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LINEAR TIME-INVARIANT CONTROL PROBLEMS WITH PERTURBATIONS

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Abstract—We deal with a class of control problems whose (uncertain) mathematical model is given by a linear differential equation with control parameters and perturbation parameters influencing the dynamics of an object under consideration. All we know about the perturbations is that their values belong to a given compact set Q. We show how to find both lower and upper bounds for the value function of our original perturbed system in terms of the value functions of two simplified unperturbed control problems. We also give explicit formulae for ε-optimal policies in the class of so-called step-guided strategies. In the last section the results above are extended to the infinite horison setting.

1. INTRODUCTION

Consider the following control problem P: Minimize a given function $g(x(\cdot))$ over all trajectories $x(\cdot)$ of the equation

$$\dot{x}(t) = Ax(t) + Bu(t) - Cv, \quad x(0) = x_0, 0 \leq t \leq T, x \in R^n, \tag{1}$$

where $u(t) \in P \subset R^p$ is a control function and $v(t) \in Q \subset R^q$ is an unknown perturbation function; no additional information on $v(t)$ is available. Our goal is to find estimates, from below and above, of the value function of problem P, as well as explicit formulae for ε-optimal strategies. Toward this end, we consider two cases articulated in assumptions (3_1), (3_2) that represent different relationships between the sets BP and CQ. Using Theorems 1 and 2 one is able to find lower and upper estimates of the value function of problem P in terms of the value functions of problems P_1, P_2, defined as follows. (P_i): Minimize $g(x(\cdot))$ over the solutions of the equation

$$\dot{w}(t) = Aw(t) + h(t), \quad h(t) \in H_i, \quad w(0) = x_0, \quad 0 \leq t \leq T \tag{2}$$

where H_i obeys condition (3_1) listed at the very beginning of Section 2 $(i = 1, 2)$.

2. ASSUMPTIONS AND STRATEGIES

$$\text{There exists a nonempty, bounded set } H_1 \subset R^n \text{ such that } H_1 + CQ \subset BP. \tag{3_1}$$

$$\text{There exists a nonempty, bounded set } H_2 \text{ such that } H_2 + CQ \supset BP. \tag{3_2}$$

Let us recall that the largest set H_1 satisfying condition (3_1), according to the terminology introduced by L.S. Pontryagin, is called the geometric difference of sets BP, CQ, and is usually denoted by $BP \stackrel{*}{-} CQ$.

The function $g(x(\cdot))$, defined on the space $C(0, T; R^n)$ of continuous mappings $x(\cdot) : [0, T] \to R^n$ equipped with the max norm, is continuous. (4)

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$$\text{The sets } P \text{ and } Q \text{ are bounded subsets of } R^p \text{ and } R^q, \text{ respectively.} \tag{5}$$

$$A \text{ is a stable matrix.} \tag{6}$$

Having listed all our assumptions, let us recall that, since all eigenvalues of A have negative real parts, there exists a positive definite quadratic form $L(x) = \langle x^T, L^* x\rangle$, $x \in R^n$, with the property that for each solution $x(\cdot)$ of equation $(*)\dot{x}(t) = Ax(t)$, the derivative of $L(x(t))$ along equation $(*)$ satisfies the relations [1, p. 347].

$$\frac{d}{dt}L(x(t))_{|(*)} = \langle \frac{d}{dt}x(t)^T, L^* x(t)\rangle + \langle x^T(t), L^* Ax(t)\rangle = x^T(t)[A^T L^* + L^* A]x(t)$$

where L satisfies the Lyapunov equation $A^T L^* + L^* A = I$ (identity matrix). To define strategies for the controller, let us introduce the following multifunctions defined on $R^n \times R^n$:

$$U(x,w) = \{\bar{u} \in P : \langle L(w-x), B(\bar{u}-u)\rangle \geq 0,\ u \in P\} \tag{7}$$

$$V(x,w) = \{\bar{v} \in Q : \langle L(x-w), C(\bar{v}-v)\rangle \geq 0,\ v \in Q\} \tag{8}$$

and, for any set H_2 satisfying condition (3_2),

$$H_2(x,w) = \{\bar{h} \in H_2 : \langle L(x-w), \bar{h}-h\rangle \geq 0,\ h \in H_2\} \tag{9}$$

It is obvious that all these multifunctions are upper semicontinuous, which follows from a known result concerning marginal maps [2]. As such, they admit Borel measurable selections $u(x,w)$, $v(x,w)$, $h_2(x,w)$ from, respectively, $U(t,x)$, $V(t,x)$, $H_2(x,w)$. In this paper, however, we do not need any measurability of selections, which, by the way, makes our results more suitable for implementation in practice (see the definition below).

