableaux of shape (1^k), such that the non-∞ entries increase strictly from top to bottom and may be followed by a string of ∞'s (i.e., the tableaux are column-strict when restricted to the alphabet $1, \bar{1}, ..., n, \bar{n}$). Observe that $\bar{e}_k(x_1^{\pm 1}, ..., x_n^{\pm}, 1) = \sum_{i=0}^{\lfloor (k/2) \rfloor} e_{k-2i}(x_1^{\pm 1}, ..., x_n^{\pm 1}, 1)$. Then one has

Theorem 5.4. *The coefficient of so_ν in $so_\lambda . \bar{e}_k$ is the number of shapes μ of length at most n, such that $\mu \supseteq \nu$, $\mu \supseteq \lambda$, μ/ν, μ/λ are vertical strips, and $|\mu/\nu| + |\mu/\lambda| = k$, plus the number of shapes μ of length exactly n, such that $\mu \supseteq \nu$, $\mu \supseteq \lambda$, μ/ν, μ/λ are vertical strips, and $|\mu/\nu| + |\mu/\lambda| < k$.*

Proof. The left-hand side enumerates pairs (P_λ^0, w) where P_λ^0 is an so-tableau of shape λ and ω is a decreasing word in $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty$ of length k (read each tableau of $\bar{e}_k$ bottom-to-top). In fact $\infty = \omega_1 = \cdots = \omega_j > \omega_{j+1} > \cdots > \omega_k$, for some $j \geqslant 0$.

Again, the argument consists of analysing the effect of $\mathscr{S}$-inserting ω into the so-tableau. By statements (4) and (5) of Lemma (5.2), since the non-∞ entries of ω are strictly decreasing, the effect of repeating a shape during the insertion is followed by another repeat or by a cancellation. By the same arguments as in [Su2], we see that the insertion of ω produces a sequence of shapes $(\lambda = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \mu = \tau^1 = \cdots = \tau^s \supset \mu = \nu^1 \supset \nu^2 \supset \cdots \nu^{k-r-s} = \nu)$, where $s > 0$ forces $l(\mu) = n$, and where the fact that

μ/λ and μ/ν are vertical strips uniquely determines the intermediate shapes in the sequence: We need to supplement the arguments in [Su2] only by statement (2) of Lemma (5.2) and the fact that insertion of successive ∞'s forms a vertical strip. The fact that the correct word can be recovered from such a sequence follows similarly. ▮

6. Conclusion

We conclude by pointing out some possible representation-theoretic uses of the insertion algorithm $\mathscr{S}$. A classical result of Weyl, also known to Littlewood (see [Li]), states the following branching rule from $Gl(2n+1)$ to $SO(2n+1)$:

Let λ be a partition of length at most $2n+1$, and let $c^{\lambda}_{\mu,\gamma}$ denote a Littlewood–Richardson coefficient (see [Macd]). Then the irreducible $Gl(2n+1)$-module indexed by λ decomposes into $SO(2n+1)$-irreducibles.

THEOREM 6.1. *If $l(\lambda)\leqslant n$, the $SO(2n+1)$-irreducible indexed by μ appears in this decomposition with multiplicity*

$$\sum_{\substack{\gamma \\ \gamma \text{ has even parts}}} c^{\lambda}_{\mu,\gamma}.$$

In [Su1], the author provided a combinatorial proof of the analogous branching rule for $Sp(2n)$, and also gave an extension of the rule to take care of the case $l(\lambda)>n$, thereby circumventing the modification rules of King [Ki2]. Since the essential tool used in that case was Berele's algorithm for $Sp(2n)$, one would hope to follow a similar programme for $SO(2n+1)$, using the algorithm $\mathscr{S}$ or some other suitable insertion scheme.

Since there are other combinatorial descriptions of tableaux which index the weights of $SO(2n+1)$, it may be of interest to uncover the connections between the different formulations, and to address the question of whether there may be other insertion algorithms.

We end with a description of the other known orthogonal tableaux:

THEOREM 6.2 [Ki1]. *For the alphabet $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$ and for a shape λ with at most n parts, define an array T_λ of shape λ with entries in $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$ to be a column-strict tableau satisfying the following additional conditions*:

(K1) *all entries in row i are larger than or equal to i*;

(K2) *in row i, $\bar{\imath}$ does not appear next to i unless there is an i in row $(i-1)$, directly above the $\bar{\imath}$; i.e., we must have the configuration of Fig.* 3.

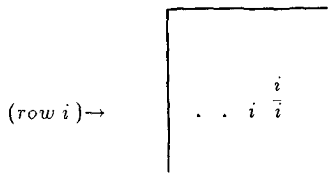

FIG. 3. King's condition for orthogonal tableaux.

Let $m(T_\lambda)$ denote the number of occurrences of i directly above $\bar{i}$ in the first column, with $\bar{i}$ in row i. Then the character of $SO(2n+1)$ indexed by the partition λ is given by the generating function

$$o_\lambda(x_1^{\pm}, \ldots, x_n^{\pm 1}) = \sum_{\text{tableaux } T_\lambda \text{ of shape } \lambda} 2^{m(T_\lambda)} (x_i)^{(\#\text{ of } i\text{'s in } T_\lambda)} (x_i^{-1})^{(\#\text{ of } \bar{i}\text{'s in } T_\lambda)},$$

where the sum is over all column-strict tableaux T_λ of shape λ satisfying (K1) *and* (K2).

Note that the weighting scheme for the alphabet is

$$i \to x_i, \qquad \bar{i} \to x_i^{-1}, \qquad \infty \to 1.$$

In the equivalent descriptions of Proctor [Pr1] and Koike and Terada [KT], one ostensibly needs an alphabet of $3n$ letters, which we order as follows:

$$1 < 1^0 < \bar{1} < 2 < \cdots < i < i^0 < \bar{i} < \cdots < n < n^0 < \bar{n}.$$

THEOREM 6.3. *The irreducible $SO(2n+1)$ character corresponding to the shape λ is the generating function for column-strict tableaux R_λ of shape λ which satisfy, in addition to* (K1) *above, the condition*

(PKT) *i^0 appears only in row i, and at most once. Thus we have*

$$o_\lambda({}_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\text{tableaux } R_\lambda \text{ of shape } \lambda} (x_i)^{(\#\text{ of } i\text{'s in } R_\lambda)} (x_i^{-1})^{(\#\text{ of } \bar{i}\text{'s in } R_\lambda)}$$

where the sum is over all column-strict tableaux R_λ of shape λ satisfying (K1) *and* (PKT).

We remark that the implicit weighting scheme for the alphabet is now

$$i \to x_i, \qquad \bar{i} \to x_i^{-1}, \qquad i^0 \to 1.$$

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