DEFINITION 1. By a step-guided strategy we mean any triplet $(u(x,w), \Delta, w_R(\cdot))$, where $u : R^n \times R^n \to R$ is an arbitrary selection from $U(x,w)$ (not necessarily measurable), $\Delta = (\tau_i)$ is a finite partition of $[0,T]$, and $w_R(\cdot)$ is a causal operator defined on the space of all trajectories of equation (1).

It means the values of the function $w_R(\cdot)$, playing the role of a guiding function, are chosen instantly in a nonanticipating fashion, according to a specified rule R. In a very particular case, when $w_R(\cdot)$ does not depend on $x(\cdot)$ at all, $w_R(\cdot)$ may be identified with a fixed function $w(t)$. In such a case, a step-guided strategy is said to be simple.

Observe that each step-guided strategy, coupled with a perturbation function $v(t) \in Q$, gives rise to the following trajectory of system (1): $\dot{x}(t) = Ax(t) + Bu(x(0)),\ w(0)) - Cv(t), x(0) = x_0$, $0 \leq t < \tau_1$, and for $t \in [\tau_i, \tau_{i+1})$, $\dot{x}(t) = Ax(t) + Bu(x(\tau_i), w(\tau_i)) - Cv(t)$.

Denote by U_{sg} the space of all step-guided strategies. In order to underline the dependence of a trajectory $x(\cdot)$ of system (1) on a step-guided strategy $u_{sg} \in U_{sg}$, and a perturbation function $v(\cdot)$, we shall often write $x(t) = x[t, x_0, u_{sg}, v(\cdot)]$. Thus, each $u_{sg} \in U_{sg}$ ensures the cost to pay by the controller will not exceed the amount

$$C(x_0, u_{sg}) = \sup\{g(x[\cdot, x_0, u_{sg}, v(\cdot)]) : v(t) \in Q\} \tag{10}$$

and the value function of problem P equals $V(x_0) = \inf\{C(x_0, u_{sg}) : u_{sg} \in U_{sg}\}$. Finally, let $V_1(x)$ (resp. $V_2(x)$) be the minimal value of $g(x(\cdot))$ over all trajectories $x(\cdot)$ of equation (2) satisfying $w(0) = x$ with H being equal to H_1 (resp. H_2) satisfying condition (3_1) (resp. 3_2).

3. MAIN RESULTS

Let us start with the following observation.

REMARK 1. The spaces of the solutions of equations (1) and (2) are equibounded ($\|x(t)\| \le K_T$, $0 \le t \le T$). They are also equicontinuous as subsets of their closures that are compact in $C(0,T,R^n)$, the space of continuous mappings from $[0,T]$ into R^n with the max norm [3]. Since A is a stable matrix, the equicontinuity and boundedness of solutions of (1), (2) do not actually depend on T.

THEOREM 1. Assume that conditions (4)–(6) hold and perturbations $v(t) \in Q$ belong to any subclass of Lebesque measurable functions (for example, piecewise constant functions). Then, under condition (3_1), each simple step-guided strategy $\bar{u}_{sg} = (\bar{u}(x,w), \Delta, \bar{w}(\cdot))$, where $\bar{u}(x,w)$ is any selection from $U(x,w)$ given by (7), $\Delta = (\tau_i)$ is any finite partition of $[0,T]$, and $\bar{w}(\cdot)$ is any optimal trajectory for the simplified problem P_1, i.e., $V_1(x_0) = g(\bar{w}(\cdot))$ (ε-optimal trajectories would also suffice), guarantees the inequality

$$V(x_0) \le C(x_0, u_{sg}) \le V_1(x_0) + m(\bar{\varepsilon}(\delta)) \tag{11}$$

where $\bar{\varepsilon}(\delta) \to 0$ as the diameter δ of the partition Δ goes to zero and $m(\cdot)$ is the modulus of continuity of $g(\cdot)$. What is important, $\bar{\varepsilon}(\delta)$ does not depend on the length of the interval $[0,T]$.

PROOF. We are going to demonstrate the estimate (l_1, l_2 and $\beta(\delta)$ will be specified later)

$$\|x(t, x_0, \bar{u}_{sg}, v(\cdot)) - \bar{w}(t)\| \le \bar{\varepsilon}(\delta) = \sqrt{\frac{l_2}{l_1}\beta(\delta)},\ \beta(\delta) \to 0 \text{ as } \delta \to 0,\ 0 \le t \le T \tag{12}$$

which does not depend on a perturbation $v(t) \in Q$ and $T > 0$. The conclusion will readily follow from (12) since, by virtue of (4) and Remark 1, $g(x(\cdot))$ is uniformly continuous on the space of trajectories of equation (1).

To prove (12), set $s(t) = x(t) - \bar{w}(t)$, $x(t) = x[t, x_0, \bar{u}_{sg}, v(\cdot)]$ and compute $d/dtL(s(t))$, where $L(s) = \langle s, L^*s\rangle$ is a positive definite quadratic form with L^* being the unique solution of the matrix Lyapunov equation (L is symmetric): $A^TL^* + L^*A = -I$ (identity matrix). Take into consideration a subinterval $[\tau_i, \tau_{i+1})$ of the partition Δ. Observe that $\dot{s}(t) = As(t) + B\bar{u}(x(\tau_i), \bar{w}(\tau_i)) - Cv(t) - \bar{h}(t), \tau_i \le t < \tau_{i+1}$, where $\bar{h}(\cdot)$ gives rise to $\bar{w}(\cdot)$ via equation (2). Denoting by $\bar{v}(\cdot)$ a concrete perturbation function that influences equation (1), we obtain $d/dtLs(t) = s^T(t)[A^TL^* + L^*A]s(t) + 2\langle L^*s(t), B\bar{u}(x(\tau_i), \bar{w}(\tau_i)) - C\bar{v}(t) - \bar{h}(t)\rangle = -s^2(t) + 2\langle L^*s(t), B\bar{u}(x(\tau_i), \bar{w}(\tau_i)) - C\bar{v}(t) - \bar{h}(t)\rangle$. By virtue of (3_1), there exists a Lebesque measurable function $\bar{u}(t) \in P$ for which $B\bar{u}(t) = C\bar{v}(t) + \bar{h}(t)$. This fact enables us to conclude

$$d/dtL(s(t)) \le -s^2(t) + 2\langle L^*s(t), B\bar{u}_i(t)\rangle \tag{13}$$

where $u_i(t) = \bar{u}(x(\tau_i), \bar{w}(\tau_i)) - \bar{u}(t), \tau \le t < \tau_{i+1}$. To estimate the second term in (13), let us observe that by the choice of $\bar{u}(x,w)$ (see(7)), we have $\langle L^*s(\tau_i), Bu_i(t)\rangle \le 0$ and consequently $\langle L^*s(t), B\bar{u}_i(t)\rangle \le \langle L^*(s(t) - s(\tau_i)), B\bar{u}_i(t)\rangle$. If we set $p' = \max\{\|u\| : u \in P\}$ and $\beta_T(r) = \max\{2\|L^*\| \cdot \|s(t) - s(t')\| \cdot \|B\| \cdot p' : |t - t'| \le r\}$ we see that $\beta_T(r)$ tends to zero as r does (independently of i and a perturbation function $v(\cdot)$), which yields the inequality $d/dtL(s(t)) \le -s^2(t) + \beta_T(\delta)$, $0 \le t \le T$. Since all eigenvalues of matrix A are negative, $\|s(t) - s(t')\|$ does not really depend on T, so the last inequality may be written as

$$\frac{d}{dt}L(s(t)) \le -s^2(t) + \beta(\delta),\ 0 \le t \le T \tag{14}$$

Setting $D = \{t \in [0,T]\colon s^2(t) \le \beta(\delta)\}$ we see that the theorem is proved for $t \in D$. Observe that $(0,T)\backslash D$ is an open set, which means it is the union of at most countable number of open intervals (α_i, β_i), $i = 1, 2, \dots$. On each of these intervals, we have $d/dtL(s(t)) < 0$ and consequently $L(s(t)) < L(s(\alpha_i))$, $\alpha_i \le t \le \beta_i$. Since $s^2(\alpha_i) = \beta(\delta) = s^2(\beta_i)$, we derive the inequality

$$0 < L(s(t)) < \max\{L(z) : \|z\| = \sqrt{\beta(\delta)}\}, \quad t \in [0,T)\backslash D,$$

where $s(t) = x(t, x_0, \bar{u}_{sg}, v(\cdot)) - \tilde{w}(t)$. Since $L(x)$ is positive definite, for some constants l_1, l_2, the following inequalities hold: $l_1 s^2 \leq L(s(t)) \leq l_2 s^2$, $s \in R^n$, which yields $l_1 s^2(t) \leq \max\{L(z) : \|z\| = \sqrt{\beta(\delta)}\} \leq l_2 \beta(\delta)$, $t \in [0, T)$ D. Finally, we obtain the estimate (12) because $\beta_T(\delta)$ does not actually depend on T.

Inequality (11) gives an upper bound on the value function $V(x)$ of problem P, namely

$$V(x_0) = \inf_{u_{sg}} \sup_{v(\cdot)} g[x(\cdot, x_0, u_{sg}, v(\cdot)] \leq V_1(x_0), \tag{15}$$

where $V_1(x_0)$ is the value function of problem P_1 defined at the end of Section 1. The theorem below provides a lower bound on $V(x_0)$.

THEOREM 2. Assume that conditions (4)–(6) hold and perturbations $v(t) \in Q$ belong to the class of piecewise constant functions. Then, under condition (3_2),

$$V(x_0) = \inf_{u_{sg}} \sup_{v(\cdot)} g[x(\cdot, x_0, u_{sg}, v(\cdot)] \geq V_2(x_0). \tag{16}$$

We shall give a constructive proof of this theorem by proving an even stronger result (Theorem 3). Toward this end, we reverse the roles of the parameters u, v, by introducing a fictitious agent A, responsible for selecting parameter v with the aim of maximizing $g(x(\cdot))$. Agent A will be allowed to employ strategies $v_{sg} = (v(x, w), \Delta, w_R(\cdot))$ in the sense of Definition 1.

THEOREM 3. Assume that conditions (3_2), (4)–(6) hold. Then each step-guided strategy $\bar{v}_{sg} = (\bar{v}(x, w), \Delta, w_R(\cdot))$ of agent A for which $\bar{v}(x, w)$ is a selection from $V(x, w)$ given by (8), $\Delta = (\tau_i)$ is a finite partition of $[0, T]$ and $w_R(\cdot) = \tilde{w}(\cdot)$ is the solution of the equation $\dot{\tilde{w}}(t) = A\tilde{w}(t) + h_2(x(\tau_i), \tilde{w}(\tau_i))$, $\tau_i \leq t < \tau_{i+1}$, $\tilde{w}(0) = x_0$, guarantees the inequality

$$\inf_{u(t) \in P} g(x[\cdot, x_0, u(\cdot), \bar{v}_{sg}]) \geq V_2(x_0) - m(\varepsilon(\delta)) \tag{17}$$

with $\varepsilon(\delta) \to 0$ as $\delta \to 0$; here $h_2(x, w)$ is any selection from $H_2(x, w)$ given by (9), $u(t)$ is any measurable selection, $m(\cdot)$ is the modulus of continuity of $g(\cdot)$, and $\varepsilon(\delta)$ does not depend on the length of the interval $[0, T]$.

Before proving this theorem, let us notice that (17) implies an essentially stronger result than (16), by giving a formula for a perturbation function $v(\cdot)$ satisfying (16) (independently of u_{sg}!). This perturbation function is piecewise constant, which follows from the definition of the step-guided strategy $\bar{v}_{sg}$. Consequently, the following inequality

$$\sup_{v_{sg}} \inf_{u_{sg}} g[x(\cdot, x_0, u_{sg}, v_{sg})] \geq V_2(x_0)$$

implying inequality (16), holds true; it is known from elementary game theory that, for any kind of payoff functional P,

$$\inf_{\alpha \in A} \sup_{\beta \in B} P(\alpha, \beta) \geq \sup_{\beta \in B} \inf_{\alpha \in A} P(\alpha, \beta)$$

provided each pair (α, β) gives rise to an outcome in whatever sense. In the proof of Theorem 3 we shall use the Filippov lemma: If $k(t, u)$: $[a, b] \times U \to R$ is a continuous function and U is a compact set, then for each Lebesque measurable function $\psi(t) \in k(t, U)$, almost everywhere (a.e.) there is a Lebesque measurable function $u(t) \in U$ for which $\psi(t) = k(t, u(t))$ a.e.

PROOF. Similarly as in the proof of Theorem 1, it is enough to show that

$$\sup_{u(t) \in P} \|x(t, x_0, u(\cdot), \bar{v}_{sg}) - \tilde{w}(t)\| \leq \varepsilon(\delta), \quad 0 \leq t \leq T \tag{18}$$

with $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Setting $s(t) = x(t, x_0, u(\cdot), \bar{v}_{sg}) - \tilde{w}(t)$, let us compute $d/dt L(s(t))$, where L is the same as in the proof of Theorem 1. Take into consideration a subinterval $[\tau_i, \tau_{i+1})$ of the partition Δ and observe that, for $t \in [\tau_i, \tau_{i+1})$, we have $d/dt L(s(t)) = s^T(t)[A^T L^* +$

$L^*A]s(t)+2\langle L^*s(t), Bu(t)-C\bar{v}(x(\tau_i),\tilde{w}(\tau_i))\rangle-2\langle L^*s(t), h_2(x(\tau_i),\tilde{w}(\tau_i))\rangle$, with $\tilde{w}(0) = x(0) = x_0$. By virtue of (3_2), we have $Bu(t) \in CQ + H_2$. Applying the Filippov lemma, we arrive at $Bu(t) = Cv(t) + h(t)$, where $v(t)$ and $h(t)$ are Lebesque measurable selections from Q and H, respectively. Hence

$$\begin{aligned} d/dt L(s(t)) \leq & - s^2(t) + 2\langle L^*s(t), C[v(t) - \bar{v}(x(\tau_i),\tilde{w}(\tau_i))]\rangle \\ & + 2\langle L^*s(t), h(t) - h_2(x(\tau_i),\tilde{w}(\tau_i))\rangle, \quad \tau_i \leq t < \tau_{i+1}. \end{aligned} \tag{19}$$

To estimate the middle term in the right-hand side of (19), let us invoke (8) to conclude $\langle L^*s(\tau_i), C[v(t) - \bar{v}(x(\tau_i),\tilde{w}(\tau_i))]\rangle \leq 0$. By virtue of Remark 1 and assumption (5), the middle term in (19) is less than or equal to $\beta_1(\tau_{i+1} - \tau_i)$, where $\beta_1(s) \to 0$ as $s \to 0$ uniformly with respect to $u(\cdot)$, $v(\cdot)$) and T. A similar conclusion, with $\beta_1(\cdot)$ replaced by a function $\beta_2(\cdot)$, refers to the last term in (19), using the boundedness of H_2 and Remark 1. Letting $\beta_3(s) = \beta_1(s) + \beta_2(s)$ we arrive at

$$d/dt L(s(t)) \leq -s^2(t) + \beta_3(\tau_{i+1} - \tau_i), \quad \tau_i \leq t < \tau_{i+1} \tag{20}$$

Arguing exactly in the same manner as in the last part of the proof of Theorem 1 (starting with inequality (14)), we obtain (18) with $\varepsilon(\delta) = \bar{\varepsilon}(\delta)$, which completes the proof.

4. CONCLUSIONS

Coupled together, the better Theorems 1, 2 (as well as Theorems 1, 3) work, the smaller the difference $|V_1(x_0) - V_2(x_0)|$. In particular, when there is a set H satisfying $H + CQ = BP$, then $V_1(x_0) = V_2(x_0)$ for each point $x_0 \in R^n$ and the pairs $(\bar{u}_{sg}, \bar{v}_{sg})$ "produce" families of saddle points, i.e.,

$$V_1(x_0) - \varepsilon_1(\delta_1) \leq g(x[\cdot, x_0, \bar{u}_{sg}, \bar{v}_{sg}]) \leq V_1(x_0) + \varepsilon_2(\delta_2)$$

where $\varepsilon_1(\delta_1)$ and $\varepsilon_2(\delta_2)$ tend to zero with the diameters δ_1, δ_2 of the corresponding partitions Δ_1, Δ_2, respectively; here $\bar{u}_{sg} = (\bar{u}(x,w), \Delta_1, \bar{w}(\cdot))$ and $\bar{v}_{sg} = (\bar{v}(x,u), \Delta_2, w_R(\cdot))$ are defined in Theorem 1 and Theorem 3, respectively. Putting the thought above in a little different terms, one may say that Theorems 1, 2, and 3 enable one to obtain both lower and upper bounds for the value function of the original problem (P) assuming (3_1) and, clearly, the regularity conditions (4)–(6). In fact, when (3_1) holds then we have an upper bound by Theorem 1. On the other hand, one can always find an $\varepsilon > 0$ (the smaller the better) with the property $H_1^\varepsilon + CQ \supset BP$, where by X^ε, $X \subset R^n$, we mean the set $\{\bar{x} \in R^n : \|\bar{x} - x\| \leq \varepsilon$ for some $x \in X\}$. Condition (3_2) will be then satisfied with $H_2 = H_1^\varepsilon$, which will enable us to apply Theorem 2 in order to get a lower bound for $V(x_0)$.

Finally, based on Remark 1, we can extend the results above to the infinite horison setting $(T = +\infty)$. Practically everything remains unchanged; small differences are needed, however. They refer to the definition of an admissible partition $\Delta = (\tau_i)$ and assumption (4). Namely, by an admissible partition, we mean in this context any partition $\Delta = (\tau_i)$ with the property that each segment $[0, T]$ contains a finite number of partition points τ_i. As far as condition (4) is concerned, we require that $g(x(\cdot))$ be continuous on the space $C(0, \infty; R^n)$ equipped with the max norm.

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